

# Reconfiguration on nowhere dense graph classes \*

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## Abstract

In the token jumping problem for a vertex subset problem  $\mathcal{Q}$  on graphs we are given a graph  $G$  and two feasible solutions  $S_s, S_t \subseteq V(G)$  of  $\mathcal{Q}$  with  $|S_s| = |S_t|$ , and imagine that a token is placed on each vertex of  $S_s$ . The problem is to determine whether there exists a sequence  $S_1, \dots, S_n$  of feasible solutions, where  $S_1 = S_s$ ,  $S_n = S_t$  and each  $S_{i+1}$  results from  $S_i$ ,  $1 \leq i < n$ , by moving exactly one token to another vertex.

We prove that for every nowhere dense class of graphs and for every  $r \geq 1$  there exists a polynomial  $p_r$  such that the token jumping problem for the distance- $r$  independent set problem and for the distance- $r$  dominating set problem admit kernels of size  $p_r(k)$ , where  $k$  is the number of used tokens. If  $k$  is equal to the size of a minimum distance- $r$  dominating set, then we even obtain a kernel of almost linear size  $\mathcal{O}(k^{1+\varepsilon})$  for any fixed  $\varepsilon > 0$ .

We then prove that if a class  $\mathcal{C}$  is somewhere dense and closed under taking subgraphs, then for some value of  $r \geq 1$  the reconfiguration problem for the above problems on  $\mathcal{C}$  is  $W[1]$ -hard (and in particular we cannot expect the existence of kernelization algorithms). Hence our results show that the limit of tractability for the reconfiguration variants of the distance- $r$  independent set problem and distance- $r$  dominating set problem on subgraph closed graph classes lies exactly on the boundary between nowhere denseness and somewhere denseness.

## 1 Introduction

In the reconfiguration framework we are not asked to find a feasible solution to an optimization problem, but rather to transform a source feasible solution  $S_s$  into a more desirable feasible target solution  $S_t$  such that each intermediate solution is also feasible. This framework allows to model real-world dynamic situations in which we need to transform one valid system state into another and it is crucial that the system keeps running in all intermediate states.

The reconfiguration question can naturally be formulated by defining the reconfiguration graph  $R_{\mathcal{Q}}(I)$  of an instance  $I$  of  $\mathcal{Q}$ . Its vertex set consists of all feasible solutions of  $\mathcal{Q}$  on instance  $I$  and there is an arc from solution  $S_1$  to solution  $S_2$  if we allow the transition from  $S_1$  to  $S_2$  by one or more reconfiguration rules. A reconfiguration sequence corresponds to a walk in the reconfiguration graph. Most results in the literature are limited to the problem of determining the existence of a reconfiguration sequence between two given solutions; an even more difficult problem is to find a (possibly minimum-length) reconfiguration sequence of solutions. Typically there are exponentially many feasible solutions to an instance  $I$ , and not surprisingly, the above problem has been shown to be PSPACE-complete for the reconfiguration variants of many important NP-complete problems.

Reconfiguration problems have received considerable attention in recent literature. The list of problems that has been studied includes VERTEX COLORING [1, 3, 4, 5, 7, 6], LIST EDGE-COLORING [22], VERTEX COVER [29, 31], INDEPENDENT SET [19, 21, 23], CLIQUE, SET COVER, INTEGER PROGRAMMING, MATCHING, SPANNING TREE, MATROID BASES [21],

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SATISFIABILITY [15, 28, 35], SHORTEST PATH [2, 25], SUBSET SUM [20], DOMINATING SET [17, 18, 30], ODD CYCLE TRANSVERSAL, FEEDBACK VERTEX SET, and HITTING SET [30]. We refer to the excellent survey by Van den Heuvel [36] for a detailed overview.

Recently, a systematic study of the parameterized complexity of reconfiguration problems was initiated by Mouawad et al. [30]. They study mostly graph theoretical vertex subset problems, that is, solutions consist of subsets  $S \subseteq V(G)$  of the input graph  $G$ . For such problems, one natural parameterization is the parameter  $k$ , a bound on the size of feasible solutions, another natural parameter is  $\ell$ , the length of the reconfiguration sequence. Note that when  $k$  is not bounded, the reconfiguration of many such problems, e.g. of the dominating set problem, becomes easy (first add the vertices of  $S_s \cup S_t$  one by one to  $S_s$  and then remove the vertices of  $S_t \setminus S_s$  to finally arrive at  $S_t$ ). They proved that FEEDBACK VERTEX SET and BOUNDED HITTING SET (where the cardinality of each input set is bounded) admit polynomial reconfiguration kernels (parameterized by  $k$ ). Concerning lower bounds, they proved that reconfiguration of DOMINATING SET is  $W[2]$ -hard when parameterized by  $k + \ell$ , as well as a general result on reconfiguration of hereditary properties and their parametric duals, implying  $W[1]$ -hardness of reconfiguration of INDEPENDENT SET, INDUCED FOREST and BIPARTITE SUBGRAPH parameterized by  $k + \ell$ , and VERTEX COVER, FEEDBACK VERTEX SET, and ODD CYCLE TRANSVERSAL parameterized by  $\ell$ .

INDEPENDENT SET reconfiguration if the solutions appearing in the reconfiguration sequence are only allowed to be of size  $k$  and  $k - 1$  is equivalent to reconfiguration under the token jumping model considered by Ito et al. [23, 24]. In the *token jumping problem* for a vertex subset problem  $\mathcal{Q}$  on graphs we are given a graph  $G$  and two feasible solutions  $S_s, S_t \subseteq V(G)$  of  $\mathcal{Q}$  with  $|S_s| = |S_t|$ , and imagine that a token is placed on each vertex of  $S_s$ . The problem is to determine whether there exists a sequence  $S_1, \dots, S_n$  of feasible solutions, where  $S_1 = S_s$ ,  $S_n = S_t$  and each  $S_{i+1}$  results from  $S_i$ ,  $1 \leq i < n$ , by moving exactly one token to another vertex. The token jumping problem for INDEPENDENT SET is known to be PSPACE-complete in this setting on graphs of bounded bandwidth [31, 37] (hence pathwidth and treewidth) and  $W[1]$ -hard on general graphs [24]. On the positive side, the problem was shown to be fixed-parameter tractable, with parameter  $k$ , for graphs of bounded degree, planar graphs, and graphs excluding  $K_{3,d}$  as a subgraph, for any constant  $d$  [23, 24]. This result was extended by Lokshantov et al. [27] to graphs of bounded degeneracy and nowhere dense graphs. Lokshantov et al. also proved that dominating set reconfiguration (in the token jumping model) is  $W[1]$ -hard on general graphs and fixed-parameter tractable, with parameter  $k$ , for graphs excluding  $K_{d,d}$  as a subgraph, for any constant  $d$  (in particular on all degenerate graph classes and nowhere dense graph classes).

