

On Wideness and Stability *

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Abstract

We study nowhere dense classes of graphs, recently introduced by Nešetřil and Ossona de Mendez [19, 20]. Firstly, we provide a new proof for the fact that these classes are uniformly quasi-wide, improving previously known bounds between the two equivalent notions. Secondly, we give a new combinatorial proof of the result of Adler and Adler [1] stating that nowhere dense classes of graphs are stable. In contrast to the original proof, our proof is completely finitistic and yields explicit bounds for ladder indices of first-order formulas on nowhere dense classes of graphs.

1 Introduction

Nowhere dense classes of graphs were introduced by Nešetřil and Ossona de Mendez [19, 20] as a very general model for uniform sparseness of graphs. These classes generalize many familiar classes of sparse graphs, such as planar graphs, graphs of bounded treewidth, graphs of bounded degree, and, in fact, all classes that exclude a fixed topological minor. Formally, a class \mathcal{C} of graphs is *nowhere dense* if there is a function $t: \mathbb{N} \rightarrow \mathbb{N}$ such that for every $r \in \mathbb{N}$, no graph G in \mathcal{C} contains the clique $K_{t(r)}$ on $t(r)$ vertices as an *r -shallow minor*, i.e., as a subgraph of a graph obtained from G by contracting mutually disjoint subgraphs of radius r to single vertices.

The concept of nowhere denseness turns out to be very robust, as witnessed by the fact that it is equivalent to multiple other concepts studied in different areas of mathematics. One can equivalently characterize nowhere dense graph classes by bounds on the density of (topological) shallow minors [19, 20], quasi-wideness [20] (a notion introduced by Dawar [3] in his study of homomorphism preservation properties), low tree-depth colorings [18], generalized coloring numbers [23], sparse neighborhood covers [11, 12], by a game called the splitter game [12], and by the model-theoretic concepts of stability and independence [1]. For a broader discussion on the graph theoretic sparsity we refer to the book of Nešetřil and Ossona de Mendez [21].

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In this work we revisit the connections between the notion of nowhere denseness and notions from model theory and finite model theory. We first consider uniform quasi-wideness, a notion introduced by Dawar [3], who proved that every quasi-wide class that is closed under taking substructures and disjoint unions has the homomorphism preservation property. Formally, a class \mathcal{C} of graphs is *uniformly quasi-wide* if there are functions $N: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and $s: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $r, m \in \mathbb{N}$ and all subsets $A \subseteq V(G)$ for $G \in \mathcal{C}$ of size $|A| \geq N(r, m)$ there is a set $S \subseteq V(G)$ of size $|S| \leq s(r)$ and a set $B \subseteq A - S$ of size $|B| \geq m$ which is r -independent in $G - S$. Recall that a set $B \subseteq V(G)$ is called r -independent in G if all distinct $u, v \in B$ are at distance larger than r in G . Nešetřil and Ossona de Mendez proved that the notions of uniform quasi-wideness and nowhere denseness coincide for classes of graphs [20]. We revisit their proof, obtaining improved bounds.

The second topic of study is the connection between nowhere denseness and stability theory. Fix a relational vocabulary Σ . Let $\varphi(\bar{x}, \bar{y})$ be a Σ -formula with the free variables partitioned into two groups \bar{x}, \bar{y} . A φ -ladder of length n in a Σ -structure \mathfrak{A} is a sequence $\bar{a}_1, \dots, \bar{a}_n, \bar{b}_1, \dots, \bar{b}_n$ of tuples of elements of \mathfrak{A} such that for all $1 \leq i, j \leq n$,

$$\mathfrak{A} \models \varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j.$$

The least n for which there is no φ -ladder of length n is the *ladder index* of $\varphi(\bar{x}, \bar{y})$ in \mathfrak{A} (which may depend on the way we split the variables).

Based on work of Podewski and Ziegler [22], Adler and Adler [1] proved that every nowhere dense class \mathcal{C} of graphs is stable, that is, the ladder index of every first-order formula $\varphi(\bar{x}, \bar{y})$ over graphs from \mathcal{C} is bounded by a constant depending only on φ and \mathcal{C} . In fact, for a subgraph-closed class \mathcal{C} , the notions of nowhere denseness and stability coincide.

We remark that the above connections between nowhere denseness and notions from model theory have recently found algorithmic applications. Both uniform quasi-wideness and stability techniques are key tools used in the study of the complexity of the DISTANCE- r DOMINATING SET problem on nowhere dense graph classes, and in particular in the design of polynomial kernelization procedures for this problem [4, 6, 8, 15].

Our contribution. Our first result is a new proof of a result of Nešetřil and Ossona de Mendez [19], which states that a class \mathcal{C} of graphs is nowhere dense if and only if it is uniformly quasi-wide. The proof of Nešetřil and Ossona de Mendez goes back to a construction of Kreidler and Seese [14] (see also Atserias et al. [2]), and uses iterated Ramsey arguments. Hence the original bounds on the function N are huge. Recently, Kreutzer, Rabinovich and the second author proved that we may always choose N to be a polynomial function of m , with the degree of the polynomial depending on r [15]. However, the exact dependence of degree of the polynomial on r and on the class \mathcal{C} itself was not specified in [15], as the existence of a polynomial bound is derived from non-constructive arguments used by Adler and Adler's in showing that every nowhere dense class of graphs is stable [1]. We give a new construction which is considerably simpler than that of [15] and which gives explicit bounds on the degree of the polynomial. More precisely, we prove the following theorem; here, the notation $\mathcal{O}_{r,t}(\cdot)$ hides computable factors depending on r and t .

Theorem 1. For all $r, t \in \mathbb{N}$ there is a polynomial $N: \mathbb{N} \rightarrow \mathbb{N}$ with $N(m) = \mathcal{O}_{r,t}(m^{(2t+1)^{2rt}})$, such that the following holds. Let G be a graph such that $K_t \not\prec_{\lfloor 5r/2 \rfloor} G$, and let $A \subseteq V(G)$ be a vertex subset of size at least $N(m)$, for a given m . Then there exists a set $S \subseteq V(G)$ of size $|S| \leq t$ and a set $B \subseteq A - S$ of size $|B| \geq m$ which is r -independent in $G - S$. Moreover, given G and A , such sets S and B can be computed in time $\mathcal{O}_{r,t}(|A| \cdot |E(G)|)$.

Let us remark that even though the techniques employed to prove [Theorem 1](#) are inspired by methods from stability theory, at the end we use only very simple graph theoretic notions. In particular, as asserted in the last sentence of the theorem, the proof can easily be turned into an efficient algorithm.

Podewski and Ziegler [22] consider *flat graphs*, a notion corresponding to uniform quasi-wideness in the infinite. They show that flat graphs are stable using an infinite Ramsey argument and compactness. The observation of Adler and Adler [1] was that this can be translated to classes \mathcal{C} of finite graphs by considering an infinite graph that is the disjoint union of graphs from \mathcal{C} . However, this inherently non-constructive approach cannot give explicit upper bounds on parameters governing the stability of \mathcal{C} , for instance, the ladder indices.

Based on the approach of Podewski and Ziegler [22], we give a combinatorial proof that every first-order formula has finite ladder index on every nowhere dense class, which does not involve infinite combinatorics and model theory. In particular, instead of compactness we use Gaifman's Locality Theorem for first-order logic [9]. The following theorem summarizes our result.

Theorem 2. There are computable functions $f: \mathbb{N}^3 \rightarrow \mathbb{N}$ and $g: \mathbb{N} \rightarrow \mathbb{N}$ with the following property. Suppose $\varphi(\bar{x}, \bar{y})$ is a formula of quantifier rank q and with d free variables, and G is a graph such that $K_t \not\prec_{g(q)} G$. Then the ladder index of $\varphi(\bar{x}, \bar{y})$ in G is at most $f(q, d, t)$.

Note that in particular, [Theorem 2](#) implies that every nowhere dense graph is stable, which was the main conclusion of Adler and Adler [1].

2 From nowhere denseness to uniform quasi-wideness

In this section we prove [Theorem 1](#). We first recall some basic notions from graph theory.

Preliminaries. All graphs in this paper are finite, undirected and simple, that is, they do not have loops or parallel edges. Our notation is standard, we refer to [5] for more background on graph theory. We write $V(G)$ for the vertex set of a graph G and $E(G)$ for its edge set. The *distance* between vertices u and v in G , denoted $\text{dist}_G(u, v)$, is the length of a shortest path between u and v in G . If there is no path between u and v in G , we put $\text{dist}_G(u, v) = \infty$. The *(open) neighborhood* of a vertex u , denoted $N(u)$, is the set of neighbors of u , excluding u itself. For a non-negative integer r , by $N_r[u]$ we denote the *(closed) r -neighborhood* of u which comprises vertices at distance at most r from u ; note that u is always contained in its closed r -neighborhood. The *radius* of a graph G is the least integer r such that there is some vertex v of G with $N_r[v] = V(G)$.

