

Lossy kernels for connected distance- r domination on nowhere dense graph classes *

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Abstract

For $\alpha: \mathbb{N} \rightarrow \mathbb{R}$, an α -approximate bi-kernel is a polynomial-time algorithm that takes as input an instance (I, k) of a problem \mathcal{Q} and outputs an instance (I', k') of a problem \mathcal{Q}' of size bounded by a function of k such that, for every $c \geq 1$, a c -approximate solution for the new instance can be turned into a $c \cdot \alpha(k)$ -approximate solution of the original instance in polynomial time. This framework of *lossy kernelization* was recently introduced by Lokshtanov et al. [21].

We prove that for every nowhere dense class of graphs, every $\alpha > 1$ and $r \in \mathbb{N}$ there exists a polynomial p (whose degree depends only on r while its coefficients depend on α) such that the connected distance- r dominating set problem with parameter k admits an α -approximate bi-kernel of size $p(k)$.

Furthermore, we show that this result cannot be extended to more general classes of graphs which are closed under taking subgraphs by showing that if a class \mathcal{C} is somewhere dense and closed under taking subgraphs, then for some value of $r \in \mathbb{N}$ there cannot exist an α -approximate bi-kernel for the (connected) distance- r dominating set problem on \mathcal{C} for any function $\alpha: \mathbb{N} \rightarrow \mathbb{R}$ (assuming the Gap Exponential Time Hypothesis).

1 Introduction

Lossy kernelization. A powerful method in parameterized complexity theory is to compute on input (I, k) a problem *kernel* in a polynomial time pre-computation step, that is, to reduce the input instance in polynomial time to an equivalent instance (I', k') of size $g(k)$ for some function g bounded in the parameter only. If the reduced instance (I', k') belongs to a different problem than (I, k) , we speak of a *bi-kernel*. It is well known that a problem is fixed-parameter tractable if and only if it admits a kernel, however, in general the function g can grow arbitrarily fast. For practical applications we are mainly interested in linear or at worst polynomial kernels. We refer to the textbooks [5, 8, 9] for extensive background on parameterized complexity and kernelization.

One shortcoming of the above notion of kernelization is that it does not combine well with approximations or heuristics. An approximate solution on the reduced instance provides no insight whatsoever about the original instance, the only statement we can derive from the definition of a kernel is that the reduced instance (I', k') is a positive instance if and only if the original instance (I, k) is a positive instance. This issue was recently addressed by Lokshtanov et al. [21], who introduced the framework of *lossy kernelization*. Intuitively, the framework combines notions from approximation and kernelization algorithms to allow for approximation preserving kernels.

Formally, a *parameterized optimization (minimization or maximization) problem* Π over finite vocabulary Σ is a computable function $\Pi: \Sigma^* \times \mathbb{N} \times \Sigma^* \rightarrow \mathbb{R} \cup \{\pm\infty\}$. A *solution*

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for an instance $(I, k) \in \Sigma^* \times \mathbb{N}$ is a string $s \in \Sigma^*$, such that $|s| \leq |I| + k$. The *value* of the solution s is $\Pi(I, k, s)$. For a minimization problem, the *optimum value* of an instance (I, k) is $\text{Opt}_\Pi(I, k) = \min_{s \in \Sigma^*, |s| \leq |I| + k} \Pi(I, k, s)$, for a maximization problem it is $\text{Opt}_\Pi(I, k) = \max_{s \in \Sigma^*, |s| \leq |I| + k} \Pi(I, k, s)$. An *optimal solution* is a solution s with $\Pi(I, k, s) = \text{Opt}_\Pi(I, k)$. If Π is clear from the context, we simply write $\text{Opt}(I, k)$.

A vertex-subset graph problem \mathcal{Q} defines which subsets of the vertices of an input graph are feasible solutions. We consider the following parameterized minimization problem associated with \mathcal{Q} :

$$\mathcal{Q}(G, k, S) = \begin{cases} \infty & \text{if } S \text{ is not a valid solution for } G \text{ as determined by } \mathcal{Q} \\ \min\{|S|, k + 1\} & \text{otherwise.} \end{cases}$$

Note that this bounding of the objective function at $k + 1$ does not make sense for approximation algorithms if one insists on k being the unknown optimum solution of the instance I . The parameterization above is by the value of the solution that we want our algorithms to output.

► **Approximate polynomial time pre-processing.** Let $\alpha: \mathbb{N} \rightarrow \mathbb{R}$ be a function and let Π be a parameterized minimization problem. An α -*approximate polynomial time pre-processing algorithm* \mathcal{A} for Π is a pair of polynomial time algorithms. The first algorithm is called the *reduction algorithm*, and computes a map $R_{\mathcal{A}}: \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$. Given as input an instance (I, k) of Π , the reduction algorithm outputs another instance $(I', k') = R_{\mathcal{A}}(I, k)$. The second algorithm is called the *solution lifting algorithm*. It takes as input an instance $(I, k) \in \Sigma^* \times \mathbb{N}$, the output instance (I', k') of the reduction algorithm, and a solution s' to the instance (I', k') . The solution lifting algorithm works in time polynomial in $|I|, k, |I'|, k'$ and s' , and outputs a solution s to (I, k) such that

$$\frac{\Pi(I, k, s)}{\text{Opt}(I, k)} \leq \alpha(k) \cdot \frac{\Pi(I', k', s')}{\text{Opt}(I', k')}.$$

► **Approximate kernelization.** An α -*approximate kernelization algorithm* is an α -approximate polynomial time pre-processing algorithm for which we can prove an upper bound on the size of the output instances in terms of the parameter of the instance to be pre-processed. We speak of a linear or polynomial kernel, if the size bound is linear or polynomial, respectively. If we allow the reduced instance to be an instance of another problem, we speak of an α -*approximate bi-kernel*.

We refer to the work of Lokshantov et al. [21] for an extensive discussion of related work and examples of problems that admit lossy kernels.