Nowhere dense graph classes, which are also the object of study in the present paper, are very general classes of uniformly sparse graphs [32, 33]. Many familiar classes of sparse graphs, like planar graphs, graphs of bounded treewidth, graphs of bounded degree, and, in fact, all classes that exclude a fixed (topological) minor, are nowhere dense. Notably, classes of bounded average degree or bounded degeneracy are not necessarily nowhere dense. In an algorithmic context this is reasonable, as every graph can be turned into a graph of degeneracy at most 2 by subdividing every edge once; however, the structure of the graph is essentially preserved under this operation. In our context, a particularly interesting algorithmic result states that every first-order definable property of graphs can be decided in almost linear time on nowhere dense graph classes [16]. This result implies that the reconfiguration variants of many of the above mentioned vertex subset problems are fixed-parameter tractable with respect to parameter  $k + \ell$  on every nowhere dense graph class (the existence of a reconfiguration sequence can be expressed with  $\mathcal{O}(k \cdot \ell)$  quantifiers in

first-order logic, whenever the property itself can be defined by a first-order formula), and by the result of [16] be decided in fixed-parameter time.

Nowhere dense graph classes play a special role for DOMINATING SET and its more general variant DISTANCE- $r$  DOMINATING SET. A distance- $r$  dominating set in a graph  $G$  (for a fixed integer parameter  $r$ ) is a set  $D \subseteq V(G)$  such that every vertex of  $G$  is at distance at most  $r$  to a vertex from  $D$ . DISTANCE- $r$  DOMINATING SET was shown to be fixed-parameter tractable on nowhere dense classes in [9] (this result is again implied by the more general result of [16] which was obtained later). It was then shown that nowhere dense classes are the limit of tractability based on sparsity methods, more precisely, it was shown in [12] that if a class  $\mathcal{C}$  is not nowhere dense and closed under taking subgraphs, then there is some  $r \geq 1$  such that DISTANCE- $r$  DOMINATING SET on  $\mathcal{C}$  is W[2]-hard. It was later shown that the problem admits a polynomial kernel [26] and in fact an almost linear kernel [13] on nowhere dense classes. A kernelization algorithm, or just a kernel, is a polynomial time algorithm which transforms every input instance  $(G, k)$  to an equivalent instance  $(G', k')$  such that  $|G'| + k' \leq f(k)$  for some function  $f$ . Hence for a token jumping problem a kernelization algorithm is a polynomial time algorithm which transforms every input instance  $(G, k, S_s, S_t)$  into an instance  $(G', k', S'_s, S'_t)$  with  $|G'| + k' \leq f(k)$  for some function  $f$  and such that there exists a valid reconfiguration sequence  $S_1 = S_s, S_2, \dots, S_n = S_t$  in  $G$  if and only if there exists a valid reconfiguration sequence  $S'_1 = S'_s, S'_2, \dots, S'_m = S'_t$  in  $G'$ . Every fixed-parameter tractable problem admits a kernel, however, possibly of exponential or worse size. For efficient algorithms it is therefore most desirable to obtain polynomial, or optimally even linear, kernels.

**Our results.** We prove that for every nowhere dense class of graphs and for every  $r \geq 1$  there exists a polynomial  $p_r$  such that the token jumping problem for the distance- $r$  independent set problem and for the distance- $r$  dominating set problem admit kernels of size  $p_r(k)$ , where  $k$  is the number of used tokens. If  $k$  is equal to the size of a minimum distance- $r$  dominating set, then we even obtain a kernel of almost linear size  $\mathcal{O}(k^{1+\varepsilon})$  for any fixed  $\varepsilon > 0$ .

Unfortunately, for DISTANCE- $r$  DOMINATION SET there is a technical subtlety that prevents us from reducing the input instance  $(G, k, I_s, I_t)$  to an equivalent instance  $(G', k, I_s, I_t)$  such that  $G'$  is a subgraph of  $G$ . Instead, we can kernelize to an annotated version of the problem, where only a given subset of vertices of  $G'$  needs to be dominated, or we can output an instance  $(G', k, I_s, I_t)$ , where  $G'$  does not belong to the class  $\mathcal{C}$  under consideration (it's density parameters are only slightly larger than those of  $G$  though). Formally, in any case, we do not reduce to the same problem, hence we compute only a so-called *bi-kernel* for the problem. Our results generalize the earlier mentioned results of Lokshtanov et al. [27], who proved that INDEPENDENT SET reconfiguration (i.e. the case  $r = 1$ ) is fixed-parameter tractable on every nowhere dense graph class and DOMINATING SET (i.e. the case  $r = 1$ ) is fixed-parameter tractable if the input graph does not contain large complete bipartite graphs (as a subgraph), in particular on all nowhere dense graph classes.

Our methods for DISTANCE- $r$  INDEPENDENT SET generalize those of Lokshtanov et al. [27] to the more general setting of distance- $r$  independence. They are strongly based on the equivalence of nowhere denseness and uniform quasi-wideness (a notion that will be defined in the next section) and polynomial bounds for the quasi-wideness functions which were recently obtained by Kreuzer et al. [26] and Pilipczuk et al. [34].

Our methods for DISTANCE- $r$  DOMINATING SET combine the approach of Lokshtanov et al. [27] for DOMINATING SET with new methods developed for the kernelization of DISTANCE- $r$  DOMINATING SET on nowhere dense graph classes by Eickmeyer et al. [13].

We then prove that if a class  $\mathcal{C}$  is somewhere dense and closed under taking subgraphs, then for some value of  $r \geq 1$  the reconfiguration variants for these problems on  $\mathcal{C}$  is  $W[1]$ -hard (and in particular we cannot expect the existence of kernelization algorithms). Hence our results show that the limit of tractability for the reconfiguration variants of DISTANCE- $r$  INDEPENDENT SET and DISTANCE- $r$  DOMINATING SET on subgraph closed graph classes lies exactly on the boundary between nowhere denseness and somewhere denseness.

Our hardness results are rather straight forward generalizations of the  $W[1]$ -hardness proofs known for INDEPENDENT SET and DOMINATING SET to their distance- $r$  variants.

## 2 Preliminaries

► **Graphs.** All graphs in this paper are finite, undirected and simple. Our notation is standard, we refer to the textbook [10] for more background on graphs. We write  $V(G)$  for the vertex set of a graph  $G$  and  $E(G)$  for its edge set. For a positive integer  $r$ , a graph  $G$  and  $v \in V(G)$  we write  $N_r(v)$  for the set of vertices of  $G$  at distance at most  $r$  from  $v$ . The radius of a graph  $G$  is the minimum integer  $r$  such that there is a vertex  $v \in V(G)$  with  $N_r(v) = V(G)$ .

► **Distance- $r$  independence and distance- $r$  domination.** Let  $G$  be a graph and let  $r \in \mathbb{N}$ . A set  $B \subseteq V(G)$  is called  *$r$ -independent* in  $G$  if for all distinct  $u, v \in B$  we have  $\text{dist}_G(u, v) > r$ . The set  $B$  is a *distance- $r$  dominating set* in  $G$  if  $N_r(B) = \bigcup_{v \in B} N_r(v) = V(G)$ .