A *minor model* of a graph H in G is a family $(I_u)_{u \in V(H)}$ of pairwise vertex-disjoint connected subgraphs of G , called *branch sets*, such that whenever uv is an edge in H , there are $u' \in I_u$ and

$v' \in I_v$ for which $u'v'$ is an edge in G . The graph H is a *depth- r minor* of G , denoted $H \preceq_r G$, if there is a minor model $(I_u)_{u \in V(H)}$ of H in G such that each I_u has radius at most r .

A class \mathcal{C} of graphs is *nowhere dense* if there is a function $t: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $r \in \mathbb{N}$ it holds that $K_{t(r)} \not\preceq_r G$ for all $G \in \mathcal{C}$, where $K_{t(r)}$ denotes the clique on $t(r)$ vertices.

A set $B \subseteq V(G)$ is called *r -independent* in a graph G if $\text{dist}_G(u, v) > r$ for all distinct $u, v \in B$. A class \mathcal{C} of graphs is *uniformly quasi-wide* if there are functions $N: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and $s: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $r, m \in \mathbb{N}$, all graphs $G \in \mathcal{C}$, and all subsets $A \subseteq V(G)$ of size $|A| \geq N(r, m)$, there is a set $S \subseteq V(G)$ of size $|S| \leq s(r)$ and a set $B \subseteq A - S$ of size $|B| \geq m$ which is r -independent in $G - S$.

General strategy. Our proof follows the same lines as the original proof of Nešetřil and Ossona de Mendez, with the difference that in the key technical lemma (Lemma 2 below), we improve the bounds significantly by replacing a Ramsey argument with a new combinatorial reasoning. The new argument essentially originates in the concept of *branching index* from stability theory, and also uses the almost linear bound on neighborhood complexity in nowhere dense graph classes, due to Gajarský et al. [10].

We first prove a restricted variant, Lemma 1 below, in which we assume that A is already $(r - 1)$ -independent. Then, in order to derive Theorem 1, we apply the lemma iteratively for r ranging from 1 to the target value.

Lemma 1. *For every pair of integers $t, r \in \mathbb{N}$ there exists an integer $d < 5r/2$ and a function $L: \mathbb{N} \rightarrow \mathbb{N}$ with $L(m) = \mathcal{O}_{r,t}(m^{(2t+1)^{2rt}})$ such that the following holds. For each $m \in \mathbb{N}$, graph G with $K_t \not\preceq_d G$, and $(r - 1)$ -independent set $A \subseteq V(G)$ of size at least $L(m)$, there is a set $S \subseteq V(G) - A$ of size at most t such that A contains a subset A' of size m which is r -independent in $G - S$. Moreover, if r is odd then S is empty, and if r is even, then every vertex of S is at distance exactly $r/2$ from every vertex of A' . Finally, given G and A , the sets A' and S can be computed in time $\mathcal{O}_{r,t}(|A| \cdot |E(G)|)$.*

We prove Lemma 1 in Section 2.2, but a very rough sketch is as follows. The case of general r reduces to the case $r = 1$ or $r = 2$, depending on the parity of r , by contracting the balls of radius $\lfloor \frac{r-1}{2} \rfloor$ around the vertices in A to single vertices. The case of $r = 1$ follows immediately from Ramsey's theorem. It turns out that the case $r = 2$ is substantially more difficult. We start by formulating and proving the main technical result needed for proving the case $r = 2$.

2.1 The main technical lemma

The main tool is the following Ramsey-like result.

Lemma 2. *For every integer $t \in \mathbb{N}$ and positive real $\alpha < \frac{1}{2}$ there is an integer $\ell_0 \in \mathbb{N}$ with the following property. Let $m, \ell \in \mathbb{N}$ be such that $\ell \geq \ell_0$. If G is a graph and A is a 1-independent set in G with at least $(m + \ell)^{2t}$ elements, then at least one of the following conditions hold:*

- $K_t \preceq_2 G$,
- A contains a 2-independent set of size m ,
- some vertex v of G has at least ℓ^α neighbors in A .

Moreover, in each case, the corresponding structure (a depth-2 minor model, a 2-independent set of size m , or a vertex v as above) can be computed in time $\mathcal{O}_t(|A| \cdot |E(G)|)$.

We remark that a statement similar to that of [Lemma 2](#) can be obtained by employing Ramsey's theorem, as has been done in [\[20\]](#). This, however, yields in place of the bound $(m + \ell)^{2t}$ a bound of the form $R(m, \underbrace{q, q, \dots, q}_{k \text{ times}})$, where $k \sim \ell^{2\alpha}$ and $R(m_1, \dots, m_c)$ is the Ramsey number for c colors. In particular, this does not give a bound which is polynomial in $m + \ell$, and thus cannot be used to prove [Theorem 1](#).

Neighborhood complexity. Let us first recall the following result of Gajarský et al. [\[10\]](#) which provides an upper bound on the number of distinct neighborhoods in a graph from a nowhere dense class.

Lemma 3 (adaptation of Lemma 4.11 in [\[10\]](#)). *Let G be a graph such that $K_t \not\leq_1 G$ for some constant $t \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists n_0 , depending only on t and ε , such that for all $A \subseteq V(G)$ with $|A| \geq n_0$ it holds that*

$$|\{N(v) \cap A : v \in V(G)\}| \leq |A|^{1+\varepsilon}.$$

We remark that the proof of [Lemma 3](#) uses only the fact that nowhere dense classes of graphs do not have dense shallow minors [\[7, 13\]](#), and does not rely on any non-constructive arguments from stability theory. In particular, n_0 depends in a computable manner on t and ε . From [Lemma 3](#) we derive the following.

Corollary 4. *Let $G, A, t, \varepsilon, n_0$ be as in [Lemma 3](#). Suppose $|A| \geq n_0$ and every pair of elements of A has a common neighbor in G . Then there exists a vertex v in G which has at least $|A|^{(1-\varepsilon)/2}$ neighbors in A .*

PROOF. Let $\mathcal{F} = \{N(v) \cap A : v \in V(G)\}$. Let k be the maximal cardinality of a set F in \mathcal{F} . Say that an unordered pair $\{a, b\}$ of distinct elements of A is *covered* by $F \in \mathcal{F}$ if $a, b \in F$. By assumption, each of the $\binom{|A|}{2}$ unordered pairs of distinct elements of A is covered by some element $F \in \mathcal{F}$, and, clearly, every $F \in \mathcal{F}$ can cover at most $\binom{k}{2}$ pairs. Hence $\frac{\binom{|A|}{2}}{\binom{k}{2}} \leq |\mathcal{F}| \leq |A|^{1+\varepsilon}$, implying $k \geq |A|^{(1-\varepsilon)/2}$. \square

Proof of Lemma 2. We proceed with the proof of [Lemma 2](#), which spans the whole remainder of this section. We will arrange the elements of A in a binary tree and prove that the tree contains a long path. From this path, we will extract the set A' . In stability theory, similar trees are called *type trees* and they are used to extract long indiscernible sequences, see e.g. [\[17\]](#).

We identify words in $\{\mathsf{D}, \mathsf{S}\}^*$ with *nodes* of the infinite rooted binary tree. The *depth* of a node w is the length of w . For $w \in \{\mathsf{D}, \mathsf{S}\}^*$, the nodes $w\mathsf{D}$ and $w\mathsf{S}$ are called, respectively, the *daughter* and the *son* of w , and w is the *parent* of both $w\mathsf{S}$ and $w\mathsf{D}$. A node w' is a *descendant* of a node w if w' is a prefix of w (possibly $w' = w$). We consider finite, labeled, rooted, binary trees, which are called simply trees below, and are defined as follows. A *tree* is a partial function $\tau : \{\mathsf{D}, \mathsf{S}\}^* \rightarrow U$ whose domain is a finite set of nodes, called the *nodes of τ* , which is closed under taking parents. If v is a node of τ , then $\tau(v)$ is called its *label*.

Let G be a graph, $A \subseteq V(G)$ be a 1-independent set in G , and \bar{a} be an enumeration of A . We define a binary tree τ which is labeled by vertices of G . The tree is defined by processing all elements of \bar{a} sequentially. We start with τ being the tree with empty domain, and for each element a of the sequence \bar{a} , processed in the order given by \bar{a} , execute the following procedure which results in adding a node with label a to τ .

When processing the vertex a , do the following. Start with w being the empty word. While w is a node of τ , repeat the following step: if the distance from a to $\tau(w)$ in the graph G is at most 2, replace w by its son, otherwise, replace w by its daughter. Once w is not a node of τ , extend τ by setting $\tau(w) = a$. In this way, we have processed the element a , and now proceed to the next element a of \bar{a} , until all elements are processed. This ends the construction of τ .