Nowhere denseness and domination. The notion of nowhere denseness was introduced by Nešetřil and Ossona de Mendez [25, 26] as a general model of *uniform sparseness* of graphs. Many familiar classes of sparse graphs, like planar graphs, graphs of bounded tree-width, graphs of bounded degree, and all classes that exclude a fixed (topological) minor, are nowhere dense. An important and related concept is the notion of a graph class of *bounded expansion*, which was also introduced by Nešetřil and Ossona de Mendez [22, 23, 24]. Before we give the formal definitions, we remark that all graphs in this paper are finite, undirected and simple. We refer to the textbook [7] for all undefined notation.

► **Shallow subdivisions.** Let H be a graph and let $r \in \mathbb{N}$. An r -*subdivision* of H is obtained by replacing all edges of H by internally vertex disjoint paths of length (exactly) r . We write H_r for the r -subdivision of H .

► **Nowhere denseness.** A class \mathcal{C} of graphs is *nowhere dense* if there exists a function $t: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $r \in \mathbb{N}$ and for all $G \in \mathcal{C}$ we do not find the r -subdivision of the complete graph $K_{t(r)}$ as a subgraph of G . Otherwise, \mathcal{C} is called *somewhere dense*.

Nowhere denseness turns out to be a very robust concept with several seemingly unrelated natural characterizations. These include characterizations by the density of shallow (topological) minors [25, 26], quasi-wideness [26], low tree-depth colorings [22], generalized coloring numbers [32], sparse neighborhood covers [16, 17], by so-called splitter games [17] and by the model-theoretic concepts of stability and independence [1]. For extensive background we refer to the textbook of Nešetřil and Ossona de Mendez [27].

► **Domination and distance- r domination.** In the parameterized *dominating set problem* we are given a graph G and an integer parameter k , and the objective is to determine the existence of a subset $D \subseteq V(G)$ of size at most k such that every vertex u of G is *dominated* by D , that is, either u belongs to D or has a neighbor in D . More generally, for fixed $r \in \mathbb{N}$, in the *distance- r dominating set problem* we are asked to determine the existence of a subset $D \subseteq V(G)$ of size at most k such that every vertex $u \in V(G)$ is within distance at most r from a vertex of D . In the *connected (distance- r) dominating set problem* we additionally demand that the (distance- r) dominating set shall be connected.

The dominating set problem plays a central role in the theory of parameterized complexity, as it is a prime example of a $W[2]$ -complete problem with the size of the optimal solution as the parameter, thus considered intractable in full generality. For this reason, the (connected) dominating set problem and distance- r dominating set problem have been extensively studied on restricted graph classes. A particularly fruitful line of research in this area concerns kernelization for the (connected) dominating set problem [2, 3, 13, 14, 15, 28]. For the more general distance- r dominating set problem we know the following results. Dawar and Kreutzer [6] showed that for every $r \in \mathbb{N}$ and every nowhere dense class \mathcal{C} , the distance- r dominating set problem is fixed-parameter tractable on \mathcal{C} . Drange et al. [10] gave a linear bi-kernel for distance- r dominating sets on any graph class of bounded expansion for every $r \in \mathbb{N}$, and a pseudo-linear kernel for dominating sets on any nowhere dense graph class; that is, a kernel of size $\mathcal{O}(k^{1+\varepsilon})$, where the \mathcal{O} -notation hides constants depending on ε . Precisely, the kernelization algorithm of Drange et al. [10] outputs an instance of an annotated problem where some vertices are not required to be dominated; this will be the case in the present paper as well. Kreutzer et al. [19] provided a polynomial bi-kernel for the distance- r dominating set problem on every nowhere dense class for every fixed $r \in \mathbb{N}$ and finally, Eickmeyer et al. [12] could prove the existence of pseudo-linear bi-kernels of size $\mathcal{O}(k^{1+\varepsilon})$, where the \mathcal{O} -notation hides constants depending on r and ε .

It is known that bounded expansion classes of graphs are the limit for the existence of polynomial kernels for the connected dominating set problem. Drange et al. [10] gave an example of a subgraph-closed class of bounded expansion which does not admit a polynomial kernel for connected dominating sets unless $\text{NP} \subseteq \text{coNP}/\text{Poly}$. They also showed that nowhere dense classes are the limit for the fixed-parameter tractability of the distance- r dominating set problem if we assume closure under taking subgraphs (in the following, classes which are closed under taking subgraphs will be called *monotone classes*).

Our results. We prove that for every nowhere dense class of graphs, every $\alpha > 1$ and $r \in \mathbb{N}$ there exists a polynomial p (whose degree depends only on r while its coefficients depend on α) such that the connected distance- r dominating set problem with parameter k admits an α -approximate bi-kernel of size $p(k)$. Our result extends an earlier result by Eiben et al. [11], who proved that the connected dominating set problem admits α -approximate bi-kernels

of linear size on classes of bounded expansion. Note that due to the before mentioned hardness result of connected dominating set on classes of bounded expansion we cannot expect to obtain an α -approximate bi-kernel of polynomial size for $\alpha = 1$, as this lossless bi-kernel would in particular imply the existence of a polynomial bi-kernel for the problem. However, our proof can easily be adapted to provide α -approximate bi-kernels for $\alpha = 1$ for the distance- r dominating set problem.

Our proof follows the approach of Eiben et al. [11] for connected dominating set ($r = 1$) on classes of bounded expansion. First, we compute a small set $Z \subseteq V(G)$ of vertices, called a (k, r) -domination core, such that every set of size at most k which r -dominates Z will also be a distance- r dominating set of G . The existence of a (k, r) -domination core on nowhere dense graph classes of size polynomial in k was recently proved by Kreuzer et al. [20]. We remark that the notion of a c -exchange domination core for a constant c , which was used by Eiben et al. [11], cannot be applied in the nowhere dense setting, as the constant c must be chosen in relation to the edge density of shallow subdivisions, an invariant that can be unbounded in nowhere dense classes.

Having found a domination core of size polynomial in k , the next step is to reduce the number of dominators, i.e. vertices whose role is to dominate other vertices, and the number of connectors, i.e. vertices whose role is to connect the solution. We apply the techniques of Eiben et al. [11] based on approximation techniques for the Steiner Tree problem. The main difficulty at this point is to find a polynomial bounding the size of the lossy kernel whose degree is independent of α .