► **Bounded-depth minors and subdivisions.** Let  $G$  be a graph and let  $r \in \mathbb{N}$ . A graph  $H$  with vertex set  $\{v_1, \dots, v_n\}$  is a *depth- $r$  minor* of  $G$ , written  $H \preceq_r G$ , if there are connected and pairwise vertex disjoint subgraphs  $H_1, \dots, H_n \subseteq G$ , each of radius at most  $r$ , such that if  $v_i v_j \in E(H)$ , then there are  $w_i \in V(H_i)$  and  $w_j \in V(H_j)$  with  $w_i w_j \in E(G)$ .

An  *$r$ -subdivision* of  $H$  is obtained by replacing edges of  $H$  by internally vertex disjoint paths of length (exactly)  $r$ . We write  $H_r$  for the  $r$ -subdivision of  $H$ .

► **Nowhere denseness.** A class  $\mathcal{C}$  of graphs is *nowhere dense* if there exists a function  $t: \mathbb{N} \rightarrow \mathbb{N}$  such that  $K_{t(r)} \not\preceq_r G$  for all  $r \in \mathbb{N}$  and for all  $G \in \mathcal{C}$ . Otherwise,  $\mathcal{C}$  is called *somewhere dense*.

We will work mainly with the following equivalent characterization of nowhere dense graph classes.

► **Quasi-wideness.** A class  $\mathcal{C}$  of graphs is called *uniformly quasi-wide* if there are functions  $N: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  and  $s: \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $r, m \in \mathbb{N}$  and all subsets  $A \subseteq V(G)$  for  $G \in \mathcal{C}$  of size  $|A| \geq N(r, m)$  there is a set  $S \subseteq V(G)$  of size  $|S| \leq s(r)$  and a set  $B \subseteq A \setminus S$  of size  $|B| \geq m$  which is  $r$ -independent in  $G - S$ . The functions  $N$  and  $s$  are called the *margins* of the class  $\mathcal{C}$ .

It was shown by Nešetřil and Ossona de Mendez [33] that a class  $\mathcal{C}$  of graphs is nowhere dense if and only if it is uniformly quasi-wide. Quasi-wideness is a very useful property for distance- $r$  domination, as large  $2r$ -independent sets are natural obstructions for small distance- $r$  dominating sets. For us it will be important that the function  $N$  can be assumed to be polynomial in  $m$  (the degree of the polynomial may depend on  $r$ ) and that the sets  $B$  and  $S$  can be efficiently computed. Polynomial bounds were first obtained by Kreutzer et al. [26], we refer to the improved bounds of Pilipczuk et al. [34].

► **Lemma 1** (Pilipczuk et al. [34]). *Let  $\mathcal{C}$  be a nowhere dense class of graphs. For all  $r \in \mathbb{N}$  there is a polynomial  $N_r: \mathbb{N} \rightarrow \mathbb{N}$  and a constant  $t_r \in \mathbb{N}$ , such that the following holds. Let  $G \in \mathcal{C}$  and let  $A \subseteq V(G)$  be a vertex subset of size at least  $N_r(m)$ , for a given  $m$ . Then there exists a set  $S \subseteq V(G)$  of size  $|S| \leq t_r$  and a set  $B \subseteq A \setminus S$  of size  $|B| \geq m$  which is  $r$ -independent in  $G - S$ . Moreover, given  $G$  and  $A$ , such sets  $S$  and  $B$  can be computed in time  $\mathcal{O}(|A| \cdot |E(G)|)$ .*

We remark that the  $\mathcal{O}$ -notation in the above lemma hides constant factors depending on  $r$  (which is considered fixed) and the class  $\mathcal{C}$ .

We need a precise description of how vertices interact with sets in their  $r$ -neighborhood.

►  **$A$ -avoiding paths.** Let  $G$  be a graph and let  $A \subseteq V(G)$  be a subset of vertices. For vertices  $v \in A$  and  $u \in V(G) \setminus A$ , a path  $P$  connecting  $u$  and  $v$  is called  *$A$ -avoiding* if all its vertices apart from  $v$  do not belong to  $A$ .

► **Projection profiles.** Let  $G$  be a graph,  $A \subseteq V(G)$  and  $r \in \mathbb{N}$ . The  *$r$ -projection* of a vertex  $u \in V(G) \setminus A$  onto  $A$ , denoted  $M_r^G(u, A)$  is the set of all vertices  $v \in A$  that can be connected to  $u$  by an  $A$ -avoiding path of length at most  $r$ . The  *$r$ -projection profile* of a vertex  $u \in V(G) \setminus A$  on  $A$  is the function  $\rho_r^G[u, A]$  mapping vertices of  $A$  to  $\{0, 1, \dots, r, \infty\}$ , defined as follows: for every  $v \in A$ , the value  $\rho_r^G[u, A](v)$  is the length of a shortest  $A$ -avoiding path connecting  $u$  and  $v$ , and  $\infty$  in case this length is larger than  $r$ . We define

$$\widehat{\mu}_r(G, A) = |\{\rho_r^G[u, A] : u \in V(G) \setminus A\}|$$

to be the number of different  $r$ -projection profiles realized on  $A$ , respectively.

► **Lemma 2** (Eickmeyer et al. [13]). *Let  $\mathcal{C}$  be a nowhere dense class of graphs. Then there is a function  $f_{\text{proj}}(r, \varepsilon)$  such that for every  $r \in \mathbb{N}$ ,  $\varepsilon > 0$ , graph  $G \in \mathcal{C}$ , and vertex subset  $A \subseteq V(G)$ , it holds that  $\widehat{\mu}_r(G, A) \leq f_{\text{proj}}(r, \varepsilon) \cdot |A|^{1+\varepsilon}$ .*

► **Parameterized complexity.** A problem is fixed-parameter tractable on a class  $\mathcal{C}$  of graphs with respect to an input parameter  $k$ , if there is an algorithm deciding whether a graph  $G \in \mathcal{C}$  admits a solution of size  $k$  in time  $f(k) \cdot |V(G)|^c$ , for some computable function  $f$  and constant  $c$ . A powerful method in parameterized complexity theory is to compute a problem kernel in a polynomial time pre-computation step, that is, to reduce the input instance to a sub-instance of size bounded in the parameter only (independently of the input graph size). Every fixed-parameter tractable problem has a kernel, however, possibly of exponential or worse size. For efficient algorithms it is therefore most desirable to obtain polynomial, or optimally even linear, kernels.

The W-hierarchy is a collection of parameterized complexity classes  $\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \dots$ . The assumption  $\text{FPT} \subsetneq \text{W}[1]$  can be seen as the analogue of the conjecture that  $\text{P} \subsetneq \text{NP}$ . Therefore, showing hardness in the parameterized setting is usually accomplished by establishing an *fpt*-reduction to a  $\text{W}[1]$ -hard problem. We refer to the textbooks [8, 11, 14] for extensive background on parameterized complexity.