Define the *depth* of τ as the maximal depth of a node of τ . For a word w , define an *alternation* to be a position i such that $w_i \neq w_{i-1}$, where w_0 is assumed to be D. The *alternation rank* of the tree τ is the maximum of the number of alternations of w , over all nodes w of τ .

Lemma 5. *Let $h, t \geq 2$. If τ has alternation rank at most $2t - 1$ and depth at most $h - 1$, then τ has fewer than h^{2t} nodes.*

PROOF. To each node w of τ assign the function $f_w: \{1, \dots, 2t\} \rightarrow \{1, \dots, h\}$ defined as follows: f_w maps each $i \in \{1, \dots, 2t\}$ to the i th alternation of w , provided i is at most the number of alternations of w , and otherwise we put $f_w(i) = |w| + 1$. It is clear that the mapping $w \mapsto f_w$ for nodes w of τ is injective and its image is contained in monotone functions, hence the domain of τ has fewer than h^{2t} elements. \square

Lemma 6. *Suppose that $K_t \not\leq_2 G$. Then τ has alternation rank at most $2t - 1$.*

PROOF. Let w be a node of τ with at least $2k$ alternations, for some $k \in \mathbb{N}$. In particular, there are vertices $a_1, b_1, \dots, a_k, b_k$ in A such that for each $i \in \{1, \dots, k\}$, the nodes in τ corresponding to $b_i, a_{i+1}, b_{i+1}, \dots, a_k, b_k$ are descendants of the son of the node which corresponds to a_i , and the nodes corresponding to $a_{i+1}, b_{i+1}, \dots, a_k, b_k$ are descendants of the daughter of the node which corresponds to b_i .

Claim 1. *For every pair a_i, b_j with $1 \leq i \leq j \leq k$, there is a vertex $z_{ij} \notin A$ which is a common neighbor of a_i and b_j , and is not a neighbor of any b_s with $s \neq j$.*

PROOF. Note that since $i \leq j$, the node corresponding to b_j is a descendant of the son of the node corresponding to a_i , hence we have $\text{dist}_G(a_i, b_j) \leq 2$ by the construction of τ . However, we also have $\text{dist}_G(a_i, b_j) > 1$ since A is 1-independent. Therefore $\text{dist}_G(a_i, b_j) = 2$, so there is a vertex z_{ij} which is a common neighbor of a_i and b_j . Suppose that z_{ij} was a neighbor of b_s , for some $s \neq j$. This would imply that $\text{dist}_G(b_j, b_s) \leq 2$, which is impossible, since the nodes corresponding to b_s and b_j in τ are such that one is a descendant of the daughter of the other, implying that $\text{dist}_G(b_s, b_j) > 2$. \lrcorner

Note that whenever $i \leq j$ and $i' \leq j'$ are such that $j \neq j'$, the vertices z_{ij} and $z_{i'j'}$ are different, because z_{ij} is adjacent to b_j but not to $b_{j'}$, and the converse holds for $z_{i'j'}$. However, it may happen that $z_{ij} = z_{i'j}$ even if $i \neq i'$. This will not affect our further reasoning.

For each $j \in \{1, \dots, k\}$, define the graph B_j as the subgraph of G induced by the set $\{a_j, b_j\} \cup \{z_{ij} : 1 \leq i \leq j\}$. By Lemma 6 and the discussion from the previous paragraph, the

graphs B_j for $j \in \{1, \dots, k\}$ are pairwise disjoint. Moreover, for all $1 \leq i \leq j \leq k$, there is an edge between B_i and B_j , namely, the edge between $z_{ij} \in B_j$ and $a_i \in B_i$. Hence, the graphs B_j , for $j \in \{1, \dots, k\}$, define a depth-2 minor model of K_k in G . Since $K_t \not\leq_2 G$, this implies that $k < t$, proving [Lemma 6](#). \square

To prove [Lemma 2](#), choose $\varepsilon = 1 - 2\alpha$ so that $\alpha = \frac{1-\varepsilon}{2}$; note that $0 < \varepsilon < 1$. Let n_0 be the integer given by [Lemma 3](#) for t and ε ; w.l.o.g. $n_0 \geq 2$. We put $\ell_0 = n_0$. Fix integers $\ell \geq \ell_0$ and m , and define $h = m + \ell$. Let A be a 1-independent set in G of size at least h^{2t} .

Suppose that the first case of [Lemma 2](#) does not hold. In particular $K_t \not\leq_2 G$, so by [Lemma 6](#), τ has alternation rank at most $2t - 1$. From [Lemma 5](#) we conclude that τ has depth at least h . As $h = m + \ell$, it follows that either τ has a node w which contains at least m letters D, or τ has a node w which contains at least ℓ letters S.

Consider the first case, i.e., there is a node w of τ which contains at least m letters D, and let X be the set of all vertices $\tau(u)$ such that uD is a prefix of w . Then, by construction, X is a 2-independent set in G of size at least m , so the second case of the lemma holds.

Finally, consider the second case, i.e., there is a node w in τ which contains at least ℓ letters S. Let Y be the set of all vertices $\tau(u)$ such that uS is a prefix of w . Then, by construction, $Y \subseteq A$ is a set of vertices which are mutually at distance exactly 2 in G . Since $K_t \not\leq_2 G$, by [Corollary 4](#) we infer that there is a vertex $v \in G$ with at least $\ell^{\frac{1-\varepsilon}{2}} = \ell^\alpha$ neighbors in Y . This finishes the proof of the first part of [Lemma 2](#).

The proof above yields an algorithm which first constructs the tree τ , by iteratively processing each vertex w of A and testing whether the distance between w and each vertex processed already is equal to 2. This amounts to running a breadth-first search from every vertex of A , which can be done in time $\mathcal{O}(|A| \cdot |E(G)|)$. Whenever a node with $2t$ alternations is inserted to τ , we can exhibit in G a depth-2 minor model of K_t . Whenever a node with least m letters D is added to τ , we have constructed an m -independent set. Whenever a node with at least ℓ letters S is added to τ , as argued, there must be some vertex $v \in V(G) - A$ with at least ℓ^α neighbors in A . To find such a vertex, scan through all neighborhoods of vertices $v \in A$ in the graph G , and then select a vertex $w \in V(G)$ which belongs to the largest number of those neighborhoods; this can be done in time $\mathcal{O}(|E(G)|)$. The overall running time is $\mathcal{O}(|A| \cdot |E(G)|)$, as required.

This finishes the proof of [Lemma 2](#).

2.2 Proof of [Lemma 1](#)

With [Lemma 2](#) proved, we can proceed with [Lemma 1](#). We start with the case $r = 1$, then we move to the case $r = 2$. Next we show how the general case reduces to one of those two cases.

Case $r = 1$. We put $d = 0$, thus we assume that $K_t \not\leq_0 G$; that is, G does not contain a clique of size t as a subgraph. By Ramsey's Theorem, in every graph every set of size $\binom{m+t-2}{t-1}$ contains an independent set of size m or a clique of size t . Therefore, taking $L(m)$ as the above binomial coefficient yields [Lemma 1](#) in case $r = 0$, for $S = \emptyset$. Note here that $\binom{m+t-2}{t-1} \in \mathcal{O}_t(m^{(2t+1)2t})$. Moreover, such independent set or clique can be computed from G and A in time $\mathcal{O}(|A| \cdot |E(G)|)$ by simulating the proof of Ramsey's theorem.

Case $r = 2$. We put $d = 2$, thus we assume that $K_t \not\leq_2 G$. Namely, we show that if A is a sufficiently large 1-independent set in a graph G such that $K_t \not\leq_2 G$, then there is a set of vertices S of size at most t such that $A - S$ contains a subset of size m which is 2-independent in $G - S$. Here, by “sufficiently large” we mean of size at least $L(m)$, for $L(m)$ emerging from the proof. To this end, we shall iteratively apply [Lemma 2](#) as long as it results in the third case, yielding a vertex v with many neighbors in A . In this case, we add v vertex to the set S , and apply the lemma again, restricting A to $A \cap N(v)$. The precise calculations follow.

Fix a number $\beta > 2t$. For $k \geq 0$, define $m_k = ((k + 1) \cdot m)^{(2\beta)^k}$. We will apply [Lemma 2](#) in the following form.