Finally, we prove that this result cannot be extended to more general classes of graphs which are monotone by showing that if a class \mathcal{C} is somewhere dense and monotone, then for some value of $r \in \mathbb{N}$ there cannot exist an α -approximate bi-kernel for the (connected) distance- r dominating set problem on \mathcal{C} for any function $\alpha: \mathbb{N} \rightarrow \mathbb{R}$ (assuming the Gap Exponential Time Hypothesis). These lower bounds are based on an equivalence between FPT-approximation algorithms and approximate kernelization which is proved in [21] and a result of Chalermsook et al. [4] stating that FPT- $\alpha(k)$ -approximation algorithms for the dominating set problem do not exist for any function α (assuming the Gap Exponential Time Hypothesis).

Organization. This paper is organized as follows. In Section 2 and Section 3 we prove our positive results. We have split the proof into one part which requires no knowledge of nowhere dense graph classes and which is proved in Section 2. In the proof we assume just one lemma which contains the main technical contribution of the paper and which requires more background from nowhere dense graphs. The lemma is proved in Section 3. In Section 4 we prove our lower bounds.

2 Building the lossy kernel

Our notation is standard, we refer to the textbook [7] for all undefined notation. In the following, we fix a nowhere dense class \mathcal{C} of graphs, $k, r \in \mathbb{N}$ and $\alpha > 1$. Furthermore, let $t = \frac{\alpha-1}{4r+2}$. As we deal with the connected distance- r dominating set problem we may assume that all graphs in \mathcal{C} are connected.

► **Domination core.** Let G be a graph. A set $Z \subseteq V(G)$ is a (k, r) -domination core for G if every set D of size at most k that r -dominates Z also r -dominates G

Domination cores of polynomial size exist for nowhere dense classes, as the following lemma shows.

► **Lemma 1** (Kreutzer et al. [20]). *There exists a polynomial q (of degree depending only on r) and a polynomial time algorithm that, given a graph $G \in \mathcal{C}$ and $k \in \mathbb{N}$ either correctly concludes that G cannot be r -dominated by a set of at most k vertices, or finds a (k, r) -domination core $Z \subseteq V(G)$ of G of size at most $q(k)$.*

We remark that the non-constructive polynomial bounds that follow from [20] can be replaced by much improved constructive bounds [29].

We will work with the following parameterized minimization variant of the connected distance- r dominating set problem.

$$\text{CDS}_r(G, k, D) = \begin{cases} \infty & \text{if } D \text{ is not a connected distance-}r \\ & \text{dominating set of } G \\ \min\{|D|, k + 1\} & \text{otherwise.} \end{cases}$$

As indicated earlier, we compute only a bi-kernel and reduce to the following annotated version of the connected distance- r dominating set problem.

$$\text{ACDS}_r((G, Z), k, D) = \begin{cases} \infty & \text{if } D \text{ is not a connected distance-}r \\ & \text{dominating set of } Z \text{ in } G \\ \min\{|D|, k + 1\} & \text{otherwise.} \end{cases}$$

The following lemma is folklore for dominating sets, its more general variant for distance- r domination is proved just as the case $r = 1$ (see e.g. Proposition 1 of [11] for a proof for the case $r = 1$).

► **Lemma 2.** *Let G be a graph, $Z \subseteq V(G)$ a connected set in G and let D be a distance- r dominating set for Z such that $G[D]$ has at most p connected components. Then we can compute in polynomial time a set Q of size at most $2rp$ such that $G[D \cup Q]$ is connected.*

The lemma implies that we may assume that our domination cores are connected.

► **Corollary 3.** *There exists a polynomial q (of degree depending only on r) and a polynomial time algorithm that, given a graph $G \in \mathcal{C}$ and $k \in \mathbb{N}$ either correctly concludes that G cannot be r -dominated by a set of at most k vertices, or finds a (k, r) -domination core $Z \subseteq V(G)$ of G of size at most $q(k)$ such that $G[Z]$ is connected.*

Proof. Assume that when applying Lemma 1, a (k, r) -domination core Y is returned, otherwise we return that no distance- r dominating set of size at most k exists.

First observe that every superset $X \supseteq Y$ is also a (k, r) -domination core of G (every set of size at most k which r -dominates X in particular r -dominates Y , and hence all of G).

Assume there is a vertex $v \in V(G)$ with distance greater than $2r$ from Y . Since Y is a (k, r) -domination core, every set of size at most k that r -dominates Y also r -dominates G . If there exists a distance- r dominator A of Y of size at most k , also $B = N_r[Y] \cap A$ (the intersection of A with the closed r -neighborhood of Y) is a distance- r dominator of Y of size at most k . However, as v has distance greater than $2r$ from Y , B cannot be a distance- r dominating set of G . Hence, if there is $v \in V(G)$ with distance greater than $2r$ from Y , we may return that G cannot be r -dominated by a set of at most k vertices. Otherwise, it follows that Y is a distance- $2r$ dominating set of G . We can hence apply Lemma 2 with

parameters $Z = V(G)$ (we assume that all graphs $G \in \mathcal{C}$ are connected) and $D = Y$ to find a connected set $X \supseteq Y$ of size at most $(2r + 1) \cdot q(k)$ which is a connected (k, r) -domination core. \blacktriangleleft

The key idea is to split connected dominating sets into parts of well controlled size. This idea will be realized by considering covering families, defined as follows.

► **Covering family.** Let G be a connected graph. A (G, t) -covering family is a family $\mathcal{F}(G, t)$ of subtrees of G such that for each $T \in \mathcal{F}(G, t)$, $|V(T)| \leq 2t$ and $\bigcup_{T \in \mathcal{F}(G, t)} V(T) = V(G)$.

► **Lemma 4** (Eiben et al. [11]). *Let G be a connected graph. There is a (G, t) -covering family $\mathcal{F}(G, t)$ with $|\mathcal{F}(G, t)| \leq |V(G)|/t + 1$, and $\sum_{T \in \mathcal{F}(G, t)} |V(T)| \leq (1 + 1/t) \cdot |V(G)| + 1$.*

To recombine the pieces we will solve instances of the (GROUP) STEINER TREE problem.

