Kernelization and approximation of distance-r independent sets on nowhere dense graphs

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Abstract

For a positive integer r, a distance-r independent set in an undirected graph G is a set $I \subseteq V(G)$ of vertices pairwise at distance greater than r, while a distance-r dominating set is a set $D \subseteq V(G)$ such that every vertex of the graph is within distance at most r from a vertex from D. We study the duality between the maximum size of a distance-$2r$ independent set and the minimum size of a distance-r dominating set in nowhere dense graph classes, as well as the kernelization complexity of the distance-r independent set problem on these graph classes. Specifically, we prove that the distance-r independent set problem admits an almost linear kernel on every nowhere dense graph class.

1. Introduction

Independence and domination. For a graph G and positive integer r, a *distance-r independent set* in G is a subset of vertices I whose members are pairwise at distance more than r. On the other hand, a *distance-r dominating set* in G is a subset of vertices D such that every vertex of G is at distance at most r from some member of D. The cases $r = 1$ correspond to the standard notions of an independent and dominating set, respectively. In this work we will consider combinatorial questions about distance-r independent and dominating sets, as well as the computational complexity of the corresponding decision problems Distance-r Independent Set and Distance-r Dominating Set: given a graph G and integer k, decide whether G has a distance-r independent set of size at least k, respectively, a distance-r dominating set of size at most k.

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In the following, we denote the minimum size of a distance-r dominating set in a graph G by $\gamma_r(G)$ and the maximum size of a distance-r independent set by $\alpha_r(G)$. Furthermore, if $A \subseteq V(G)$, we write $\gamma_r(G, A)$ for the minimum size of a *distance-r dominating set of A*, i.e., we only require that each vertex of A is at distance at most r from the dominating set. Similarly, we write $\alpha_r(G, A)$ for the maximum size of a distance-r independent subset of A. Observe that for every graph G, vertex subset A, and $r \in \mathbb{N}$ it holds that

$$\alpha_{2r}(G, A) \leqslant \gamma_r(G, A),$$

because every member of a set that distance-r dominates A can dominate at most one member of a distance-$2r$ independent subset of A. The study of a reverse inequality (in the approximate sense) for certain graph classes is the main combinatorial goal of this work.

Regarding computational complexity, both Independent Set and Dominating Set are NP-hard [26] and this even holds in very restricted settings, e.g., on planar graphs of maximum degree 3 [21, 22]. Even worse, under the assumption that $\mathrm{P} \neq \mathrm{NP}$, for every $\varepsilon > 0$, the size of a maximum independent set of an n-vertex graph cannot be approximated in polynomial time within a factor better than $\mathcal{O}(n^{1-\varepsilon})$ [25]. Under the assumption $\mathrm{P} \neq \mathrm{NP}$, the domination number of a graph cannot be approximated in polynomial time within a factor better than $\mathcal{O}(\log n)$ [32]. However, it turns out that in several restricted graph classes the problems can be approximated much better. For instance, for fixed r the distance-r variants of both problems admit a polynomial-time approximation scheme (PTAS) on planar graphs [4] and, more generally, in graph classes with polynomial expansion [24]. We will discuss further approximation results later.

Abstract notions of sparsity. In this paper we are going to study Distance-r Independent Set and Distance-r Dominating Set on *nowhere dense* graph classes. The notions of *nowhere denseness* and *bounded expansion* are the fundamental definitions of the sparsity theory introduced by Nešetřil and Ossona de Mendez [28, 29]. Many familiar classes of sparse graphs, like classes of bounded treewidth, planar graphs, classes of bounded degree, and all classes that exclude a fixed minor or topological minor have bounded expansion and are nowhere dense. In order to facilitate further discussion, we now recall basic definitions.