Claim 2. *There is an integer \widehat{m} , depending only on t and β , such that for all $m \geq \widehat{m}$ and $k \geq 1$, the following holds. If G is a graph such that $K_t \not\leq_2 G$, and $A \subseteq V(G)$ is an 1-independent set in G which does not contain a 2-independent set of size m and satisfies $|A| \geq m_k$, then there exists a vertex v of G such that $|N_G(v) \cap A| \geq m_{k-1}$.*

PROOF. Let $\alpha = t/\beta$, so that $\alpha < 1/2$. Put $\widehat{m} = \ell^\alpha = \ell_0^{t/\beta}$, where ℓ_0 is the constant given by [Lemma 2](#) for t and α . Let $\ell = (k \cdot m)^{(2\beta)^{k-1}/\alpha}$. Then $m \geq \widehat{m}$ implies that $\ell \geq \ell_0$. Observe that

$$|A| \geq ((k + 1) \cdot m)^{(2\beta)^k} = \left((m + k \cdot m)^{(2\beta)^{k-1}/\alpha} \right)^{2t} \geq (m + (k \cdot m)^{(2\beta)^{k-1}/\alpha})^{2t} = (m + \ell)^{2t}.$$

Therefore, we may apply [Lemma 2](#), yielding a vertex v with at least $\ell^\alpha = (k \cdot m)^{(2\beta)^{k-1}} = m_{k-1}$ neighbors in A . ┘

Let \widehat{m} be the number given by [Claim 2](#). In the following we assume that $m \geq \max(t, \widehat{m})$, for we may always ask for finding a 2-independent set of size $\max(t, \widehat{m}, m)$ instead of m . We will find a subset of A of size m which is 2-independent in $G - S$, where $|S| \leq t$.

Assume that $|A| \geq m_t$. By induction, we construct a sequence $A = A_0 \supseteq A_1 \supseteq \dots$ of 1-independent subsets of G of length at most t such that $|A_i| \geq m_{t-i}$, as follows. Start with $A_0 = A$. We maintain a set S of vertices of G which is initially empty, and we maintain the invariant that A_i is disjoint with S at each step of the induction.

For $i = 0, 1, 2, \dots$, do as follows. If A_i contains a 2-independent set of size m in $G - S$, terminate. Otherwise, apply [Claim 2](#) to the graph $G - S$ with 1-independent set A_i of size $|A_i| \geq m_{t-i}$. This yields a vertex v_{i+1} of $G - S$ whose neighborhood in $G - S$ contains at least m_{t-i-1} vertices of A_i . Let A_{i+1} consist of those vertices, and add v_{i+1} to the set S . If $i \leq t$, proceed by replacing i by $i + 1$.

Claim 3. *The construction halts after less than t steps.*

PROOF. Suppose that the construction proceeds for $k \leq t$ steps. By construction, each vertex v_i , for $i \leq k$, is adjacent in G to all the vertices of A_j , for each $i \leq j \leq k$. In particular, all the vertices v_1, \dots, v_k are adjacent to all the vertices of A_k and $|A_k| \geq m_{t-k} \geq m \geq t$. Choose any pairwise distinct vertices $w_1, \dots, w_k \in A_k$ and observe that the connected subgraphs $\{w_i, v_i\}$ of G yield a depth-1 minor model of K_k in G . Since $K_t \not\leq_2 G$, we must have $k < t$. ┘

Therefore, at some step $k < t$ of the construction we must have obtained a 2-independent subset A' of $G - S$ of size m . Moreover, $|S| \leq k < t$.

This proves [Lemma 1](#) in the case $r = 2$, for the function $L(m) = m_t = ((t + 1) \cdot m)^{\beta^{2t}}$ for $m \geq \max(t, \widehat{m})$ and $L(m) = m_{\max(t, \widehat{m})}$ for $m < \max(t, \widehat{m})$, where $\beta > 2 \cdot t$ is any fixed constant and \widehat{m} is obtained from [Claim 2](#). It is easy to see that then $L_t(m) = L(m) \in \mathcal{O}_t(m^{(2t+1)^{2t}})$, provided we put $\beta = 2t + 1$. Also, the proof easily yields an algorithm constructing the sets A' and S , which amounts to applying at most t times the algorithm of [Lemma 2](#). Hence, its running time is $\mathcal{O}_{r,t}(|A| \cdot |E(G)|)$, as required.

Odd case. We prove [Lemma 1](#) in the case when $r = 2s + 1$, for some integer $s \geq 1$. We put $d = s = \frac{r-1}{2}$. Let G be a graph such that $K_t \not\prec_s G$, and let A be a $2s$ -independent set in G . Consider the graph G' obtained from G by contracting the (pairwise disjoint) balls of radius s around each vertex $v \in A$. Let A' denote the set of vertices of G' corresponding to the contracted balls. There is a natural bijection between A and A' . From $K_t \not\prec_s G$ it follows that G' does not contain K_t as a subgraph. Applying the already proved case $r = 1$ to G' and A' , we conclude that if $|A| = |A'| \geq \binom{m+t-2}{t-1}$, then A' contains a 1-independent subset B of size m , which corresponds to a $(2s + 1)$ -independent set in G that is contained in A . Hence, the obtained bound is $L(m) = \binom{m+t-2}{t-1}$, and we have already argued that then $L(m) \in \mathcal{O}_{r,t}(m^{(2t+1)^{2t}})$.

Even case. Finally, we prove [Lemma 1](#) in the case $r = 2s + 2$, for some integer $s \geq 1$. We put $d = 5s + 2 = 5r/2 - 3$. Let G be such that $K_t \not\prec_d G$, and let A be a $(2s + 1)$ -independent set in G . Consider the graph G' obtained from G by contracting the (pairwise disjoint) balls of radius s around each vertex $v \in A$. Let A' denote the set of vertices of G' corresponding to the contracted balls. Again, there is a bijection between A and A' . Note that this time, A' is a 1-independent set in G' . From $K_t \not\prec_{5s+2} G$ it follows that $K_t \not\prec_2 G'$. Apply the already proved case $r = 2$ to G' and A' . Then, if $|A| = |A'| \geq L_t(m)$, where $L_t(m)$ is the function as defined in the case $r = 2$, then A' contains a subset B' of size m which is 2-independent in $G' - S'$, for some $S' \subseteq V(G') - A'$. The set S' corresponds to some set of vertices $S \subseteq V(G)$ which are at distance at least $s + 1$ from each vertex in A , and the set B' corresponds to some subset B of A which is $(2s + 2)$ -independent in $G - S$. Moreover, as each vertex of S' is a neighbor of each vertex of B' , each vertex of S has distance exactly $s + 1 = r/2$ from each vertex of B .

An algorithm computing the sets B and S (in either the odd or even case) can be given as follows: simply run a breadth-first search from each vertex of A to compute the graph G' with the balls of radius $\lfloor \frac{r-1}{2} \rfloor$ around the vertices in A contracted to single vertices, and then run the algorithm for the case $r = 1$ or $r = 2$. This yields a running time of $\mathcal{O}_{r,t}(|A| \cdot |E(G)|)$.

This finishes the proof of [Lemma 1](#).

Remark 7. A more careful analysis of the case $r = 2s + 2$, in particular of the construction of G' and the properties of G' actually used in the proof of [Claim 1](#), shows that the bound of $d = 5s + 2$ may be decreased to $d = 3s + 2$. We refrain from giving the details for the sake of clarity of the argument.

2.3 Proof of [Theorem 1](#)

We now wrap up the proof of [Theorem 1](#) by iteratively applying [Lemma 1](#).

PROOF (OF [THEOREM 1](#)). Fix integers r, t , and a graph G such that $K_t \not\leq_d G$, for $d = \lfloor 5r/2 \rfloor$. Let $\beta > 2 \cdot t$ be a fixed real, and let \widehat{m} be the number from [Claim 2](#). As in the proof of [Lemma 1](#), without loss of generality suppose $m \geq \max(t, \widehat{m})$. Denote $\gamma = \beta^{2t}$, and define the function $L(m)$ as $L(m) = ((t+1) \cdot m)^\gamma$.

Define the sequence m_0, m_1, \dots, m_r as follows:

$$m_i = (t+1)^{\gamma^{r-i}} \cdot m^{\gamma^{r-i}}.$$

Note that $m_r \geq m$ and $m_i = L(m_{i+1})$, for all $i \in \{0, \dots, r-1\}$.

Suppose that A is a set of vertices of G such that $|A| \geq m_0 = (t+1)^{\gamma^r} \cdot m^{\gamma^r}$. We inductively construct sequences of sets $A = A_0 \supseteq A_1 \supseteq \dots \supseteq A_r$ and $\emptyset = S_0 \subseteq S_1 \subseteq S_2 \dots$ satisfying the following conditions:

- $|A_i| \geq m_i \geq L(m_{i+1})$,
- $A_i \cap S_i = \emptyset$ and A_i is i -independent in $G - S_i$.

To construct A_{i+1} out of A_i , apply [Lemma 1](#) to the graph $G - S_i$ and the i -independent set A_i of size at least $L(m_{i+1})$. This yields a set $S \subseteq V(G)$ which is disjoint from $S_i \cup A_i$, and a subset A_{i+1} of A_i of size at least m_{i+1} which is $(i+1)$ -independent in $G - S_{i+1}$, where $S_{i+1} = S \cup S_i$. This completes the inductive construction.