► **Steiner tree.** Let G be a graph and let $Y \subseteq V(G)$ be a set of *terminals*. A *Steiner tree* for Y is a subtree of G spanning Y . We write $\text{st}_G(Y)$ for the order of (i.e. the number of vertices of) the smallest Steiner tree for Y in G (including the vertices of Y).

Let G be a graph and let $\mathcal{Y} = \{V_1, \dots, V_s\}$ be a family of vertex disjoint subsets of G . A *group Steiner tree* for \mathcal{Y} is a subtree of G that contains (at least) one vertex of each group V_i . We write $\text{st}_G(\mathcal{Y})$ for the order of the smallest group Steiner tree for \mathcal{Y} .

When recombining the pieces, we have to preserve their domination properties. For this, we will need precise a description of how vertices interact with the domination core.

► **A -avoiding path.** Let G be a graph and let $A \subseteq V(G)$ be a subset of vertices. For vertices $v \in A$ and $u \in V(G) \setminus A$, a path P connecting u and v is called *A -avoiding* if all its vertices apart from v do not belong to A .

► **Projection profile.** The *r -projection* of a vertex $u \in V(G) \setminus A$ onto A , denoted $M_r^G(u, A)$ is the set of all vertices $v \in A$ that can be connected to u by an A -avoiding path of length at most r . The *r -projection profile* of a vertex $u \in V(G) \setminus A$ on A is a function $\rho_r^G[u, A]$ mapping vertices of A to $\{0, 1, \dots, r, \infty\}$, defined as follows: for every $v \in A$, the value $\rho_r^G[u, A](v)$ is the length of a shortest A -avoiding path connecting u and v , and ∞ in case this length is larger than r . We define

$$\widehat{\mu}_r(G, A) = |\{\rho_r^G[u, A] : u \in V(G) \setminus A\}|$$

to be the number of different r -projection profiles realized on A .

► **Lemma 5** (Eickmeyer et al. [12]). *There is a function f_{proj} such that for every $G \in \mathcal{C}$, vertex subset $A \subseteq V(G)$, and $\varepsilon > 0$ we have $\widehat{\mu}_r(G, A) \leq f_{\text{proj}}(r, \varepsilon) \cdot |A|^{1+\varepsilon}$.*

The following lemma is immediate from the definitions.

► **Lemma 6.** *Let G be a graph and let $X \subseteq V(G)$. Let D be a distance- r dominating set of X . Then every set D' such that for each $u \in D$ there is $v \in D'$ with $\rho_r^G[u, X] = \rho_r^G[v, X]$ is a distance- r dominating set of X .*

The following generalization of the *Tree Closure Lemma* (Lemma 4.7 of Eiben et al. [11]) shows that we can re-combine the pieces in nowhere dense graph classes.

► **Lemma 7.** *There exists a function f such that the following holds. Let $G \in \mathcal{C}$, let $X \subseteq V(G)$, and let $\varepsilon > 0$. Define an equivalence relation $\sim_{X, r}$ on $V(G)$ by*

$$u \sim_{X, r} v \Leftrightarrow \rho_r^G[u, X] = \rho_r^G[v, X].$$

Then we can compute in time $\mathcal{O}(|X|^{t(1+\varepsilon)} \cdot n^{1+\varepsilon})$ a subgraph $G' \subseteq G$ of G such that

- 1) $X \subseteq V(G')$,
- 2) for every $u \in V(G)$ there is $v \in V(G')$ with $\rho_r^G[u, X] = \rho_r^{G'}[v, X]$,
- 3) for every set \mathcal{Y} of at most $2t$ projection classes (i.e., equivalence classes of $\sim_{X,r}$),
if $\text{st}_G(\mathcal{Y}) \leq 2t$, then $\text{st}_{G'}(\mathcal{Y}) = \text{st}_G(\mathcal{Y})$, and
- 4) $|V(G')| \leq f(r, t, \varepsilon) \cdot |X|^{2+\varepsilon}$.

Note that in item 3), due to item 2), every class of $\sim_{X,r}$ which is non-empty in G is also a non-empty class of $\sim_{X,r}$ in G' .

We defer the proof of the lemma to the next section.

► **Lemma 8.** *Let $\varepsilon > 0$ and let q be the polynomial from [Corollary 3](#). There exists an algorithm running in time $\mathcal{O}(q(k)^{t(1+\varepsilon)} \cdot n^{1+\varepsilon})$ that, given an n -vertex graph $G \in \mathcal{C}$ and a positive integer k , either returns that there exists no connected distance- r dominating set of G , or returns a subgraph $G' \subseteq G$ and a vertex subset $Z \subseteq V(G')$ with the following properties:*

- 1) Z is a (k, r) -domination core of G ,
- 2) $\text{Opt}_{\text{ACDS}_r}((G', Z), k) \leq \alpha \cdot \text{Opt}_{\text{CDS}_r}(G, k)$, and
- 3) $|V(G')| \leq p(k)$, for some polynomial p whose degree depends only on r .

Proof. Using [Corollary 3](#), we first conclude that G cannot be r -dominated by a connected set of at most k vertices, or we find a connected (k, r) -domination core $Z \subseteq V(G)$ of G of size at most $q(k)$. In the first case, we reject the instance, otherwise, let $G' \subseteq G$ be the subgraph that we obtain by applying [Lemma 7](#) with parameters G, Z, t and ε . Let $p := f(r, t, \varepsilon) \cdot q^{2+\varepsilon}$ (where f is the function from [Lemma 7](#)), which is a polynomial of degree depending only on r , only the coefficients depend on α .

It remains to show that $\text{Opt}_{\text{ACDS}_r}((G', Z), k) \leq \alpha \cdot \text{Opt}_{\text{CDS}_r}(G, k)$. Let D^* be a minimum connected distance- r dominating set of G of size at most k (if $|D^*| > k$, then $\text{Opt}_{\text{ACDS}_r}((G', Z), k) \leq \alpha \cdot \text{Opt}_{\text{CDS}_r}(G, k)$ trivially holds). Let $\mathcal{F} = \mathcal{F}(G[D^*], t) = \{T_1, \dots, T_\ell\}$ be a covering family for the connected graph $G[D^*]$ obtained by [Lemma 4](#). Note that by the lemma we have $\ell \leq |D^*|/t + 1$ and $\sum_{1 \leq i \leq \ell} V(T_i) \leq (1 + 1/t)|D^*| + 1$. Moreover, the size of each subtree T_i is at most $2t$. By construction of G' (according to item 3) of [Lemma 7](#)), for each $T \in \mathcal{F}$ there exists a tree T' in G' of size at most $|V(T)|$ which contains for each $u \in V(T)$ a vertex v with $\rho_r^G[u, Z] = \rho_r^{G'}[v, Z]$.