► **Reconfiguration.** We consider the token jumping variants of the  $\text{DISTANCE-}r$  INDEPENDENT SET RECONFIGURATION problem ( $r$ -ISR) and  $\text{DISTANCE-}r$  DOMINATING SET RECONFIGURATION problem ( $r$ -DSR) which are formally defined as follows.

**DISTANCE- $r$  INDEPENDENT SET RECONFIGURATION**

*Input:* Graph  $G$ , positive integer  $k$ , distance- $r$  independent sets  $I_s, I_t \subseteq V(G)$  with  $|I_s| = |I_t| = k$ ,

*Parameter:*  $k$

*Problem:* Does there exist a sequence  $I_1, \dots, I_\ell$  for some  $\ell$  with

1.  $I_1 = I_s$  and  $I_\ell = I_t$ ,
2.  $I_j$  is a distance- $r$  independent set in  $G$  for all  $1 \leq j \leq \ell$ ,
3.  $|(I_j \setminus I_{j+1}) \cup (I_{j+1} \setminus I_j)| \leq 1$  for all  $1 \leq j \leq \ell - 1$ , and
4.  $k - 1 \leq |I_j| \leq k$  for all  $1 \leq j \leq \ell$ .

The DISTANCE- $r$  DOMINATING SET RECONFIGURATION problem is defined analogously, we only demand that in the forth item we have  $k \leq |D_j| \leq k + 1$  for the appearing distance- $r$  dominating sets  $D_j$ ,  $1 \leq j \leq \ell$ . We obtain positive results also for the variants where for  $r$ -ISR we get as input two integer parameters  $k, k'$  and we replace the fourth condition by  $k \leq |I_j| \leq k'$  for all  $1 \leq j \leq \ell$ . For  $r$ -DSR we may remove the condition on a lower bound completely, that is, in the forth condition demand only that  $|D_j| \leq k + 1$  for all  $i \leq j \leq \ell$ .

### 3 Distance- $r$ independent set reconfiguration

#### 3.1 Polynomial kernel

Our approach for kernelization of  $r$ -ISR is similar to that of Lokshtanov et al. [27]. We iteratively remove *irrelevant vertices* from the input instance, until this is no longer possible, in which case the resulting instance will be small.

► **Irrelevant vertex.** Let  $(G, I_s, I_t, k)$  be an instance of the distance- $r$  independent set reconfiguration problem. A vertex  $v \in V(G) \setminus (I_s \cup I_t)$  is called an *irrelevant vertex* if  $(G, k, I_s, I_t)$  is a positive instance if and only if  $(G - v, k, I_s, I_t)$  is a positive instance.

► **Lemma 3.** *Let  $\mathcal{C}$  be a nowhere-dense class of graphs and let  $r \in \mathbb{N}$ . Let  $N = N_{2r} : \mathbb{N} \times \mathbb{N}$  be the function and  $t = t_{2r} \in \mathbb{N}$  be the constant for  $\mathcal{C}$  describing  $\mathcal{C}$  as uniformly quasi-wide (for parameter  $2r$ ) as defined in Lemma 1. Let  $(G, k, I_s, I_t)$  for  $G \in \mathcal{C}$  be an instance of the distance- $r$  independent set reconfiguration problem and let  $R := V(G) \setminus (I_s \cup I_t)$ . For  $u, v \in R$  define  $u \cong v$  if  $\rho_r^G[u, I_s \cup I_t] = \rho_r^G[v, I_s \cup I_t]$ . If there is an equivalence class  $\kappa$  with  $|\kappa| > N_{2r}(r + 2)^t(k + 2)$ , then  $\kappa$  contains an irrelevant vertex.*

**Proof.** Assume there is an equivalence class  $\kappa$  with  $|\kappa| > N((r + 2)^t(k + 2))$ . Using Lemma 1, we compute a set  $S \subseteq V(G)$  with  $|S| \leq t$  and a set  $B \subseteq \kappa$  with  $|B| \geq (r + 2)^t(k + 2)$  which is  $2r$ -independent in  $G - S$ . Now, for  $u, v \in B$ , let  $u \cong_S v$  if  $\rho_r^G[u, S] = \rho_r^G[v, S]$ . Note that there are at most  $(r + 2)^t$  distinct projection profiles to  $S$ , as  $|S| \leq t$  and  $\rho_r^G[u, S]$  is a mapping from  $S$  to  $\{0, 1, \dots, r, \infty\}$ .

Since  $|B| \geq (r + 2)^t(k + 2)$ , we know that at least one equivalence class  $\kappa_S$  (of equivalence relation  $\cong_S$ ) contains at least  $k + 2$  vertices of  $S$ . Let  $v \in \kappa_S$ . We claim that  $v$  is an irrelevant vertex. Note that  $v$  does not belong to  $I_s \cup I_t$ .

Consider a reconfiguration sequence  $I_s = I_1, I_2, \dots, I_t = I_\ell$  from  $I_s$  to  $I_t$  in  $G$  with a minimum number of occurrences of  $v$ . We want to prove that  $v$  does not occur at all in the sequence, as this proves that  $v$  is irrelevant. Towards a contradiction assume that  $v$  does occur in the sequence and let  $p, 1 < p < \ell$ , be the first index at which  $v$  appears in  $I_p$  (that is,  $v \in I_p$  and  $v \notin I_i$  for all  $i < p$ ). Let  $q + 1, p < q + 1 \leq \ell$  be the first index after  $p$  at which  $v$  is removed (that is,  $v \in I_q$  and  $v \notin I_{q+1}$ ). We will modify the sub-sequence  $I_p, \dots, I_q$  such that it does not use  $v$ , contradicting our choice of a reconfiguration sequence with a minimum number of occurrences of  $v$ .

Enumerate the vertices of  $\kappa_S \setminus \{v\}$  as  $w_1, w_2, \dots, w_{k+1}, \dots$ . Fix some  $j, p \leq j \leq q$ , and let  $I = I_j \setminus \{v\}$ . Assume there is  $z \in I$  with  $\text{dist}_{G-S}(w_i, z) \leq r$ . We claim that in this case  $\text{dist}_G(w_\ell, z) > r$  for all  $\ell \neq j$ . Because  $B$  is  $2r$ -independent in  $G - S$  and  $\text{dist}_{G-S}(w_i, z) \leq r$  we have  $\text{dist}_{G-S}(w_\ell, z) > r$  for  $\ell \neq j$ . Hence, if  $\text{dist}_G(w_\ell, z) \leq r$ , then there are  $s_1, s_2 \in S$  (possibly  $s_1 = s_2$ ) such that  $\text{dist}_{G-(S \setminus \{s_1\})}(z, s_1) + \text{dist}_G(s_1, s_2) + \text{dist}_{G-(S \setminus \{s_2\})}(s_2, w_\ell) \leq r$ . However, as  $\rho_r^G[v, S] = \rho_r^G[w_2, S]$  we have  $\text{dist}_{G-(S \setminus \{s_1\})}(z, s_1) + \text{dist}_G(s_1, s_2) + \text{dist}_{G-(S \setminus \{s_2\})}(s_2, v) \leq r$ , which implies  $\text{dist}_G(z, v) \leq r$ , contradicting that  $I_j$  is a distance- $r$  independent set. Similarly, there does not exist  $z \in I$  with both  $\text{dist}_G(z, w_i) \leq r$

for some  $i$  and  $\text{dist}_{G-S}(z, w_i) > r$  for all  $i$ , as again in this case  $\text{dist}_G(z, v) \leq r$ . As the set  $I$  contains at most  $k - 1$  elements we conclude by the pigeon hole principle that there is  $w \in \kappa_S \setminus \{v\}$  with  $I \cap N_r(w) = \emptyset$ .