In particular, $|A_r| \geq m$ and A_r is a subset of A which is r -independent in $G - S_r$. Observe that by construction, $|S_r| \leq rt/2$, as in the odd steps, the constructed set S is empty, and in the even steps, it has at most t elements. We show that in fact we have $|S_r| < t$ using the following argument, similar to the one used in [Claim 3](#).

By the last part of the statement of [Lemma 1](#), at the i th step of the construction, each vertex of the set S has distance exactly $i/2$ from all vertices in A_{i+1} in the graph $G - S_i$. For $a \in A_r$, let $\overline{N}(a)$ denote the $\lfloor r/2 \rfloor$ -neighborhood of a in $G - S_r$; note that sets $\overline{N}(a)$ are pairwise disjoint. The above remark implies that each vertex v of the final set S_r has a neighbor in the set $\overline{N}(a)$ for each $a \in A_r$. Indeed, suppose v belonged to the set S added to S_r in the i th step of the construction; i.e. $v \in S_{i+1} - S_i$. Then there exists a path in $G - S_i$ from v to a of length exactly $i/2$, which traverses only vertices at distance less than $i/2$ from a . Since in this and further steps of the construction we were removing only vertices at distance at least $i/2$ from a , this path stays intact in $G - S_r$ and hence is completely contained in $\overline{N}(a)$.

By assumption that $m \geq t$, we may choose pairwise different vertices $a_1, \dots, a_t \in A_r$. To reach a contradiction, suppose that S_r contains t distinct vertices s_1, \dots, s_t . By the above, the sets $\overline{N}(a_i) \cup \{s_i\}$ form a minor model of K_t in G at depth- $(\lfloor r/2 \rfloor + 1)$. This contradicts the assumption that $K_t \not\leq_d G$ for $d = \lfloor 5r/2 \rfloor$. Hence, $|S| < t$.

Define the function $N : \mathbb{N} \rightarrow \mathbb{N}$ as $N(m) = ((t+1)m)^{\gamma^r} = ((t+1)m)^{\beta^{2rt}}$ for $m \geq \max(t, \widehat{m})$ and $N(m) = N(\widehat{m})$ for $m < \max(t, \widehat{m})$. Note that $N(m) \in \mathcal{O}_{r,t}(m^{(2t+1)^{2rt}})$, provided we put $\beta = 2t + 1$. The argument above shows that if $|A| \geq N(m)$ then there is a set $S \subseteq V(G)$, equal to S_r above, and a set $B \subseteq A$, equal to A_r above, so that B is r -independent in $G - S$. Given G and A , the sets S and B can be computed by applying the algorithm given by [Lemma 1](#) at most r times, so in time $\mathcal{O}_{r,t}(|A| \cdot |E(G)|)$. This finishes the proof of [Theorem 1](#). \square

3 From uniform quasi-wideness to stability

In this section we focus on proving the following result, observed earlier by Adler and Adler [1].

Theorem 3. *Let \mathcal{C} be a uniformly quasi-wide class of graphs. Then \mathcal{C} is stable.*

Recall that a class \mathcal{C} is stable if and only if for every first-order formula $\varphi(\bar{x}, \bar{y})$, its ladder index over graphs from \mathcal{C} is bounded by a constant depending only on \mathcal{C} and φ ; see Section 1 to recall the background on stability. Thus Theorem 3 is implied by Theorem 2 (stated in Section 1), and is weaker in the following sense: Theorem 2 asserts in addition that there is a computable bounds on the ladder index of any formula that depends only on the size of an excluded clique minor on a levels bounded in terms of formula's quantifier rank and number of free variables. At the end of the proof we argue that the obtained bounds in fact also imply the stronger statement of Theorem 2, but for the clarity of presentation we find it more instructive to work with the cleaner formulation of Theorem 3.

The plan is as follows. In Section 3.1 we formulate a variant of uniform quasi-wideness tailored to tuples of vertices. Using this and Gaifman's Locality Theorem, we prove Theorem 3 in Section 3.2.

3.1 Uniform quasi-widness for tuples

Fix a graph G , the dimension $d \in \mathbb{N}$, and the radius $r \in \mathbb{N}$. If $S \subseteq V(G)$ is a set of vertices and $A \subseteq V(G)^d$ is a set of d -tuples of vertices, then we say that A is *mutually r -independent* in $G - S$ if for every two distinct $(u_1, \dots, u_d), (v_1, \dots, v_d) \in A$ and for all $1 \leq i, j \leq d$, the distance between the vertices u_i and v_j in the graph $G - S$ is larger than r . Throughout this section we use the convention that if $x \in S$, then the distance in $G - S$ between x and any vertex, including x itself, is infinity. For instance, in the definition above, the tuples from A may contain vertices of S , and such a vertex is considered infinitely far from every vertex.

We now prove the following proposition, which can be viewed as an extension of uniform quasi-wideness to tuples. The proof is based on translating the approach of Podewski and Ziegler [22] to the finite.

Proposition 8. *Let \mathcal{C} be a uniformly quasi-wide class of graphs and $d \in \mathbb{N}$ be an integer. Then there are functions $N^d: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and $s^d: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $r, m \in \mathbb{N}$ and all subsets $A \subseteq V(G)^d$ with $|A| \geq N^d(r, m)$ there is a set $S \subseteq V(G)$ of size $|S| \leq s^d(r)$ and a subset $B \subseteq A$ of size $|B| \geq m$ which is mutually r -independent in $G - S$.*

The rest of this section is devoted to the proof of Proposition 8. Fix a uniformly quasi-wide class \mathcal{C} and functions $N: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and $s: \mathbb{N} \rightarrow \mathbb{N}$ as in the definition of uniform quasi-wideness. Let $d \in \mathbb{N}$ be a fixed dimension. For a fixed graph $G \in \mathcal{C}$ and coordinate $i \in \{1, \dots, d\}$, let π_i denote the projection from $V(G)^d$ onto the i th coordinate.

Lemma 9. *For any $r, m \in \mathbb{N}$ there is an integer $K(r, m)$ such that for any given $A \subseteq V(G)^d$ with $|A| \geq K(r, m)$, there is a set $B \subseteq A$ with $|B| \geq m$ and a set $S \subseteq V(G)$ with $|S| \leq d \cdot s(r)$, such that for each coordinate $i \in \{1, \dots, d\}$ and all distinct $\bar{x}, \bar{y} \in B$, $\pi_i(\bar{x})$ and $\pi_i(\bar{y})$ are at distance greater than r in $G - S$.*

PROOF. We will iteratively apply the following claim.

Claim 4. Fix a coordinate $i \in \{1, \dots, d\}$, an integer $m' \in \mathbb{N}$, and a set $A' \subseteq V(G)^d$ with $|A'| \geq N(r, m') \cdot m'$. Then there is a set $B' \subseteq D$ with $|B'| \geq m'$ and a set $S' \subseteq V(G)$ with $|S'| \leq s(r)$, such that for all distinct $\bar{x}, \bar{y} \in B'$, $\pi_i(\bar{x})$ and $\pi_i(\bar{y})$ are at distance greater than r in $G - S'$.

PROOF. We consider two cases.

If $\pi_i(A') \subseteq V(G)$ has at least $N(r, m')$ elements, then we apply the definition of uniform quasi-widness to $\pi_i(A') \subseteq V(G)$. This yields sets $S' \subseteq V(G)$ and $B'' \subseteq \pi_i(A')$ such that $|B''| \geq m'$, $|S'| \leq s(r)$, and B'' is r -independent in $G - S'$. Let $B' \subseteq A'$ be a subset of tuples constructed as follows: for each $u \in B''$, include in B' one arbitrarily chosen tuple $\bar{x} \in A'$ such that $\pi_i(\bar{x}) = u$. Clearly $|B'| = |B''| \geq m'$ and for all distinct $\bar{x}, \bar{y} \in B'$, we have that $\pi_i(\bar{x})$ and $\pi_i(\bar{y})$ are distinct and at distance greater than r in $G - S'$; this is because B'' is r -independent in $G - S'$. Hence B' and S' satisfy all the required properties.