We construct a new family \mathcal{F}' which we obtain by replacing each $T \in \mathcal{F}$ by the tree T' described above. Let $D' := \bigcup_{T' \in \mathcal{F}'} V(T')$ in G' . We have $\sum_{T' \in \mathcal{F}'} |V(T')| \leq (1 + 1/t)|D^*| + 1$ and since D' contains vertices from the same projection classes as D^* , according to [Lemma 6](#), D' is a distance- r dominating set of Z . Moreover, $G[D']$ has at most $\ell \leq |D^*|/t + 1$ components. We apply [Lemma 2](#), and obtain a set Q of size at most $2r(|D^*|/t + 1)$ such that $D'' = D' \cup Q$ is a connected distance- r dominating set of Z . We hence have

$$|D''| \leq 2r(|D^*|/t + 1) + (1 + 1/t)|D^*| + 1 = \left(1 + \frac{2r+1}{t}\right)|D^*| + 2r + 1 \leq \left(1 + \frac{4r+2}{t}\right)|D^*|$$

(we may assume that $2r + 1 \leq \frac{2r+1}{t}|D^*|$, as otherwise we can simply run a brute force algorithm in polynomial time). We conclude by recalling that $t = \frac{\alpha-1}{4r+2}$. ◀

► **Theorem 9.** *There exists a polynomial p whose degree depends only on r such that the connected distance- r dominating set problem on \mathcal{C} admits an α -approximate bi-kernel with $p(k)$ vertices.*

Proof. The α -approximate polynomial time pre-processing algorithm first calls the algorithm of Lemma 8. If it returns that there exists no distance- r dominating set of size at most k for G , we return a trivial negative instance. Otherwise, let $((G', Z), k)$ be the annotated instance returned by the algorithm. The solution lifting algorithm, given a connected distance- r dominating set of Z in G' simply returns D .

By construction of G' we have $M_r^{G'}(u, Z) \subseteq M_r^G(u, Z)$ for all $u \in V(G')$. Hence every distance- r Z -dominator in G' is also a distance- r Z -dominator in G . In particular, since Z is a (k, r) -domination core, D is also a connected distance- r dominating set for G .

Finally, by Lemma 8 we have $\text{Opt}_{\text{ACDS}_r}((G', Z), k) \leq \alpha \cdot \text{Opt}_{\text{CDS}_r}(G, k)$, which implies

$$\frac{\text{CDS}_r(G, k, D)}{\text{Opt}_{\text{CDS}_r}(G, k)} \leq \alpha(k) \cdot \frac{\text{ACDS}_r((G', Z), k, D)}{\text{Opt}_{\text{ACDS}_r}((G', Z), k)}. \quad \blacktriangleleft$$

Observe that we obtain a 1-approximate bi-kernel for the distance- r dominating set problem by just taking one vertex from each projection class of the (k, r) -domination core.

3 The proof of Lemma 7

Lemma 7 is the most technical contribution of this paper. This whole section is devoted to its proof. We will mainly make use of a characterization of nowhere dense graph classes by the so-called *weak coloring numbers*.

► **Weak coloring numbers.** For a graph G , by $\Pi(G)$ we denote the set of all linear orders of $V(G)$. For $u, v \in V(G)$ and any $s \in \mathbb{N}$, we say that u is *weakly s -reachable* from v with respect to L , if there is a path P of length at most s connecting u and v such that u is the smallest among the vertices of P with respect to L . By $\text{WReach}_s[G, L, v]$ we denote the set of vertices that are weakly s -reachable from v with respect to L . For any subset $A \subseteq V(G)$, we let $\text{WReach}_s[G, L, A] = \bigcup_{v \in A} \text{WReach}_s[G, L, v]$. The *weak s -coloring number* $\text{wcol}_s(G)$ of G is defined as

$$\text{wcol}_s(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} |\text{WReach}_s[G, L, v]|.$$

The weak coloring numbers were introduced by Kierstead and Yang [18] in the context of coloring and marking games on graphs. As proved by Zhu [32], they can be used to characterize both bounded expansion and nowhere dense classes of graphs. In particular, we use the following.

► **Theorem 10 (Zhu [32]).** *Let \mathcal{C} be a nowhere dense class of graphs. There is a function f_{wcol} such that for all $s \in \mathbb{N}$, $\varepsilon > 0$, and $H \subseteq G \in \mathcal{C}$ we have $\text{wcol}_s(H) \leq f_{\text{wcol}}(s, \varepsilon) \cdot |V(H)|^\varepsilon$.*

One can define artificial classes where the functions f_{wcol} grow arbitrarily fast, however, on many familiar sparse graph classes they are quite tame, e.g. on bounded tree-width graphs [16], graphs with excluded minors [31] or excluded topological minors [19]. Observe that in any case the theorem allows to pull polynomial blow-ups on the graph size to the function f_{wcol} . More precisely, for any $\varepsilon > 0$, if we deal with a subgraph of size n^x for some $x \in \mathbb{N}$, by re-scaling ε to $\varepsilon' = \varepsilon/x$, we will get a bound of $f_{\text{wcol}}(s, \varepsilon') \cdot (n^x)^{\varepsilon'} = f_{\text{wcol}}(s, \varepsilon') \cdot n^\varepsilon$ for the weak s -coloring number.

Our second application of the weak coloring numbers is described in the next lemma, which shows that they capture the local separation properties of a graph.

