Partial differential equations I, 2011/2012

IV: Canonical forms.

Find diffeomorphisms $(x, y) \mapsto (\xi, \eta)$ such that the following equations

a) $2u_{xx} + 3u_{xy} + u_{yy} + 7u_x + 4u_y - 2u = 0$
b) $u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 15u_y + 27x = 0$
c) $(1 + x^2)^2 u_{xx} + u_{yy} + 2x(1 + x^2)u_x = 0$
d) $y^2 u_{xx} + 2xyu_{xy} + x^2 u_{yy} = 0$
e) $u_{xx} - (1 + y^2)^2 u_{yy} - 2y(1 + y^2)u_y = 0$
f) $xy^2 u_{xx} - 2x^2 yu_{xy} + x^3 u_{yy} - y^2 u_x = 0$
g) $e^{2x} u_{xx} + 2e^{x+y} u_{xy} + e^{2y} u_{yy} - xu = 0$
h) $u_{xx} + 2 \sin x u_{xy} - (\cos^2 x - \sin^2 x)u_{yy} + \cos x u_y = 0$
i) $u_{xx} + xyu_{yy} = 0$
j) $yu_{xx} + u_{yy} = 0$

take the canonical form when expressed in the variables (ξ, η). For each equation determine its type (elliptic, parabolic or hyperbolic). Note that the type may change at different points of $\mathbb{R}^2$.

Having an equation of the form

$$au_{xx} + 2bu_{xy} + cu_{yy} + f(u_x, u_y, u, x, y) = 0 \tag{1}$$

we set $\Delta = b^2 - ac$. For $\Delta < 0$, $\Delta = 0$ or $\Delta > 0$ we say that (1) is elliptic, parabolic or hyperbolic respectively. If $(x, y) \mapsto (\xi, \eta)$ is a diffeomorphism and $u(x, y) = v(\xi, \eta)$ then

$$\begin{aligned} au_{xx} + 2bu_{xy} + cu_{yy} + f(u_x, u_y, u, x, y) &= v_{\xi\xi}\left(a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2\right) \\ &+ v_{\xi\eta}\left(a\xi_x\eta_x + 2b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y\right) \\ &+ v_{\eta\eta}\left(a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2\right) + \tilde{f}(v_\xi, v_\eta, v, \xi, \eta)\,. \end{aligned}$$

If (1) is elliptic ($\Delta < 0$) then we have to solve

$$\begin{aligned} a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 &= 1 \\ a\xi_x\eta_x + 2b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y &= 0 \\ a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 &= 1\,, \end{aligned}$$

which is equivalent to

$$\begin{aligned} a\xi_x + b\xi_y + \sqrt{-\Delta}\eta_y &= 0 \\ a\eta_x + b\eta_y - \sqrt{-\Delta}\xi_y &= 0\,. \end{aligned}$$

If (1) is hyperbolic ($\Delta > 0$) then we have to solve

$$\begin{aligned} a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 &= 0 \\ a\xi_x\eta_x + 2b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y &= 1 \\ a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 &= 0\,, \end{aligned}$$

which should be equivalent to

$$
\begin{aligned}
a\xi_x + (b + \sqrt{\Delta})\xi_y &= 0 \\
a\eta_x + (b - \sqrt{\Delta})\eta_y &= 0\,.
\end{aligned}
$$

This way we obtain an equation of the form

$$
v_{\xi\eta} + \hat{f}(v_\xi, v_\eta, v, \xi, \eta) = 0\,,
$$

which in turn can be transformed to the canonical form by the substitution $\xi = \alpha + \beta$ and $\eta = \alpha - \beta$.

If (1) is parabolic ($\Delta = 0$) then we have to solve

$$
\begin{aligned}
a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 &= 1 \\
a\xi_x\eta_x + 2b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y &= 0 \\
a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 &= 0\,,
\end{aligned}
$$

so it suffices to find $\eta(x, y)$ such that

$$
a\eta_x + b\eta_y = 0
$$

and then ξ may be any function such that

$$
a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 \neq 0\,.
$$