If $\pi_i(A')$ has less than $N(r, m')$ elements, then choose the element $a \in \pi_i(A')$ whose inverse image $\pi_i^{-1}(\{a\}) \cap A'$ has the largest cardinality. Let $S' = \{a\}$ and let $B' = \pi_i^{-1}(\{a\})$. Then

$$|B'| \geq \frac{|A'|}{|\pi_i(A')|} \geq \frac{|A'|}{N(r, m')} \geq \frac{N(r, m') \cdot m'}{N(r, m')} = m',$$

and $|S'| = 1$. Observe that $\pi_i(\bar{x}) = a$ for all $\bar{x} \in B'$. As $a \in S'$, by the adopted convention, $\pi_i(\bar{x})$ and $\pi_i(\bar{y})$ are at infinite distance for all distinct $\bar{x}, \bar{y} \in B'$. \lrcorner

We proceed with the proof of [Lemma 9](#). Let $f(m') = N(r, m') \cdot m'$ for $m' \in \mathbb{N}$; by f^k we denote the k -fold composition of f with itself. Let $A \subseteq V(G)^d$ be such that $|A| \geq f^d(m)$. Define $B_0 = A$, $S_0 = \emptyset$, and for $i = 1, \dots, d$, let B_i and S_i be the B' and S' obtained from [Claim 4](#) applied to the set of tuples $B_{i-1} \subseteq V(G)^d$, the coordinate i , and $m' = f^{d-i}(m)$. The invariant is that $|B_i| \geq f^{d-i}(m)$. In particular, taking $B = B_d$ and $S = S_1 \cup \dots \cup S_d$, we obtain that $|B| \geq m$ and $|S| \leq d \cdot s(r)$, and, by construction, $\pi_i(B)$ is r -independent in $G - S$ for every coordinate $i \in \{1, \dots, d\}$. Letting $K(r, m) = f^d(m)$ yields the lemma. \square

Lemma 10. Let $B \subseteq V(G)^d$ and $S \subseteq V(G)$ be such that for all $i \in \{1, \dots, d\}$, $\pi_i(B)$ is $2r$ -independent in $G - S$. Then there is a set C with $C \subseteq B$ such that C is mutually r -independent in $G - S$ and $|C| \geq \frac{|B|}{d^2+1}$.

PROOF. We construct a sequence $C_0 \subseteq C_1 \subseteq \dots$ of subsets of B which are mutually r -independent in $G - S$, as follows.

We start with $C_0 = \emptyset$. Suppose that $C_s \subseteq B$ is already constructed for some $s \geq 0$ and is mutually r -independent in $G - S$; we construct C_{s+1} . With each element $a \in B - C_s$, we associate an arbitrarily chosen function $f_a: \{1, \dots, d\}^2 \rightarrow C_s \cup \{\perp\}$ with the following properties:

- If $f_a(i, j) = b$ then the i th coordinate of a and the j th coordinate of b are at distance at most r in $G - S$.
- If $f_a(i, j) = \perp$ then there is no element $b \in C_s$ such that the i th coordinate of a and the j th coordinate of b are at distance at most r in $G - S$.

Observe that whenever a_1, a_2 are two distinct elements of $B - C_s$, then for all $i, j \in \{1, \dots, d\}^2$, the values $f_{a_1}(i, j)$ and $f_{a_2}(i, j)$ cannot be equal to the same element $b \in C_s$: otherwise, we would have that the i th coordinate of a_1 and the i th coordinate of a_2 are at distance at most $2r$ in $G - S$, which is impossible by the assumption on B . In particular, if $|B - C_s| > |C_s| \cdot d^2$ then there must be some element $a \in B - C_s$ such that $f_a(i, j) = \perp$ for all $i, j \in \{1, \dots, d\}$. Let $C_{s+1} = C_s \cup \{a\}$. By construction, C_{s+1} is mutually r -independent in $G - S$.

We may repeat the construction as long as $|B| > |C_s| \cdot (d^2 + 1) = s \cdot (d^2 + 1)$, and we stop when this inequality no longer holds. Define the set C as the last constructed set C_s . By construction, $|C_s| = s \geq \frac{|B|}{d^2 + 1}$. \square

To finish the proof of [Proposition 8](#), given a set $A \subseteq V(G)^d$ and integers $r, m \in \mathbb{N}$, first apply [Lemma 9](#) with $r' = 2r$ and $m' = m \cdot (d^2 + 1)$. Assuming that $|A| \geq K(r', m')$, we obtain a set $B \subseteq A$ with $|B| \geq m \cdot (d^2 + 1)$ and a set $S \subseteq V(G)$ with $|S| \leq s(2r)$. Apply [Lemma 10](#) to B and S , yielding a set $C \subseteq B$ which is mutually r -independent in $G - S$ and has size at least m . This concludes the proof of [Proposition 8](#), where the obtained bounds are $N^d(r, m) = K(r', m') = K(2r, m \cdot (d^2 + 1))$ and $s^d(r) = d \cdot s(2r)$.

3.2 Excluding long ladders

We consider only finite sets of variables, and any mathematical object can be considered a variable (formally, whenever we want to treat an object a as a variable, we introduce a variable x_a , where x is a fixed special symbol). If φ is a formula then it has a specified set of free variables. By abuse of language, if X is a set of variables and φ is a formula, when we say that φ *has free variables* X we allow the set of free variables of φ to be a subset of X .

If X is a finite set and V is a set, then V^X denotes the set of functions from X to V , and an element $v \in V^X$ will be sometimes called a *tuple* or, in the case X is a set of variables, a *valuation* of X . For two tuples $v \in V^Y$ and $w \in W^Z$, where Y and Z are disjoint, by $v \oplus w$ we denote the set-theoretic union of the functions v, w , which is a tuple $v \oplus w \in (V \cup W)^{Y \cup Z}$. The image of a valuation v is denoted by $\text{im}(v)$. If $V = V(G)$ for some graph G , then we can treat valuations of X as tuples of length $|X|$ consisting of vertices of G , ordered in the same way according to any fixed enumeration of X . Then we may say that a set of valuations is mutually r -independent in G , or in $G - S$ for some $S \subseteq V(G)$, meaning the notion discussed in the previous section.

Fix a formula φ with free variables X . Given a valuation $v \in V(G)^X$, we write $G, v \models \varphi$ to denote that φ is satisfied in the graph G with valuation v , which is defined as usual in logic, by induction on the structure of φ .

In this language, assuming that X is partitioned into disjoint sets Y, Z , a φ -ladder of length n in a graph G consists of two sequences of valuations $u_1, \dots, u_n \in V(G)^Y$ and $v_1, \dots, v_n \in V(G)^Z$ such that $G, u_i \oplus v_j \models \varphi$ if and only if $i \leq j$.

Let $u, v \in V(G)^X$, u_Y and u_Z be the restrictions of u to Y and Z , respectively, and v_Y and v_Z be the restrictions of v to Y and Z , respectively. We say that evaluations u and v are *confusing* for φ if

$$G, u_Y \oplus v_Z \models \varphi \iff G, v_Y \oplus u_Z \models \varphi.$$

The key technical lemma of this section is the following.

Proposition 11. *Let φ be a formula with quantifier rank q and free variables X partitioned as $X = Y \cup Z$. Further, let G be a graph and $S \subseteq V(G)$ be a set of vertices. Then one can associate to each tuple $v \in V(G)^X$ its φ -type, so that the following conditions hold:*

- (1) *The set of φ -types of all tuples $v \in V(G)^X$ is finite and has size bounded by $T(|S|, q)$, where $T: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is some computable function.*
- (2) *If $u, v \in V(G)^X$ are different, have the same φ -types, and $\{u, v\}$ is mutually $2r(q)$ -independent in $G - S$, then u and v are confusing for φ , where $r: \mathbb{N} \rightarrow \mathbb{N}$ is some computable function.*

Before proving [Proposition 11](#), we show how it yields [Theorem 3](#).

PROOF (OF [THEOREM 3](#)). Fix a formula φ of quantifier rank q and with free variables X , partitioned into disjoint sets Y, Z . For $d = |X|$, let $N^d(\cdot, \cdot)$ and $s^d(\cdot)$ be functions yielded by [Proposition 8](#). Let $T(\cdot, \cdot)$ and $r(\cdot)$ be the functions given by [Proposition 11](#). We assume without loss of generality that the function $T(\cdot, \cdot)$ is weakly increasing in the first argument, by redefining $T(s, q)$ as $\max_{s' \leq s} T(s', q)$. Denote $r = r(q)$, $m = T(s^d(2r), q) + 1$, and $\ell = N^d(2r, m)$.

We show that every φ -ladder in a graph $G \in \mathcal{C}$ has length smaller than ℓ . For the sake of contradiction, assume that there is a graph $G \in \mathcal{C}$, a number $k \geq \ell$, and valuations $u_1, \dots, u_k \in V(G)^Y$ and $v_1, \dots, v_k \in V(G)^Z$ which form a φ -ladder in G . Denote $w_i = u_i \oplus v_i$, for $i = 1, \dots, k$. Let $A = \{w_i: i = 1, \dots, k\} \subseteq V(G)^X$; note that $|A| = k$. Applying [Proposition 8](#) to the set A , radius $2r$, and target size m yields a set $S \subseteq V(G)$ with $|S| \leq s^d(2r)$ and a set $B \subseteq A$ with $|B| \geq m$ which is mutually $2r$ -independent in $G - S$. With each tuple $w_i \in B$ we associate its φ -type, as described in [Proposition 11](#). Since B has at least $m > T(|S|, q)$ elements, by the pigeonhole principle, there are i and j with $i < j$ such that w_i and w_j have the same φ -types. Since $\{w_i, w_j\} \subseteq B$ is mutually $2r$ -independent in $G - S$, w_i and w_j are confusing for φ , i.e., $G, u_i \oplus v_j \models \varphi$ if and only if $G, u_j \oplus v_i \models \varphi$. This contradicts the assumption that u_1, \dots, u_k and v_1, \dots, v_k form a φ -ladder, and finishes the proof of [Theorem 3](#). \square

It remains to prove [Proposition 11](#). First we recall some notions from logic, namely quantifier ranks, Gaifman's Locality Theorem, and some simple manipulations on formulas.