► **Lemma 11 (see Reidl et al. [30]).** *Let G be a graph and let $L \in \Pi(G)$. Let $X \subseteq V(G)$, $y \in V(G)$ and let P be a path of length at most r between a vertex $x \in X$ and y . Then*

$$(\text{WReach}_r[G, L, X] \cap \text{WReach}_r[G, L, y]) \cap V(P) \neq \emptyset.$$

Proof. Let z be the minimal vertex of P with respect to L . Then both $z \in \text{WReach}_r[G, L, x]$ and $z \in \text{WReach}_r[G, L, y]$. ◀

We are now ready to define the graph G' whose existence we claimed in the previous section.

► **The graph G' .** Let $G \in \mathcal{C}$ and fix a subset $X \subseteq V(G)$. Define an equivalence relation $\sim_{X,r}$ on $V(G)$ by

$$u \sim_{X,r} v \Leftrightarrow \rho_r^G[u, X] = \rho_r^G[v, X].$$

For each subset \mathcal{Y} of projection classes of size at most $2t$, if $\text{st}_G(\mathcal{Y}) \leq 2t$, fix a Steiner tree $T_{\mathcal{Y}}$ for \mathcal{Y} of minimum size. For such a tree $T_{\mathcal{Y}}$ call a vertex $u \in \kappa \cap V(T_{\mathcal{Y}})$ with $\kappa \in \mathcal{Y}$ a *terminal* of $T_{\mathcal{Y}}$. We let $C = \{u \in V(G) : u \text{ is a terminal of some } T_{\mathcal{Y}}\}$.

Let G' be a subgraph of G which contains X , all $T_{\mathcal{Y}}$ as above, and a set of vertices and edges such that $\rho_r^G[u, X] = \rho_r^{G'}[u, X]$ for all $u \in C$.

► **Lemma 12.** *There exist functions f and g such that for every $G \in \mathcal{C}$, $X \subseteq V(G)$ and $\varepsilon > 0$ we can compute a graph G' as described above of size $f(r, \varepsilon) \cdot |X|^{2t(1+\varepsilon)}$ in time $g(r, t, \varepsilon) \cdot |X|^{2t(1+\varepsilon)}$.*

Proof. According to Lemma 5 there is a function f_{proj} such that for every $G \in \mathcal{C}$, vertex subset $A \subseteq V(G)$, and $\varepsilon > 0$ we have $\widehat{\mu}_r(G, A) \leq f_{\text{proj}}(r, \varepsilon) \cdot |A|^{1+\varepsilon}$. We now apply the lemma to $A = X$.

We compute for each $v \in X$ the first r levels of a breadth-first search (which terminates whenever another vertex of X is encountered, as to compute X -avoiding paths). For each visited vertex $w \in V(G)$ we remember the distance to v . In this manner, we compute in time $\mathcal{O}(|X| \cdot n^{1+\varepsilon})$ the projection profile of every vertex $w \in V(G)$. Observe that Theorem 10 applied to $r = 1$ implies that an n -vertex graph $G \in \mathcal{C}$ is n^ε -degenerate and in particular has only $\mathcal{O}(n^{1+\varepsilon})$ many edges. Hence a breadth-first search can be computed in time $\mathcal{O}(n^{1+\varepsilon})$.

We now decide for each subset \mathcal{Y} of at most $2t$ projection classes whether $\text{st}_G(\mathcal{Y}) \leq 2t$. If this is the case, we also compute a Steiner tree $T_{\mathcal{Y}}$ of minimum size in time $h(t, \varepsilon) \cdot n^{1+\varepsilon}$ for some function h . To see that this is possible, observe that the problem is equivalent to testing whether an existential first-order sentence holds in a colored graph, which is possible in the desired time on nowhere dense classes [17, 27].

Finally, for each sub-tree $T_{\mathcal{Y}}$ and each $\kappa \in \mathcal{Y}$ fix some terminal $u \in \kappa \cap V(T_{\mathcal{Y}})$. Compute the first r levels of an X -avoiding breadth-first search with root u and add the vertices and edges of the bfs-tree to ensure that $\rho_r^G[u, X] = \rho_r^{G'}[u, X]$. Observe that by adding these vertices we add at most $|X| \cdot r$ vertices for each vertex u .

As we have $\mathcal{O}\left((|X|^{(1+\varepsilon)})^{2t}\right) = \mathcal{O}(|X|^{2t(1+\varepsilon)})$ many subsets of projection classes of size at most $2t$, we can conclude by defining f and g appropriately. ◀

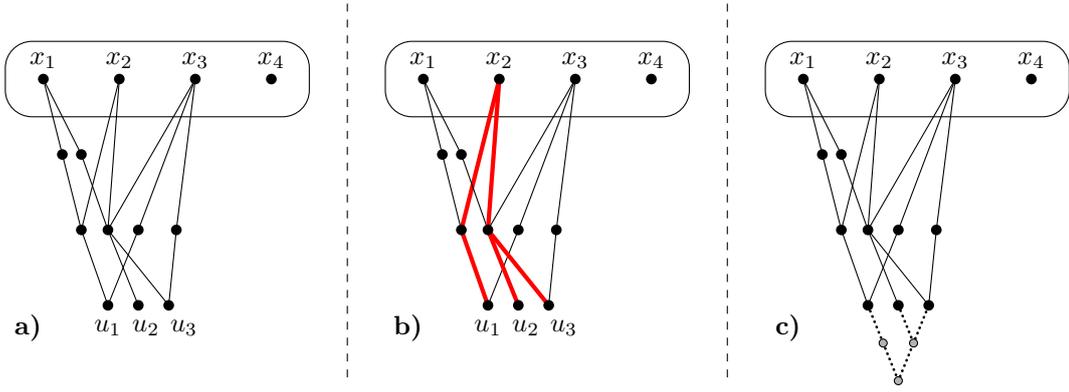
It remains to argue that the graph G' is in fact much smaller than our initial estimation in Lemma 12. First, as outlined earlier, we do not care about polynomial blow-ups when bounding the weak coloring numbers.

