Partial differential equations I, 2011/2012

Homework 1

Deadline: March 21, 2012

1. **[2p.]** Solve the equation
$$xuu_x - yuu_y = y^2 - x^2$$
knowing that the graph of the solution contains the curve[1] parameterized by
$$\phi(t) = \left(t, \tfrac{t}{2}, 0\right) .$$

2. **[2p.]** Solve the boundary value problem
$$\begin{aligned} u_{tt} - u_{xx} &= \cos x && \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) &= \tfrac{1}{2}x^2 && \text{for } x \in \mathbb{R} \\ u_t(x,0) &= \sin x && \text{for } x \in \mathbb{R} . \end{aligned}$$

3. **[3p.]** Solve the boundary value problem
$$\begin{aligned} u_{tt} - u_{xx} &= \cos(3x) && \text{for } (x,t) \in (0,\pi) \times \mathbb{R}_+ \\ u_x(0,t) = u_x(\pi,t) &= 0 && \text{for } t > 0 \\ u(x,0) &= 2\cos^2 x && \text{for } x \in (0,\pi) \\ u_t(x,0) &= 5\cos(2x) && \text{for } x \in (0,\pi) . \end{aligned}$$

4. **[4p.]** Find canonical forms[2] of
$$\begin{aligned} u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 15u_y + 27x &= 0 \\ xy^2u_{xx} - 2x^2yu_{xy} + x^3u_{yy} - y^2u_x &= 0 . \end{aligned}$$
Hint: Try not to calculate too much. Maybe you can guess the right change of variables.

5. **[5p.]** Let u be a solution to the following boundary value problem
$$\begin{aligned} u_{tt} - \Delta_x u &= 0 && \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ \\ u(x,0) &= g(x) && \text{for } x \in \mathbb{R}^3 \\ u_t(x,0) &= h(x) && \text{for } x \in \mathbb{R}^3 , \end{aligned}$$
where g and h are functions $\mathbb{R}^3 \to \mathbb{R}$ with compact support. Prove that there exists a constant $C > 0$ such that
$$|u(x,t)| \leqslant \tfrac{C}{t} \quad \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ .$$

[1] Erratum: Such solution does not exist. If $u(t, t/2) = 0$ then $LHS = 0$ and $RHS = -\frac{3}{4}$. A better boundary condition would be e.g. $u(t,t) = t$.

[2] A canonical form for an elliptic equation is $u_{xx} + u_{yy} + \text{l.o.t.} = 0$, for hyperbolic equation $u_{xx} - u_{yy} + \text{l.o.t.} = 0$ and for parabolic equation $u_x - u_{yy} + \text{l.o.t.} = 0$, where "l.o.t." stands for "lower order terms".