Nowhere dense classes and classes of bounded expansion are defined by imposing restrictions on the graphs that can be found as *bounded depth minors* in the class. Formally, for a positive integer r, a graph H with vertex set $\{v_1, \ldots, v_n\}$ is a *depth-r minor* of a graph G, written $H \preccurlyeq_r G$, if there are connected and pairwise vertex disjoint subgraphs $H_1, \ldots, H_n \subseteq G$, each of radius at most r, such that if $v_i v_j \in E(H)$, then there are $w_i \in V(H_i)$ and $w_j \in V(H_j)$ with $w_i w_j \in E(G)$. Now, a class $\mathscr{C}$ of graphs has bounded expansion if for every positive integer r and every $H \preccurlyeq_r G$ for $G \in \mathscr{C}$, the edge density $|E(H)|/|V(H)|$ of H is bounded by some constant $d(r)$. Furthermore, $\mathscr{C}$ is nowhere dense if for every positive integer r there exists a constant $t(r)$ such that $K_{t(r)} \not\preccurlyeq_r G$ for all $G \in \mathscr{C}$, where K_t denotes the complete graph on t vertices.

We call $\mathscr{C}$ *effectively nowhere dense*, respectively, of *effectively bounded expansion*, if the function $t(r)$, respectively $d(r)$, is computable; such effectiveness is enjoyed by essentially all natural classes of sparse graphs. Clearly, every class of bounded expansion is nowhere dense, but the converse is not true. For example the class consisting of all graphs G with $\mathrm{girth}(G) \geqslant \Delta(G)$ is nowhere dense, however, it does not have bounded average degree and in particular does not have bounded expansion, see [30].

The duality between independence and domination numbers on classes of bounded expansion was studied by Dvořák [15], who proved that for such classes, there is a constant-factor multiplicative gap between them. More precisely, Dvořák [15] proved that for every class $\mathscr{C}$ of bounded expansion and every positive integer r, there exists a constant $c(r)$ such that every graph $G \in \mathscr{C}$ satisfies

$$\alpha_{2r}(G) \leqslant \gamma_r(G) \leqslant c(r) \cdot \alpha_{2r}(G).$$

A by-product of this combinatorial result is a pair of constant-factor approximation algorithms, for the Distance-r Independent Set and Distance-r Dominating Set problems on any class of bounded expansion. One of goals of this work is to investigate to what extent the above duality can be lifted to the more general setting of nowhere dense graph classes.

Fractional parameters. It will be convenient to study the relation between $\gamma_r(G)$ and $\alpha_{2r}(G)$ through the lenses of their fractional relaxations. For a graph G and positive integer r, consider the following linear programs; here, $N_r(u)$ denotes the set of vertices at distance at most r from u.

$$\gamma_r^\star(G) \coloneqq \min \sum_{v \in V(G)} x_v \quad \text{subject to} \quad \begin{aligned} \sum_{v \in N_r(u)} x_v &\geqslant 1 \quad \text{for all } u \in V(G), \text{ and} \\ x_v &\geqslant 0 \quad \text{for all } u \in V(G). \end{aligned}$$

and

$$\alpha_{2r}^\star(G) \coloneqq \max \sum_{v \in V(G)} y_v \quad \text{subject to} \quad \begin{aligned} \sum_{v \in N_r(u)} y_v &\leqslant 1 \quad \text{for all } u \in V(G), \text{ and} \\ y_v &\geqslant 0 \quad \text{for all } u \in V(G). \end{aligned}$$

The two above LPs are dual to each other, and requiring the variables to be integral yields the values $\gamma_r(G)$ and $\alpha_{2r}(G)$, respectively. Hence we have

$$\alpha_{2r}(G) \leqslant \alpha_{2r}^\star(G) = \gamma_r^\star(G) \leqslant \gamma_r(G).$$

VC-dimension. Consider a ground set U and a set system (family) $\mathcal{F}$ consisting of subsets of U. A subset $X \subseteq U$ is *shattered* by $\mathcal{F}$ if for every subset $Y \subseteq X$ there exists $F \in \mathcal{F}$ such that $F \cap X = Y$. The *Vapnik-Chervonenkis dimension*, short *VC-dimension*, of $\mathcal{F}$ is the maximum size of a set shattered by $\mathcal{F}$ [10]. We also define the notions of a 2-*shattered* set and the *2VC-dimension* of a set system by restricting subsets $Y \subseteq X$ considered in the above definition only to

subsets of size exactly 2. Clearly, the VC-dimension of a set system is upper bounded by its 2VC-dimension.