► **Lemma 13.** *There is a function h such that for all $s \in \mathbb{N}$ and $\varepsilon > 0$ we have*

$$\text{wcol}_s(G') \leq h(r, s, t, \varepsilon) \cdot |X|^\varepsilon.$$

Proof. Choose $\varepsilon' := \varepsilon/(3t)$. According to Lemma 12, G' has size at most $f(r, 1/2) \cdot |X|^{3t}$ (apply the lemma with $\varepsilon = 1/2$). According to Theorem 10, we have

$$\text{wcol}_s(G') \leq f_{\text{wcol}}(s, \varepsilon') \cdot (f(r, 1/2) \cdot |X|^{3t})^{\varepsilon'}.$$



■ **Figure 1** a) The vertices u_1, u_2, u_3 realize the same projection profile $\rho_r^G[u_1, X] = (3, 2, 2, \infty)$. b) We have chosen x_2 as x_κ , which results in the indicated tree T_κ . c) A subdivided copy of T_κ is added to \dot{G} .

Conclude by defining $h(r, s, t, \varepsilon) = f_{\text{wcol}}(s, \varepsilon') \cdot f(r, 1/2)^{\varepsilon'}$. ◀

Our next aim is to decompose the group Steiner trees into single paths which are then analyzed with the help of the weak coloring numbers. We need a few more auxiliary lemmas.

► **Definition 14.** The *lexicographic product* $G \bullet H$ of two graphs G and H is defined by $V(G \bullet H) = V(G) \times V(H)$ and $E(G \bullet H) = \{\{(x, y), (x', y')\} : \{x, x'\} \in E(G) \text{ or } (x = x' \text{ and } \{y, y'\} \in E(H))\}$.

The following two lemmas are easy consequences of the definitions.

► **Lemma 15.** Let G, H be graphs and let $s \in \mathbb{N}$. Then $\text{wcol}_s(G \bullet H) \leq |V(H)| \cdot \text{wcol}_s(G)$.

► **Lemma 16.** Let G be a graph and let $r', s \in \mathbb{N}$. Let H be any graph obtained by replacing some edges of G by paths of length r . Then $\text{wcol}_{r'}(H) \leq s + \text{wcol}_{r'}(G)$.

To estimate the size of G' we reduce the group Steiner tree problems to simple Steiner tree problems in a super-graph \dot{G} of G' .

► **The graph \dot{G} .** See Figure 1 for an illustration of the following construction. Let G' with distinguished terminal vertices C be as described above. For each equivalence class κ represented in C , fix some vertex $x_\kappa \in M_r^G(u, X)$ for $u \in \kappa$ which is of minimum distance to u among all such choices (for our purpose we may assume that the empty class with $M_r^G(u, X) = \emptyset$ is not realized in G).

Let T_κ be a tree which contains for each $u \in \kappa \cap C$ an X -avoiding path of minimum length between u and x_κ (e.g. obtained by an X -avoiding breadth-first search with root x_κ). Note that the vertices of $\kappa \cap C$ appear as leaves of T_κ and all leaves have the same distance from the root x_κ . To see this, note that if a vertex u of $\kappa \cap C$ lies on a shortest path from x_κ to another vertex v of $\kappa \cap C$, then the X -avoiding distance between u and x_κ is smaller than the X -avoiding distance between v and x_κ , contradicting that all vertices of $\kappa \cap C$ have the same projection profile. Recall that by construction projection profiles are preserved for each vertex of $\kappa \cap C$.

Let \dot{G} be the graph obtained by adding to G' for each equivalence class $\kappa \cap C$ a copy of T_κ , with each edge subdivided $2r$ times. Then identify the leaves of this copy T_κ with the respective vertices of κ .

► **Lemma 17.** *There exists a function f_\bullet such that for all $r' \in \mathbb{N}$ and all $\varepsilon > 0$ we have $\text{wcol}_{r'}(\dot{G}) \leq f_\bullet(r', t, \varepsilon) \cdot |X|^{1+\varepsilon}$.*

Proof. Let $\varepsilon' := \varepsilon/2$. According to Lemma 5, there is a function f_{proj} such that there are at most $f_{\text{proj}}(r, \varepsilon') \cdot |X|^{1+\varepsilon'} =: x$ distinct projection profiles. When constructing the graph \dot{G} , we hence create at most so many trees T_κ . These can be found as disjoint subgraphs in $G' \bullet K_x$. Hence, \dot{G} is a subgraph of a $2r$ -subdivision of $G' \bullet K_x$. According to Lemma 13, Lemma 15 and Lemma 16 we have $\text{wcol}_{r'}(\dot{G}) \leq h(r, r', t, \varepsilon') \cdot |X|^{\varepsilon'} \cdot f_{\text{proj}}(r, \varepsilon') \cdot |X|^{1+\varepsilon'} + r'$, where h is the function from Lemma 13. Assuming that each of these terms is at least 1, we can define $f_\bullet(r', t, \varepsilon) := r' \cdot h(r', t, \varepsilon') \cdot f_{\text{proj}}(r, \varepsilon')$. ◀

► **Lemma 18.** *With each group Steiner tree problem for \mathcal{Y} , we associate the Steiner tree problem for the set Y which contains exactly the roots of the subdivided trees T_κ for each $\kappa \in \mathcal{Y}$. Denote this root by v_κ (it is a copy of x_κ). Denote by d_κ the distance from v_κ to x_κ . Then every group Steiner tree T_Y for \mathcal{Y} of size $s \leq 2t$ in G gives rise to a Steiner tree for Y of size $s + \sum_{\kappa \in \mathcal{Y}} d_\kappa$ in \dot{G} . Vice versa, every Steiner tree for a set Y of the above form of size $s + \sum_{\kappa \in \mathcal{Y}} d_\kappa$ in \dot{G} gives rise to a group Steiner tree of size s for \mathcal{Y} in G .*

Proof. The forward direction is clear. Conversely, let T_Y be a Steiner tree for a set Y which contains only roots of subdivided trees T_κ of size $s + \sum_{\kappa} d_\kappa$ in \dot{G} . We claim that T_Y uses exactly d_κ vertices of T_κ , more precisely, T_Y connects exactly one vertex $u \in \kappa$ with v_κ . Assume T_Y contains two paths P_1, P_2 between v_κ and vertices u_1, u_2 from κ . Because we work with a $2r$ -subdivision of T_κ , we have $|V(P_1) \cup V(P_2)| \geq d_\kappa + 2r$. However, there is a path between u_1 and u_2 via x_κ of length at most $2r$ (which uses only $2r - 1$ vertices) in \dot{G} , contradicting the fact that T_Y uses a minimum number of vertices. ◀