We now modify the sequence  $I_p, \dots, I_q$  as follows. For each  $j$ ,  $p \leq j \leq q$ , choose  $w^j \in \kappa_S \setminus \{v\}$  such that  $I_j \setminus \{v\} \cap N_r(w^j) = \emptyset$  (which exists by our above argument). Note that we have  $|I_p| = |I_q| = k$ , as  $v$  was introduced at  $I_p$  and removed at  $I_{q+1}$ . We replace each sub-sequence  $I_j, I_{j+1}$ ,  $j = p, p + 2, \dots, q$ , by the sub-sequence  $J_j, J_j^1, J_j^2, J_{j+1}$ , where

- $J_j = (I_j \setminus \{v\}) \cup \{w^j\}$ ,
- $J_j^1 = J_j \setminus \{w^j\}$ ,
- $J_j^2 = J_j^1 \cup \{w^{j+1}\}$ ,
- $J_{j+1} = J_j^2 \setminus (I_j \setminus I_{j+1})$ .

By our above argument, each of the intermediate configurations is a distance- $r$  independent set. Furthermore, the transition  $I_{p-1}, J_p$  is valid, so is every intermediate transition and the transition  $J_q, I_{q+1}$ . This finishes the proof of the lemma. ◀

► **Theorem 4.** *Let  $\mathcal{C}$  be a nowhere dense class of graphs and let  $r \in \mathbb{N}$ . Let  $(G, k, I_s, I_t)$  be an instance of the distance- $r$  independent set reconfiguration problem, where  $G \in \mathcal{C}$ . Then we can compute in polynomial time a subgraph  $G' \subseteq G$  with  $I_s, I_t \subseteq G'$  such that  $(G, k, I_s, I_t)$  is a positive instance if and only if  $(G', k, I_s, I_t)$  is a positive instance and  $G'$  has order polynomial in  $k$ .*

**Proof.** Iteratively apply Lemma 3 until this is no longer possible. Fix some  $\varepsilon > 0$  and let  $f_{\text{proj}}(r, \varepsilon)$  be the function from Lemma 2. According to the lemma there are only  $f_{\text{proj}}(r, \varepsilon) \cdot (k^{1+\varepsilon})$  projection classes and according to Lemma 3 we can find an irrelevant vertex as long as some class contains more than  $N((r + 2)^t(k + 2))$  vertices. We hence find a kernel of size  $f_{\text{proj}}(\varepsilon, r) \cdot N((r + 2)^t(k + 2)) \cdot k^{1+\varepsilon}$ , which is polynomial in  $k$  for each fixed value of  $r$ . ◀

It is easy to see that we can carry out the same proof for the reconfiguration variant where we get as input two integer parameters  $k, k'$  and we replace the fourth condition by  $k \leq |I_j| \leq k'$  for all  $1 \leq j \leq \ell$ . The kernel will have size polynomial in  $k'$ .

### 3.2 Lower bounds

Recall that for a graph  $G$  and  $r \in \mathbb{N}$ ,  $G_r$  denotes the  $r$ -subdivision of  $G$ .

► **Lemma 5** (Nešetřil and Ossona de Mendez [33], see also [12]). *Let  $\mathcal{C}$  be somewhere dense and closed under taking subgraphs. Then there is  $r \in \mathbb{N}$  such that for all graphs  $G$  we have  $G_r \in \mathcal{C}$ .*

► **Lemma 6** (Ito et al. [24]). *The token jumping variant of DISTANCE-1 INDEPENDENT SET RECONFIGURATION is  $W[1]$ -hard.*

► **Theorem 7.** *Let  $\mathcal{C}$  be somewhere dense and closed under taking subgraphs. Then there is  $r \in \mathbb{N}$  such that DISTANCE- $r$  INDEPENDENT SET RECONFIGURATION is  $W[1]$ -hard on  $\mathcal{C}$ .*

**Proof.** According to Lemma 5, there is  $s \in \mathbb{N}$  such that for all graphs  $G$  we have  $G_s \in \mathcal{C}$ . We reduce 1-ISR to  $(4s - 1)$ -ISR on  $\mathcal{C}$  by establishing the following. For each graph  $G$  there exists a polynomial time computable graph  $H \in \mathcal{C}$  such that  $V(G) \subseteq V(H)$  and such that

1. every independent set  $I$  in  $G$  is a  $(4s - 1)$ -independent set in  $H$  and

2. every  $(4s - 1)$ -independent set  $I$  of size at least 2 in  $H$  consists only of vertices which are also vertices of  $G$  and  $I$  is an independent set in  $G$ .

Note that we may assume that the parameter  $k$  is always at least 2. The above properties guarantee that every reconfiguration sequence  $I_1, \dots, I_\ell$  of independent sets in  $G$  corresponds uniquely to a reconfiguration sequence of distance- $(4s-1)$  independent sets in  $H$  and vice versa. Hence, we can conclude the statement of the theorem for  $r = 4s - 1$  by applying [Lemma 6](#). Note that the reduction also establishes  $W[1]$ -hardness of DISTANCE- $r$  INDEPENDENT SET on somewhere dense graph classes which are closed under taking subgraphs.

Let  $G$  be a graph with vertex set  $\{v_1, \dots, v_n\}$  and edge set  $\{e_1, \dots, e_m\}$ . We remark that the hardness result of [Lemma 6](#) works also if we assume that all input graphs do not have isolated vertices, so we may assume that  $G$  does not contain isolated vertices. We define the new graph  $J$  with vertex set

$$\{v_1, \dots, v_n, e_1, \dots, e_m, w\}$$

and edge set

$$\{ve : v \in V(G), e \in E(G), v \in e\} \cup \{ew : e \in E(G)\}.$$

We claim that  $H = J_s$  satisfies the above claimed properties.

Let  $I$  be a distance-1 independent set in  $G$ . By construction of  $J$ , if  $u, v \in V(G)$  are adjacent in  $G$ , then they have distance  $2s$  in  $J_s$ , otherwise they have distance  $4s$  in  $J_s$  (via the vertex  $w$ ). Hence,  $I$  is a distance- $(4s - 1)$  independent set in  $J_s$ .