A fundamental result about VC-dimension is that in set systems of bounded VC-dimension the gap between integral and fractional hitting sets is bounded. A *hitting set* of a set system $\mathcal{F}$ over U is a subset $H \subseteq U$ that intersects every member of $\mathcal{F}$, while a *fractional hitting set* is a distribution of weights from $[0,1]$ among elements of U so that every member of $\mathcal{F}$ has total weight at least 1. Let $\tau(\mathcal{F})$ and $\tau^\star(\mathcal{F})$ denote the minimum size, respectively weight, of an integral, respectively fractional, hitting set of $\mathcal{F}$.

Theorem 1 (see e.g. [8, 17]). *There exists a universal constant C such that for every set system $\mathcal{F}$ of VC-dimension at most d, we have*

$$\tau(\mathcal{F}) \leqslant C \cdot d \cdot \tau^\star(\mathcal{F}) \cdot \ln \tau^\star(\mathcal{F}).$$

Moreover, there exists a polynomial-time algorithm that computes a hitting set of $\mathcal{F}$ of size bounded as above.

As proved in [1], any nowhere dense class $\mathscr{C}$ of graphs is *stable*; see also [31] for a combinatorial proof of this fact. This, in particular, implies the following assertion: for every $r \in \mathbb{N}$ there exists a constant $d(r)$ such that for every $G \in \mathscr{C}$ the family of distance-r balls

$$\mathrm{Balls}_r(G) \coloneqq \{\{v \colon \mathrm{dist}_G(u,v) \leqslant r\} \colon u \in V(G)\},$$

treated as a set system over $V(G)$, has VC-dimension at most $d(r)$. Combining this with Theorem 1 shows that

$$\gamma_r(G) \leqslant C \cdot d(r) \cdot \gamma_r^\star(G) \cdot \ln \gamma_r^\star(G)$$

for every graph $G \in \mathscr{C}$. However, both [1] and [31] only prove the statement about stability, and consequentely do not provide explicit bounds on the constant $d(r)$ in the above inequality.

Observe that VC-dimension is a hereditary measure, i.e., for any subset $A \subseteq U$ of the universe, the VC-dimension of the system $\mathcal{F} \cap A \coloneqq \{F \cap A \colon F \in \mathcal{F}\}$ is not larger than the VC-dimension of $\mathcal{F}$. Hence, Theorem 1 applied to the set system $\mathcal{F} \cap A$ yields $\tau(\mathcal{F} \cap A) \leqslant C \cdot d \cdot \tau^\star(\mathcal{F} \cap A) \cdot \ln \tau^\star(\mathcal{F} \cap A)$, and we can make the same conclusion about the system stemming from the r-neighborhoods of graphs from a nowhere dense class $\mathscr{C}$. That is, if $A \subseteq V(G)$ for $G \in \mathscr{C}$, then $\gamma_r(G,A) \leqslant C \cdot d(r) \cdot \gamma_r^\star(G,A) \cdot \ln \gamma_r^\star(G,A)$, and the algorithm provided by Theorem 1 can be applied to compute a vertex subset that distance-r dominates A with this size guarantee.

Contribution: duality in nowhere dense classes. We first study the VC-dimension of systems of radius-r balls in graphs from nowhere dense graph classes. By following the lines of a recent result of Bousquet and Thomassé [7], we are able to provide explicit bounds for the VC-dimension, in fact even for the 2VC-dimension, of the rth powers of graphs for any nowhere dense class. More precisely, we prove the following theorem.

Theorem 2. *Let $r \in \mathbb{N}$ and let G be a graph. If $K_t \not\preccurlyeq_r G$, then the 2VC-dimension of the set system* $\mathrm{Balls}_r(G)$ *is at most $t-1$.*

We immediately derive the following; here, C is the constant provided by Theorem 1.