Fix a finite set of colors C . By a *colored graph* we mean a graph in which every vertex is assigned zero or more colors from C . We view a colored graph as a relational structure as usual, by treating each color as a unary predicate.

The *quantifier rank* of a formula φ is the maximal number of nested quantifiers in φ . For $i = 1, 2$, let G_i be a colored graph and $v_i: X \rightarrow V(G_i)$ be a valuation from a common set of variables X . We say that (G_1, v_1) and (G_2, v_2) have the same *quantifier rank q type* if for every formula φ with free variables X and of quantifier rank q ,

$$G_1, v_1 \models \varphi \quad \iff \quad G_2, v_2 \models \varphi.$$

If G is a graph colored with colors from C and $v: X \rightarrow V(G)$ is a valuation, then the equivalence class of (G, v) under the above equivalence relation is called the *quantifier rank q type* of (G, v) , and the set of *quantifier rank q types with free variables X* is the set of all equivalence classes, denoted $\text{Tp}_X^{q, C}$.

If G is a colored graph, $r \in \mathbb{N}$ is an integer, and v is a valuation of a set of variables in G , then $N_r^G[v]$ denotes the pair (H, v) , where H is the colored subgraph of G induced by the set

of all vertices which are in distance at most r from some vertex in the image of v . If r is an integer, then the (r, q) -local type of (G, v) is the quantifier rank q type of $N_r^G[v]$. The q -local type of (G, v) is the (r, q) -local type, where r is the value described in the second item of the following proposition, summarizing several well-known properties of types and local types.

Proposition 12. *Fix a positive integer q , a finite set of colors C , and a set of variables X . Let G be any C -colored graph. Then the following conditions hold:*

- (1) (Computability of types) *The set of types $\text{Tp}_X^{q,C}$ is finite and computable from C , X , and q .*
- (2) (Locality of first order logic) *There is an integer r computable from q such that for every formula φ in the signature of C -colored graphs of quantifier rank q and with free variables X , if (G, v_1) and (G, v_2) have the same (r, q) -local types, then*

$$G_1, v_1 \models \varphi \quad \iff \quad G_2, v_2 \models \varphi.$$

- (3) (Independence) *Suppose X is partitioned into $Y \cup Z$, and that $u_1, u_2: Y \rightarrow V(G)$ and $v_1, v_2: Z \rightarrow V(G)$ are valuations such that each of the sets $\{u_1, v_1\}$ and $\{u_2, v_2\}$ is mutually $2r$ -independent in G , for some r . Moreover, suppose that the (r, q) -local types of $(G, \text{im}(u_1))$ and $(G, \text{im}(u_2))$ are equal, and that the (r, q) -local types of $(G, \text{im}(v_1))$ and $(G, \text{im}(v_2))$ are equal. Then the (r, q) -local types of $(G, u_1 \oplus v_1)$ and $(G, u_2 \oplus v_2)$ are equal.*

PROOF. *Computability of types* is standard (see e.g. Lemma 3.13 in [16]).

Locality of first order logic is an immediate consequence of Gaifman's Locality Theorem (the Main Theorem in [9]), where it is shown that one can take $r = 7^q$. It is also known (cf. Corollary 4.13 in [16]) that it suffices to take $r = \frac{3^{q+1}-1}{2}$.

For *Independence*, write $G \oplus H$ for the disjoint union of colored graphs G, H . We use the following claim, whose proof is a standard application of Ehrenfeucht-Fraisse games.

Claim 5. *For $i = 1, 2$, let G_i, H_i be colored graphs, and $v_i: Y \rightarrow V(G_i)$ and $w_i: Z \rightarrow V(H_i)$ be valuations. Suppose that (G_1, v_1) and (G_2, v_2) have the same quantifier rank q type, and (H_1, w_1) and (H_2, w_2) have the same quantifier rank q type. Then the quantifier rank q type of the disjoint union $(G_1 \oplus H_1, v_1 \oplus w_1)$ is equal to the one of $(G_2 \oplus H_2, v_2 \oplus w_2)$.*

PROOF (SKETCH). The proof proceeds by applying the well-known characterization of quantifier rank q types using Ehrenfeucht-Fraisse games (see e.g. Theorem 3.9 in [16]). By assumption, duplicator has a winning strategy γ in the q -round game on (G_1, v_1) and (G_2, v_2) , and a winning strategy η in the q -round game on (H_1, w_1) and (H_2, w_2) . The strategies γ and η can be combined into a winning strategy on $(G_1 \oplus H_1, v_1 \oplus w_1)$ and $(G_2 \oplus H_2, v_2 \oplus w_2)$. \lrcorner

The *Independence* property is now almost immediate. Since $\{u_1, v_1\}$ is mutually $2r$ -independent in G , the subgraph of G induced by the r -neighborhood of $\text{im}(u_1 \oplus v_1)$ is isomorphic to the disjoint union of the subgraphs of G induced by the r -neighborhoods of $\text{im}(u_1)$ and $\text{im}(v_1)$. The same holds also for the r -neighborhoods of $\text{im}(u_2 \oplus v_2)$, $\text{im}(u_2)$, and $\text{im}(v_2)$. It now suffices to apply Claim 5 and use the assumed equality of (r, q) -local types. \square

We now prove [Proposition 11](#). Until the end of the proof, fix a graph G and a set of vertices $S \subseteq V(G)$. We now introduce some notation allowing to translate a formula φ talking about G into an equivalent formula φ' talking about a suitably colored graph G with the set of vertices S removed. Precisely, define the structure G^S as the graph $G - S$ colored with colors $\{C_s : s \in S\}$ as follows. For each $s \in S$, a vertex $v \in V(G) - S$ is colored with color C_s in G^S if and only if v is a neighbor of s in G .

Fix a formula φ with free variables X . If G is a (colored) graph, $S \subseteq V(G)$ is a set of vertices, and Y is a set of variables disjoint from $V(G)$, then a *pre-valuation* in $Y \cup S$ is a function $\alpha : X \rightarrow Y \cup S$. If $v : Y \rightarrow V(G)$ is a valuation and α is as above, then by $\alpha \cdot v$ we denote the valuation of X in $V(G)$ which maps $x \in X$ to $\alpha(x)$ if $\alpha(x) \in S$ and to $v(\alpha(x))$ if $\alpha(x) \in Y$. The pair (φ, α) can be treated as a syntactic object denoted φ^α , whose semantics is defined so that for a valuation $v : Y \rightarrow V(G)$,

$$G, v \models \varphi^\alpha \quad \iff \quad G, \alpha \cdot v \models \varphi.$$

Intuitively φ^α is the formula φ with variable x substituted by $\alpha(x)$, which can be either a variable in Y or a vertex in S , treated as a constant.

Lemma 13. *Let G, S be as above. For every formula φ with free variables X and every pre-valuation $\alpha : X \rightarrow Y \cup S$ there is a formula φ' with free variables Y of the same quantifier rank as φ and over the signature of G^S such that for every valuation v of Y in $G - S$, we have*

$$G, v \models \varphi^\alpha \quad \iff \quad G^S, v \models \varphi'.$$

PROOF. The proof proceeds by induction on the structure of the formula φ .

If φ is an atomic formula $E(x, x')$ or $x = x'$, then the formula φ' is constructed by case analysis. If $\alpha(x), \alpha(x') \in Y$ then φ' is obtained from φ by substituting the variables x, x' with variables from Y according to α . If $\alpha(x), \alpha(x') \in S$ then φ' is the truth value \perp or \top of the formula φ in the graph G under the valuation which maps x to $\alpha(x)$ and x' to $\alpha(x')$. Finally, suppose that $\alpha(x) = y \in Y$ and $\alpha(x') = s \in S$. If φ is $E(x, x')$ then φ' is the formula $C_s(y)$, and if φ is $x = x'$ then φ' is the formula \perp .

For the inductive step, we consider two cases. If φ is a boolean combination of formulas $\varphi_1, \dots, \varphi_k$, then apply the inductive assumption to each formula φ_i , yielding formulas $\varphi'_1, \dots, \varphi'_k$. Then let φ' be the analogous boolean combination of the formulas $\varphi'_1, \dots, \varphi'_k$.