Conversely, let  $I$  be a distance- $(4s - 1)$  independent set in  $J_s$  of size at least 2. First observe that  $I$  consists only of vertices which are also vertices of  $G$ . All other vertices have mutual distance at most  $4s - 1$  via the vertex  $w$ . As seen above, the elements of  $I$  have distance  $4s$  in  $J_s$  and therefore distance at least 2 in  $H$ , that is,  $I$  is an independent set in  $H$ . This finishes the proof.  $\blacktriangleleft$

## 4 Distance- $r$ dominating set reconfiguration

### 4.1 Polynomial kernel

The kernelization for the reconfiguration of distance- $r$  domination strongly depends on the following notion of a domination core which was (in different variants) also used in the earlier kernelization results for distance- $r$  dominating sets [[9](#), [12](#), [13](#)].

► **Domination core.** Let  $G$  be a graph and  $k, r \geq 1$ . A set  $Z \subseteq V(G)$  is a  $(k, r)$ -domination core if every set  $D$  of size at most  $k$  that  $r$ -dominates  $Z$  also  $r$ -dominates  $G$ .

Clearly,  $V(G)$  is a  $(k, r)$ -domination core. Hence, starting with  $Z = V(G)$ , using the next lemma, we can gradually remove vertices from  $Z$  while maintaining the invariant that  $Z$  is a  $(k, r)$ -domination core. The proof of the lemma is the same as the proof of [Lemma 11](#) in [[9](#)] and [Lemma 4.1](#) in [[26](#)], we just use the better bounds from [Lemma 1](#).

► **Lemma 8.** *Let  $\mathcal{C}$  be a nowhere dense class of graphs and let  $k, r \geq 1$ . Let  $N = N_{2r}$  and  $t = t_{2r}$  be the functions characterizing  $\mathcal{C}$  as uniformly quasi-wide according to [Lemma 1](#) with parameter  $2r$ . There is an algorithm that, given a graph  $G \in \mathcal{C}$ ,  $k \geq 1$  and  $Z \subseteq V(G)$  with  $|Z| > N((k + 2)(2r + 1)^t) =: \ell$  runs in time  $\mathcal{O}(t \cdot \ell \cdot |E(G)|)$ , and returns a vertex  $w \in Z$  such that for any set  $X \subseteq V(G)$  with  $|X| \leq k$ , it holds that  $X$  is an  $r$ -dominating set of  $Z$  if, and only if,  $X$  is an  $r$ -dominating set of  $Z \setminus \{w\}$ .*

We iteratively apply [Lemma 8](#) for at most  $n$  times, until this is no longer possible. This yields the following lemma.

► **Lemma 9.** *Let  $\mathcal{C}$  be a nowhere dense class of graphs and let  $k, r \geq 1$ . There exists a polynomial  $q_r$  and a polynomial time algorithm that, given a graph  $G \in \mathcal{C}$  and  $k \geq 1$  either correctly concludes that  $G$  cannot be  $r$ -dominated by a set of at most  $k$  vertices, or finds a  $(k, r)$ -domination core  $Z \subseteq V(G)$  of  $G$  of size at most  $q_r(k)$ .*

As the source dominating set  $D_s$  witnesses that  $G$  can be  $r$ -dominated by a set of  $k$  vertices, we conclude the existence of a domination core from the lemma.

We now define the annotated problem  $\text{DISTANCE-}r \text{ Z-DOMINATING SET}$ ,  $r$ -ZDS, as the problem to find on input  $(G, Z, k)$  a set  $D$  with  $Z \subseteq N_r(D)$ . Such a set is called a  $(Z, r)$ -dominator. By definition, if  $Z$  is a  $(k, r)$ -domination core, then every  $(Z, r)$ -dominator of size at most  $k$  corresponds to a distance- $r$  dominating set of  $G$ . Similarly, if  $Z$  is a distance- $r$  domination core, then every minimum  $(Z, r)$ -dominator corresponds to a distance- $r$  dominating set of  $G$ . We define the reconfiguration variant of the problem,  $r$ -ZDSR, in the obvious way.

► **Theorem 10.** *Let  $\mathcal{C}$  be a nowhere dense class of graphs and let  $r \in \mathbb{N}$ . Let  $(G, D_s, D_t, k)$  be an instance of  $r$ -DSR, where  $G \in \mathcal{C}$ . We can compute in polynomial time a subgraph  $G' \subseteq G$  with  $D_s, D_t \subseteq G'$  and  $Z \subseteq V(G')$  such that  $(G, k, D_s, D_t)$  is a positive instance of  $r$ -DSR if and only if  $((G', Z), k, D_s, D_t)$  is a positive instance of  $r$ -ZDSR and  $G'$  has order polynomial in  $k$ .*

**Proof.** We compute a  $(k, r)$ -domination core  $Z \subseteq V(G)$  of size at most  $q_r(k)$  using [Lemma 9](#). Let  $\varepsilon > 0$  and let  $f_{\text{proj}}(r, \varepsilon)$  be the function from [Lemma 2](#). According to the lemma there are at most  $f_{\text{proj}}(r, \varepsilon) \cdot |Z|^{1+\varepsilon}$  different  $r$ -projections to  $Z$ . We classify the elements of  $V(G) \setminus Z$  according to their  $r$ -projections to  $Z$ , that is, we define  $u \cong v \Leftrightarrow \rho_r^G[u, Z] = \rho_r^G[v, Z]$ . Now for each projection class  $\kappa$  we choose a representative  $v_\kappa$  from that class.

► **Claim 1.** For all  $u, v \in V(G)$ ,  $\rho_r^G[u, Z] = \rho_r^G[v, Z]$  implies  $N_r^G(u) \cap Z = N_r^G(v) \cap Z$ .

To see this, let  $z \in N_r^G(u) \cap Z$  and let  $P$  be a shortest path between  $u$  and  $z$ . If  $P$  is  $Z$ -avoiding we conclude from  $\rho_r^G[u, Z] = \rho_r^G[v, Z]$  that there exists also a  $Z$ -avoiding path  $P'$  of the same length as  $P$  between  $v$  and  $z$ , which implies  $z \in N_r^G(v) \cap Z$ . Otherwise, let  $z'$  be the first vertex of  $Z$  on  $P$  and let  $Q$  be the initial part of  $P$  between  $u$  and  $z'$ . Note that  $Q$  is a shortest path between  $u$  and  $z'$ . By the same argument as above, we find a  $Z$ -avoiding path  $Q'$  of the same length as  $Q$  between  $v$  and  $z'$ . By replacing  $Q$  in  $P$  by  $Q'$  we obtain a path of the same length as  $P$  between  $v$  and  $z$ , which again proves  $z \in N_r^G(v) \cap Z$ . ◻

We now construct  $G'$  such that it contains  $D_s, D_t$ , the set  $Z$ , all the representatives  $v_\kappa$  and furthermore a small set  $T$  of vertices such that  $N_r^G(v_\kappa) \cap Z = N_r^{G'}(v_\kappa) \cap Z$ . The set  $T$  is constructed as follows. For each  $v \in V(G)$ , let  $T_v$  be a breadth-first search tree with root  $v$  of depth  $r$  which has elements of  $Z$  as its leaves. Clearly,  $\text{dist}_G(v, z) = \text{dist}_{T_v}(v, z)$  for all  $z \in N_r(v) \cap Z$ . Let  $T$  be the set  $\bigcup_\kappa V(T_{v_\kappa}) \cup \bigcup_{v \in D_s \cup D_t} V(T_v)$ . Hence for each  $v_\kappa$  we have  $N_r^{G'}(v_\kappa) \cap Z = N_r^G(v_\kappa) \cap Z$ , which immediately implies the next claim.