► **Lemma 19.** *There is a function f such that for every $\varepsilon > 0$ the graph \dot{G} contains at most $f(r, t, \varepsilon) \cdot |X|^{2+\varepsilon}$ vertices.*

Proof. Let $\varepsilon' := \varepsilon/2$. Every Steiner tree T_Y that connects a subset Y decomposes into paths P_{uv} between pairs $u, v \in Y$. According to Lemma 11, each such path P_{uv} contains a vertex z which is weakly $(4r^2 + 2t)$ -reachable from u and from v . This is because each Steiner tree in \dot{G} connecting u and v contains a path of length at most $2r^2$ between u and some leaf $u_\kappa \in \kappa \cap C$ (and analogously a path of length at most $2r^2$ between v and some leaf $v_\kappa \in \kappa \cap C$). Now u_κ and v_κ are connected by a path of length at most $2t$ by construction.

Denote by Q_u and Q_v , respectively, the sub-path of P_{uv} between u and z , and v and z , respectively. We charge the vertices of Q_u to vertex u and the vertices of Q_v to vertex v (and the vertex z to one of the two). According to Lemma 17, each vertex weakly $(4r^2 + 2t)$ -reaches at most $f_\bullet(4r^2 + 2t, t, \varepsilon') \cdot |X|^{1+\varepsilon'}$ vertices which can play the role of z . According to Lemma 5 we have at most $f_{\text{proj}}(r, \varepsilon') \cdot |X|^{1+\varepsilon'}$ choices for $u, v \in Y$. Hence we obtain that all Steiner trees add up to at most $f_{\text{proj}}(r, \varepsilon') \cdot |X|^{1+\varepsilon'} \cdot f_\bullet(4r^2 + 2t, t, \varepsilon') \cdot |X|^{1+\varepsilon'} =: f(r, t, \varepsilon) \cdot |X|^{2+\varepsilon}$ vertices. ◀

As G' is a subgraph of \dot{G} , we conclude that also G' is small.

► **Corollary 20.** *There is a function f such that for every $\varepsilon > 0$ the graph \dot{G} has size at most $f(r, t, \varepsilon) \cdot |X|^{2+\varepsilon}$.*

This was the last missing statement of Lemma 7, which finishes the proof.

4 Lower bounds

Our lower bound is based on Proposition 3.2 of [21] which establishes equivalence between FPT-approximation algorithms and approximate kernelization.

► **Lemma 21** (Proposition 3.2 of [21]). *For every function α and decidable parameterized optimization problem Π , Π admits a fixed parameter tractable α -approximation algorithm if and only if Π has an α -approximate kernel.*

We will use a reduction from set cover to the distance- r dominating set problem. Recall that the instance of the Set Cover problem consists of (U, \mathcal{F}, k) , where U is a finite universe, $\mathcal{F} \subseteq 2^U$ is a family of subsets of the universe, and k is a positive integer. The question is whether there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size k such that every element of U is covered by \mathcal{G} , i.e., $\bigcup G = U$. The following result states that under complexity theoretic assumptions for the set cover problem on general graphs there does not exist a fixed-parameter tractable α -approximation algorithm for any function α .

► **Lemma 22** (Chalermsook et al. [4]). *If the Gap Exponential Time Hypothesis (gap-ETH) holds, then there is no fixed parameter tractable α -approximation algorithm for the set cover problem, for any function α .*

By definition of nowhere dense graph classes, if \mathcal{C} is somewhere dense (that is, not nowhere dense), then for some $r \in \mathbb{N}$ we find the r -subdivision of every graph as a subgraph of a graph in \mathcal{C} . For $p \geq 0$, let \mathcal{H}_p be the class of p -subdivisions of all simple graphs, that is, the class comprising all the graphs that can be obtained from any simple graph by replacing every edge by a path of length p . As our definition of nowhere denseness in the introduction is not the standard definition but tailored to the following hardness reduction, we give reference to the following lemma.

► **Lemma 23** (Nešetřil and Ossona de Mendez [26]). *For every monotone somewhere dense graph class \mathcal{C} , there exists $r \in \mathbb{N}$ such that $\mathcal{H}_r \subseteq \mathcal{C}$.*

Based on the above lemma, in the arxiv-version of [10], a parameterized reduction from set cover to the distance- r dominating set problem is presented which preserves the parameter k exactly. In that paper, the reduction is used to prove W[2]-hardness of the distance- r dominating set problem.

► **Lemma 24** (Drange et al. [10]). *Let (U, \mathcal{F}, k) be an instance of set cover and let $r \in \mathbb{N}$. There exists a graph $G \in \mathcal{H}_r$ such that (U, \mathcal{F}, k) is a positive instance of the set cover problem if and only if (G, k) is a positive instance of the distance- r dominating set problem.*

Combining Lemma 21, Lemma 22, Lemma 23 and Lemma 24 now gives the following theorem.

► **Theorem 25.** *If the Gap Exponential Time Hypothesis holds, then for every monotone somewhere dense class of graphs \mathcal{C} there is no $\alpha(k)$ -approximate kernel for the distance- r dominating set problem on \mathcal{C} for any function $\alpha: \mathbb{N} \rightarrow \mathbb{N}$.*

The same statement holds for the connected distance- r dominating set problem, as every graph that admits a distance- r dominating set of size k also admits a connected distance- r dominating set of size at most $3k$.

5 Conclusion

The study of computationally hard problems on restricted classes of inputs is a very fruitful line of research in algorithmic graph structure theory and in particular in parameterized complexity theory. This research is based on the observation that many problems such as DOMINATING SET, which are considered intractable in general, can be solved efficiently on restricted graph classes. Of course it is a very desirable goal in this line of research to identify the most general classes of graphs on which certain problems can be solved efficiently. In this work we were able to identify the exact limit for the existence of lossy kernels for the connected distance- r dominating set problem. One interesting open question is whether our polynomial bounds on the size of the lossy kernel can be improved to pseudo-linear bounds. The first step to achieve this is to prove the existence of a (k, r) -domination core of pseudo-linear size on every nowhere dense class of graphs, or to avoid the use of such cores in the construction.

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