Finally, suppose that φ is of the form $\exists x. \psi$, where Y are the free variables of φ and $x \notin Y$. For w being either the variable x or an element $s \in S$, let ψ^w be the formula obtained from the inductive assumption applied to the formula ψ and pre-valuation α extended to a valuation which maps x to w . Then let φ' be the formula $\exists x. \psi^x \vee \bigvee_{s \in S} \psi^s$. The case of \forall is dual.

In each case, it follows from the inductive assumption that φ' satisfies the required condition. \square

Let X be a set of variables. For a valuation v of X in G , we introduce the notion of an *S-decomposition* of v , which is the (essentially unique) pair (α, v^S) such that $\alpha : X \rightarrow Y \cup S$ is a pre-valuation for some set of variables Y , and $v^S : Y \rightarrow (\text{im}(v) - S)$ is a bijective valuation such that $\alpha \cdot v^S = v$. The formal definition is as follows. Let $Y = \{v^{-1}(\{u\}) : u \in \text{im}(v) - S\}$. We treat Y as a set of variables. Define the pre-valuation $\alpha : X \rightarrow Y \cup S$ by letting $\alpha(x)$ be $v(x)$ if $v(x) \in S$, and $v^{-1}(\{u\})$ if $v(x) = u$ for some $u \in \text{im}(v) - S$. Finally, let v^S be the valuation of Y

in G which maps $v^{-1}(\{u\})$ to u , for $u \in \text{im}(v) - S$. It is easy to see that $v = \alpha \cdot v^S$ and v^S is a bijection from Y to $\text{im}(v) - S$. We call the pair (α, v^S) the S -decomposition of v .

For a number q and valuation $v: X \rightarrow V(G)$, the (q, S) -local type of v is the pair (α, τ) , where (α, v^S) is the S -decomposition of v and τ is the q -local type of (G^S, v^S) . Note that there are at most $(s + d)^d$ possible functions α , where $s = |S|$ and $d = |X|$. In particular, by the computability of types (cf. [Proposition 12](#)), the number of (q, S) -local types of valuations from X in arbitrary graphs is bounded by a number computable from s, d , and q .

Lemma 14. *Let φ be a formula with free variables X and of quantifier rank q . Suppose that u and v are two valuations of X in G with the same (q, S) -local types. Then*

$$G, u \models \varphi \iff G, v \models \varphi.$$

PROOF. Let (α, τ) be the (q, S) -local type of the valuations u and v , where $\alpha: X \rightarrow Y \cup S$ for some set of variables Y . Let (α, u^S) be the S -decomposition of u and let (α, v^S) be the S -decomposition of v . Consider the formulas φ^α and φ' as described in [Lemma 13](#), both with free variables Y . In particular, the following equivalences hold:

$$\begin{aligned} G, u \models \varphi &\iff G, u^S \models \varphi^\alpha \iff G^S, u^S \models \varphi', \\ G, v \models \varphi &\iff G, v^S \models \varphi^\alpha \iff G^S, v^S \models \varphi'. \end{aligned}$$

Note that φ' has the same quantifier rank as φ , that is, q . Since u and v have the same (q, S) -local type τ , it follows that (G^S, u^S) and (G^S, v^S) have the same q -local type. By the locality of first order logic (cf. [Proposition 12](#)) applied to G^S, φ', u^S , and v^S , we infer that $G^S, u^S \models \varphi'$ if and only if $G^S, v^S \models \varphi'$. The lemma follows by combining this with the above equivalences. \square

We are now ready to prove [Proposition 11](#).

PROOF (OF [PROPOSITION 11](#)). Let φ be a formula of quantifier rank q whose free variables X are partitioned into Y and Z . Let C be the set of colors $\{C_s : s \in S\}$. Let r be the number given by the locality of first order logic (cf. [Proposition 12](#)).

For a valuation $v \in V(G)^X$, define the φ -type of v as the ordered pair consisting of two (q, S) -local types: of $v|_Y$ and of $v|_Z$. It is clear that condition (1) of [Proposition 11](#) is satisfied. We are left with verifying condition (2).

Let $u, v \in V(G)^X$ be such that $\{u, v\}$ is mutually $2r$ -independent in $G - S$, and moreover u and v have the same φ -type. We show that u, v are confusing for φ . Let u_Y, u_Z and v_Y, v_Z denote the restrictions of u and v to Y and Z , respectively.

Claim 6. *Valuations $u_Y \oplus v_Z$ and $v_Y \oplus u_Z$ have the same (q, S) -local types.*

PROOF. Let $(\alpha_Y, u_Y^S), (\alpha_Z, u_Z^S), (\beta_Y, v_Y^S), (\beta_Z, v_Z^S)$ denote the S -decompositions of u_Y, u_Z, v_Y, v_Z , respectively. From the assumption that u, v are mutually $2r$ -independent in $G - S$ it follows in particular that $\text{im}(u_Y) \cap \text{im}(v_Z) \subseteq S$ and $\text{im}(v_Y) \cap \text{im}(u_Z) \subseteq S$. This implies that the S -decompositions of $u_Y \oplus v_Z$ and of $v_Y \oplus u_Z$ can be computed in a component wise fashion, and are as follows:

$$u_Y \oplus v_Z: (\alpha_Y \oplus \beta_Z, u_Y^S \oplus v_Z^S), \tag{1}$$

$$v_Y \oplus u_Z: (\beta_Y \oplus \alpha_Z, v_Y^S \oplus u_Z^S). \tag{2}$$

The assumption that u, v have the same φ -type implies the following:

- $\alpha_Y = \beta_Y$ and $\alpha_Z = \beta_Z$. In particular, the S -decompositions (1) and (2) have the first components equal.
- Valuations u_Y^S and v_Y^S have the same q -local types, and similarly, v_Z^S and u_Z^S have the same q -local types. Since $\{u, v\}$ is mutually $2r$ -independent in $G - S$, it follows that both $\{u_Y^S, v_Z^S\}$ and $\{v_Y^S, u_Z^S\}$ are mutually $2r$ -independent in G^S . By the independence property of [Proposition 12](#), the second components of the S -decompositions (1) and (2) have the same q -local type.

The two observations above yield the conclusion of the claim. \square

By [Claim 6](#) and [Lemma 14](#) we infer that u and v are confusing for φ . This finishes the proof. \square

We conclude by proving [Theorem 2](#), promised in [Section 1](#), which follows by tracking the precise dependencies on the parameters in the proof of [Theorem 3](#).

PROOF (OF [THEOREM 2](#)). Let $r(\cdot)$ be the function described in [Proposition 11](#). We define the function $g(\cdot)$ from the statement of the theorem by $g(q) = 10 \cdot r(q)$. Let φ be the given formula of quantifier rank q and with free variables X , partitioned into Y and Z . Denote $r = r(q)$. From now on we consider only graphs G such that $K_t \not\preceq_{10r} G$.

We first examine [Proposition 8](#), and in particular the dependence of the yielded functions $N^d(\cdot, \cdot)$ and $s^d(\cdot)$ on the assumed quasi-wideness properties of the class \mathcal{C} . Precisely, having assumed that the underlying class \mathcal{C} is quasi-wide with functions $N(\cdot, \cdot)$ and $s(\cdot)$, we have obtained:

$$N^d(2r, m) = K(4r, m(d^2 + 1)) \quad \text{and} \quad s^d(2r) = d \cdot s(4r),$$

where $K(4r, m')$ is the d -fold composition of the function $f(m') = N(4r, m') \cdot m'$. Thus, when establishing the values of $N^d(2r, m)$ and $s^d(2r)$, we refer to the quasi-wideness of \mathcal{C} only by using numbers $s(4r)$ and $N(4r, m')$ for $m' \in \mathbb{N}$. By [Theorem 1](#), it suffices to assume that $K_t \not\preceq_{10r} G$ to have $s(4r) \leq t$ and $N(4r, m') \leq c(r, t) \cdot (m')^{(2t+1)^{8rt}}$ for some computable function $c(r, t)$. Hence, this supposition alone, instead of full quasi-wideness of \mathcal{C} , is sufficient to claim that the conclusion of [Theorem 1](#) holds with $s^d(2r)$ bounded by a computable function of t , d , and q , and $N^d(2r, m)$ bounded by a computable function of m , t , d , and q .

Finally, in the proof of [Theorem 3](#) we have argued that the ladder index of φ is bounded by $N^d(2r, m)$, where $d = |X|$, $m = T(s^d(2r), q) + 1$ and $T(\cdot, \cdot)$ is a computable function, and the only reference to the quasi-wideness of \mathcal{C} in the proof is by invoking [Proposition 8](#). As we argued above, in [Proposition 8](#) it suffices to assume $K_t \not\preceq_{10r} G$ to claim that $N^d(2r, m)$ is bounded by a computable function of m , t , d , and q . Since m is bounded by a computable function of t , d , and q , the obtained upper bound on the ladder index of φ depends in a computable way on t , d , and q , and to derive it we only need to assume that $K_t \not\preceq_{10r} G$. This concludes the proof. \square

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