► **Claim 2.** Let  $D'$  be a  $(Z, r)$ -dominator in  $G'$  which contains only representative vertices  $v_\kappa$ . Then  $D'$  is also a  $(Z, r)$ -dominator in  $G$ .

We will always find  $(Z, r)$ -dominators of the above form.

► **Claim 3.** Let  $v \in V(G')$ . Then there is  $v_\kappa \in G'$  such that  $N_r^{G'}(v) \cap Z \subseteq N_r^{G'}(v_\kappa) \cap Z$ .

Let  $\kappa$  be the equivalence class of  $v$  in  $G$ . Then  $N_r^{G'}(v) \cap Z \subseteq N_r^{G'}(v_\kappa) \cap Z$ .  $\lrcorner$

Conversely,  $(Z, r)$ -dominators in  $G$  can be translated to  $(Z, r)$ -dominators in  $G'$ .

► **Claim 4.** Let  $D$  be a distance- $r$  dominating set in  $G$ . Then  $D' = \{v_\kappa : v \in D\}$  is a  $(Z, r)$ -dominator in  $G'$ .

As  $v_\kappa$  is chosen so that  $\rho_r^G[v, Z] = \rho_r^G[v_\kappa, Z]$ , by Claim 1 it holds that  $N_r^G(v) \cap Z = N_r^G(v_\kappa) \cap Z$ . Hence  $Z \subseteq N_r^G(D)$  and  $N_r^{G'}(v_\kappa) \cap Z = N_r^G(v_\kappa) \cap Z$  implies that also  $Z \subseteq N_r^{G'}(D')$ .  $\lrcorner$

We can now prove that the instance  $((G', Z), k, D_s, D_t)$  of  $r$ -ZDSR is equivalent to  $(G, k, D_s, D_t)$ . If  $D_1, \dots, D_n$  is a valid reconfiguration sequence in  $G$ , then according to Claim 4 it is also a valid reconfiguration sequence of  $(Z, r)$ -dominators in  $G'$ .

Conversely, Let  $D'_1, \dots, D'_n$  be a reconfiguration sequence of  $(Z, r)$ -dominators in  $G'$ . We first modify  $D'_i$  such that it uses only representative vertices  $v_\kappa$ , using Claim 3. Now according to Claim 2,  $D'_i$  is also a  $(Z, r)$ -dominator in  $G$ . By definition of a  $(k, r)$ -domination core,  $D'_i$  is a distance- $r$  dominating set in  $G$ .

It remains to estimate the size of  $G'$ . According to Lemma 9,  $Z$  has polynomial size at most  $q_r(k)$ . According to Lemma 2 there are at most  $f_{\text{proj}}(r, \varepsilon) \cdot |Z|^{1+\varepsilon}$  projection classes, hence we add at most so many vertices  $v_\kappa$  to  $G'$ . Furthermore, each spanning tree  $T_{v_\kappa}$  has order at most  $r \cdot |Z|$ . Together with the  $2k$  spanning trees we add for  $D_s$  and  $D_t$ , we have  $|V(G')| \leq (f_{\text{proj}}(r, \varepsilon) + 2k) \cdot q_r(k)^{2+\varepsilon} \cdot r$ , which is polynomial for every fixed value of  $r$  and  $\varepsilon$ .  $\blacktriangleleft$

The annoying fact that we reduce to an annotated version of the problem can be dealt with by introducing a simple gadget to  $G'$ . The same problem occurred also in the kernelization algorithms for DISTANCE- $r$  DOMINATING SET on bounded expansion and nowhere dense graph classes [12, 13]. We refer to these papers for the (very simple) details.

We can find much smaller domination cores if we make a further assumption on the dominating set size. The following definition was first given in [12] and is also the basis for the kernelization of DISTANCE- $r$  DOMINATING SET in [13].

► **Domination core for minimum dominating sets.** Let  $G$  be a graph and  $r \geq 1$ . A set  $Z \subseteq V(G)$  is a *distance- $r$  domination core* if every set  $D$  of minimum size that  $r$ -dominates  $Z$  also  $r$ -dominates  $G$ .

The little change in the definition makes a large difference for the sizes of the respective cores, as the next lemma shows.

► **Lemma 11** (Eickmeyer et al. [13]). *Let  $\mathcal{C}$  be a nowhere dense class of graphs. There exists a function  $f_{\text{core}}(r, \varepsilon)$  and a polynomial-time algorithm that, given a graph  $G \in \mathcal{C}$ , integer  $k \in \mathbb{N}$  and  $\varepsilon > 0$ , either correctly concludes that  $G$  cannot be  $r$ -dominated by  $k$  vertices, or finds a distance- $r$  domination core  $Z \subseteq V(G)$  of  $G$  of size at most  $f_{\text{core}}(r, \varepsilon) \cdot k^{1+\varepsilon}$ .*

If we make the assumption that the source and target sets  $D_s$  and  $D_t$  are of minimum size, we can work with the improved bounds of Lemma 11 instead of the polynomial bounds of Lemma 9. The final obstacle is to better control the sizes of the trees  $T_{v_\kappa}$  that we add to the graph  $G'$  in the construction of Theorem 10. For this, we need the following two lemmas.

► **Lemma 12** (Lemma 2.9 of [12], adjusted (see Lemma 8 of [13])). *There is a function  $f_{\text{cl}}(r, \varepsilon)$  and a polynomial-time algorithm that, given  $G \in \mathcal{C}$ ,  $X \subseteq V(G)$ ,  $r \in \mathbb{N}$ , and  $\varepsilon > 0$ , computes the  $r$ -closure of  $X$ , denoted  $\text{cl}_r(X)$  with the following properties.*

- $X \subseteq \text{cl}_r(X) \subseteq V(G)$ ;
- $|\text{cl}_r(X)| \leq f_{\text{cl}}(r, \varepsilon) \cdot |X|^{1+\varepsilon}$ ; and
- $|M_r^G(u, \text{cl}_r(X))| \leq f_{\text{cl}}(r, \varepsilon) \cdot |X|^\varepsilon$  for each  $u \in V(G) \setminus \text{cl}_r(X)$ .

We can now compute a breadth-first search tree with root  $v_\kappa$  of depth at most  $r$  which stops whenever it first encounters a vertex of  $Z$ . This gives us a tree  $T_{v_\kappa}$  of size at most  $f_{\text{cl}}(r, \varepsilon) \cdot |Z|^\varepsilon \cdot r$ . As breadth-first search does not continue when meeting  $Z$ , we now have to connect the vertices of  $Z$  with minimum length paths (up to length  $r$ ) to preserve all distances. This is possible as the next lemma shows.

