Partial differential equations I, 2011/2012

V: Uniqueness of solutions

1. Let $U \subseteq \mathbb{R}^n$ be an open, bounded set with smooth boundary. For any $T > 0$ we set $U_T = U \times (0, T]$ and $\Gamma_T = U \times \{t = 0\} \cup \partial U \times [0, T)$. Show that the problem

$$u_{tt} - \Delta_x u = f \text{ in } U_T\,, \quad u = g \text{ on } \Gamma_T \quad \text{and} \quad u_t = h \text{ in } U \times \{t = 0\}$$

has at most one solution.

2. Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded set with smooth boundary. Show that the problem

$$\begin{aligned} u_{tt} - c^2 \Delta_x u &= -u \text{ in } \Omega \times \mathbb{R}_+\,, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega \times \{t = 0\} \\ \text{and} \quad u(x, 0) = u_t(x, 0) &= 0 \text{ for } x \in \Omega \end{aligned}$$

has at most one solution.

3. Let $u \in C([0, \infty) \times \mathbb{R}) \cap C^2((0, \infty) \times \mathbb{R})$ be the solution to the problem

$$u_{tt} - u_{xx} = 0 \text{ for } x \in \mathbb{R}, t > 0 \quad u(x, 0) = g(x) \quad \text{and} \quad u_t(x, 0) = h(h) \text{ for } x \in \mathbb{R}\,.$$

Assume that g and h have compact supports. We define

$$\begin{aligned} k(t) &= \frac{1}{2} \int_R u_x^2(x, t)\; dx \\ \text{and} \quad p(t) &= \frac{1}{2} \int_R u_t^2(x, t)\; dx\,. \end{aligned}$$

Show that

(a) k and p are well defined,

(b) $k(t) = p(t)$ for large t,

(c) $E(t) = k(t) + p(t)$ is constant.

4. Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded set with smooth boundary. Let $A \in \mathbb{R}^{n\times n}$ be a real, symmetric and positive definite matrix. Let u be a smooth solution to the problem

$$\begin{aligned} u_{tt} - \operatorname{div}(A\nabla u) = 0 \text{ in } \Omega \times \mathbb{R}_+\,, &\quad u = 0 \text{ on } \partial\Omega \times (0, \infty) \\ \text{and} \quad u(x, 0) = g(x)\,, &\quad u_t(x, 0) = h(x) \text{ for } x \in \Omega\,. \end{aligned}$$

Show that for each $t > 0$

$$\int_\Omega u_t^2(x, t) + \langle A\nabla u(x, t), \nabla u(x, t)\rangle\; dx = \int_\Omega h^2(x) + \langle A\nabla g(x), \nabla g(x)\rangle\; dx\,.$$

Then prove that the solution is unique.

5. Show that the problem

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } (0, \pi) \times \mathbb{R}_+\,, \\ u(\pi, t) = u(0, t) &= 0 \quad \text{for } t > 0 \\ \text{and} \quad u(x, 0) &= g(x) \quad \text{for } x \in (0, \pi) \end{aligned}$$

has at most one solution.

6. Show that the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{in } (0,\pi)\times\mathbb{R}_+\,, \\ u_x(\pi,t) = u_x(0,t) &= 0 \quad \text{for } t>0 \\ \text{and} \quad u(x,0) = g(x)\,, &\quad u_t(x,0) = h(x) \quad \text{for } x\in(0,\pi) \end{aligned}$$

has at most one solution.

7. Let u be a solution to

$$\begin{aligned} u_t - \Delta_x u &= |\nabla u|^2 \quad \text{for } x\in\mathbb{R}^n, t>0\,, \\ \text{and} \quad u(x,0) &= g(x) \quad \text{for } x\in\mathbb{R}^n\,, \end{aligned}$$

where g is a continuous and bounded function on $\mathbb{R}^n$.

(a) Find a function $\phi:\mathbb{R}\to\mathbb{R}$ so that $w=\phi\circ u$ is a solution to the heat equation. Find an explicit formula for u.

(b) Assume that $f>0$. Is it true that $u\geqslant 0$? Is the solution unique in this case?

Quick cheat sheet

1. (*d'Alemebert formula*) The solution to

$$u_{tt} - u_{xx} = 0\,, \quad u(x,0)=g(x) \quad \text{and} \quad u_t(x,0)=h(x) \quad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}_+$$

$$\text{is} \quad u(x,t) = \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ dy\,.$$

2. The solution to

$$u_{tt} - u_{xx} = f\,, \quad u(x,0)=0 \quad \text{and} \quad u_t(x,0)=0 \quad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}_+$$

$$\text{is} \quad u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\ dy\ ds\,.$$

3. (*Kirchhoff formula*) The solution to

$$u_{tt} - \Delta_x u = 0\,, \quad u(x,0)=g(x) \quad \text{and} \quad u_t(x,0)=h(x) \quad \text{for } (x,t)\in\mathbb{R}^3\times\mathbb{R}_+$$

$$\text{is} \quad u(x,t) = \fint_{\partial\mathbb{B}(x,t)} th(y) + g(y) + \langle Dg(y), y-x\rangle\ dS(y)\,.$$

4. (*Poisson formula*) The solution to

$$u_{tt} - \Delta_x u = 0\,, \quad u(x,0)=g(x) \quad \text{and} \quad u_t(x,0)=h(x) \quad \text{for } (x,t)\in\mathbb{R}^2\times\mathbb{R}_+$$

$$\text{is} \quad u(x,t) = \fint_{\mathbb{B}(x,t)} \frac{tg(y) + t^2h(y) + t\langle Dg(y), y-x\rangle}{\sqrt{t^2-|y-x|^2}}\ dy\,.$$

5. The solution to

$$u_t - \Delta_x u = 0\,, \quad u(x,0)=g(x) \quad \text{for } (x,t)\in\mathbb{R}^n\times\mathbb{R}_+$$

$$\text{is} \quad u(x,t) = (\Phi_t * g)(x) = \frac{1}{\sqrt{(4\pi t)^n}}\int_{\mathbb{R}^n} \exp\left(-\tfrac{|x-y|^2}{4t}\right) g(y)\ dy\,.$$

6. The solution to

$$u_t - \Delta_x u = f\,, \quad u(x,0)=0 \quad \text{for } (x,t)\in\mathbb{R}^n\times\mathbb{R}_+$$

$$\begin{aligned} \text{is} \quad u(x,t) &= \int_0^t (\Phi_{t-s} * f)(x)\ ds \\ &= \int_0^t \frac{1}{\sqrt{(4\pi(t-s))^n}}\int_{\mathbb{R}^n} \exp\left(-\tfrac{|x-y|^2}{4(t-s)}\right) f(y,s)\ dy\ ds\,. \end{aligned}$$