► **Lemma 13** (Lemma 2.11 of [12], adjusted (see Lemma 9 of [13])). *There is a function  $f_{\text{pth}}(r, \varepsilon)$  and a polynomial-time algorithm which on input  $G \in \mathcal{C}$ ,  $X \subseteq V(G)$ ,  $r \in \mathbb{N}$ , and  $\varepsilon > 0$ , computes a superset  $X' \supseteq X$  of vertices with the following properties:*

- whenever  $\text{dist}_G(u, v) \leq r$  for  $u, v \in X$ , then  $\text{dist}_{G[X']}(u, v) = \text{dist}_G(u, v)$ ; and
- $|X'| \leq f_{\text{pth}}(r, \varepsilon) \cdot |X|^{1+\varepsilon}$ .

We can now prove the following theorem.

► **Theorem 14.** *Let  $\mathcal{C}$  be a nowhere dense class of graphs and let  $r \in \mathbb{N}$ . Let  $(G, D_s, D_t, k)$  be an instance of  $r$ -DSR, where  $G \in \mathcal{C}$  and where  $D_s$  and  $D_t$  are minimum distance- $r$  dominating sets in  $G$ . There is a function  $f_{\text{ker}}(r, \varepsilon)$  and a polynomial-time algorithm which on input  $(G, D_s, D_t, k)$  computes a subgraph  $G' \subseteq G$  with  $D_s, D_t \subseteq G'$  and  $Z \subseteq V(G')$  such that  $(G, k, D_s, D_t)$  is a positive instance of  $r$ -DSR if and only if  $((G', Z), k, D_s, D_t)$  is a positive instance of  $r$ -ZDSR and  $G'$  has order at most  $f_{\text{ker}}(r, \varepsilon) \cdot k^{1+\varepsilon}$ .*

**Proof.** The proof parallels that of Theorem 10. We compute a distance- $r$  domination core  $Z \subseteq V(G)$  using Lemma 11 with parameter  $\varepsilon'$  which will be determined in the course of the proof instead of Lemma 9. Now, using Lemma 12, we compute  $Z' = \text{cl}_r(Z)$  and using Lemma 13 we compute  $Z'' \supseteq Z'$  such that whenever  $\text{dist}_G(u, v) \leq r$  for  $u, v \in Z'$ , then  $\text{dist}_{G[Z'']}(u, v) = \text{dist}_G(u, v)$ . Again, we classify the elements of  $V(G) \setminus Z$  according to their  $r$ -projections to  $Z$ , that is, we define  $u \cong v \Leftrightarrow \rho_r^G[u, Z] = \rho_r^G[v, Z]$ .

We now construct  $G'$  such that it contains  $D_s, D_t$ , the set  $Z''$ , all the representatives  $v_\kappa$  and furthermore a small set  $T$  of vertices such that  $N_r^G(v_\kappa) \cap Z = N_r^{G'}(v_\kappa) \cap Z$ . The set  $T$  is constructed as follows. For each  $v \in V(G)$ , let  $T_v$  be a breadth-first search tree with root  $v$  of depth  $r$  which does not continue when meeting  $Z$  for the first time. The crucial claim is the following.

► **Claim 5.** Let  $v_\kappa$  be a representative vertex. Then  $\text{dist}_G(v_\kappa, z) = \text{dist}_{G'}(v_\kappa, z)$  for all  $z \in N_r^G(v_\kappa) \cap Z$ .

Let  $z \in N_r^G(v_\kappa) \cap Z$  and let  $P$  be a minimum length path between  $v_\kappa$  and  $z$ . Let  $z'$  be the first vertex on  $P$  which belongs to  $Z'$ . Then we have  $\text{dist}_G(v_\kappa, z') = \text{dist}_{T_{v_\kappa}}(v_\kappa, z')$ . Now by construction of  $Z''$  we have  $\text{dist}_G(z', z) = \text{dist}_{G'}(z', z)$ , which implies the claim. ◻

The rest of the proof works exactly as the proof of Theorem 10. Let us determine a bound on the size of  $G'$ , which also determines our initial choice of  $\varepsilon'$ . The set  $Z$  has size at most  $f_{\text{core}}(r, \varepsilon') \cdot k^{1+\varepsilon'}$ . According to Lemma 2 there are at most  $f_{\text{proj}}(r, \varepsilon') \cdot |Z|^{1+\varepsilon'}$  projection classes, hence we add at most so many vertices  $v_\kappa$  to  $G'$ . The set  $Z'$  has size at most  $f_{\text{proj}}(r, \varepsilon') \cdot |Z|^{1+\varepsilon'}$  according to Lemma 12 and the set  $Z''$  has size at most  $f_{\text{pth}}(r, \varepsilon') \cdot |Z'|^{1+\varepsilon}$  according to Lemma 13. Now, each tree  $T_v$  has order at most  $|Z'|^{\varepsilon'} \cdot r$ . Hence in total we have

$$|V(G')| \leq |Z''| + (f_{\text{proj}}(r, \varepsilon') \cdot |Z|^{1+\varepsilon'} + 2) \cdot |Z'|^{\varepsilon'} \cdot r$$

$$=: f_{\text{ker}}(r, \varepsilon) \cdot k^{1+\varepsilon} \text{ for an appropriately chosen function } f_{\text{ker}} \text{ and } \varepsilon'. \quad \blacktriangleleft$$

## 4.2 Lower bounds

► **Theorem 15.** *Let  $\mathcal{C}$  be somewhere dense and closed under taking subgraphs. Then there is  $r \in \mathbb{N}$  such that the distance- $r$  dominating set reconfiguration problem is  $W[1]$ -hard on  $\mathcal{C}$ .*

**Proof.** The proof works in principle as the proof of [Theorem 7](#). Again, let  $s \in \mathbb{N}$  be the number such that according to [Lemma 5](#) for all graphs  $G$  we have  $G_s \in \mathcal{C}$ . We reduce 1-DSR to  $(3s)$ -DSR on  $\mathcal{C}$  by finding an appropriate subdivision of a graph in which dominating sets are translated 1-to-1 to distance- $(3s)$  dominating sets. Here we can directly use the reduction from set cover to distance- $r$  dominating set from [\[12\]](#), where we use the fact that dominating set and set cover are equivalent problems (just define the set system consisting of the neighborhoods of all vertices). Now use that the token swapping version of dominating set reconfiguration is  $W[1]$ -hard [\[27\]](#). ◀

## 5 Conclusion

The study of computationally hard problems on restricted classes of inputs is a very fruitful line of research in algorithmic graph structure theory and in particular in parameterized complexity theory. This research is based on the observation that many problems such as DOMINATING SET, which are considered intractable in general, can be solved efficiently on restricted graph classes. Of course it is a very desirable goal in this line of research to identify the most general classes of graphs on which certain problems can be solved efficiently. In this work we were able to identify the exact limit of tractability for the reconfiguration variants of the distance- $r$  independent set problem and distance- $r$  dominating set problem on subgraph closed graph classes. Clearly, the main open question is to identify the most general graph classes which are not subgraph closed on which these problems admit efficient solutions.

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