Corollary 3. *Let $\mathscr{C}$ be a nowhere dense class of graphs such that $K_{t(r)} \not\preccurlyeq_r G$ for all $r \in \mathbb{N}$. Then for every $r \in \mathbb{N}$, every $G \in \mathscr{C}$ and every $A \subseteq V(G)$ we have*

$$\alpha_{2r}(G,A) \leqslant \alpha^\star_{2r}(G,A) = \gamma^\star_r(G,A) \leqslant \gamma_r(G,A) \leqslant C \cdot t(r) \cdot \gamma^\star_r(G,A) \cdot \ln \gamma^\star_r(G,A).$$

Moreover, there exists a polynomial-time algorithm that computes a distance-r dominating set of A in G of size bounded as above.

Corollary 3 gives an upper bound of $\mathcal{O}(\log \gamma^\star_r(G,A))$ on the multiplicative gap between $\gamma_r(G,A)$ and $\gamma^\star_r(G,A)$. For a lower bound, we prove that one cannot be expect that this gap can be bounded by a constant on every nowhere dense class; recall that this is the case for classes of bounded expansion [15].

Theorem 4. *There exists a nowhere dense class $\mathscr{C}$ of graphs with the property that for every $r \in \mathbb{N}$ we have*

$$\sup_{G \in \mathscr{C}} \frac{\gamma_r(G)}{\gamma^\star_r(G)} = +\infty.$$

Finally, we want to investigate the multiplicative gap between $\gamma_r(G,A)$ and $\alpha_{2r}(G,A)$. While the lower bound of Theorem 4 asserts that in some nowhere dense class this gap cannot be bounded by any constant, the upper bound of Corollary 3 does not provide any upper bound in terms of $\alpha_{2r}(G,A)$. To this end, we leverage the kernelization results for Distance-r Dominating Set in nowhere dense classes of [16] to prove the following.

Theorem 5. *Let $\mathscr{C}$ be a nowhere dense class of graphs. There exists a function $f_{\mathrm{dual}} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $G \in \mathscr{C}$, $A \subseteq V(G)$, $r \in \mathbb{N}$, and $\varepsilon > 0$, we have*

$$\gamma_r(G,A) \leqslant f_{\mathrm{dual}}(r,\varepsilon) \cdot \alpha_{2r+1}(G,A)^{1+\varepsilon} \leqslant f_{\mathrm{dual}}(r,\varepsilon) \cdot \alpha_{2r}(G,A)^{1+\varepsilon}.$$

Furthermore, there is a polynomial-time algorithm that given G, A, r, ε as above, computes a distance-r dominating set of A in G of size bounded as above.

Thus, the multiplicative gap is $\mathcal{O}(\alpha_{2r}(G,A)^\varepsilon)$ (and even $\mathcal{O}(\alpha_{2r+1}(G,A)^\varepsilon)$) for any $\varepsilon > 0$.

Contribution: kernelization. In the second part of the paper we turn to the parameterized complexity of Distance-r Independent Set on nowhere dense classes. Also from the view of parameterized complexity both Independent Set and Dominating Set are hard: parameterized by the target size k, Independent Set is W[1]-complete and Dominating Set is W[2]-complete [13]. Hence both problems are not likely to be *fixed-parameter tractable*, i.e., solvable in time $f(k) \cdot n^c$ on instances of input size n, where $f(k)$ is a computable function, depending only on the value of the parameter k and c is a fixed constant.

Again, it turns out that in several restricted graph classes the problems become easier to handle. As far as classes of sparse graphs are concerned, both Distance-r Dominating Set and Distance-r Independent Set are expressible in first-order logic (for fixed r), and hence fixed-parameter tractable on any nowhere dense class of graphs by the meta-theorem of Grohe et al. [23]. This was earlier proved in the particular case of Distance-r Dominating Set by Dawar and Kreutzer [11].

Once fixed-parameter tractability of a problem on a certain class of graphs is established, we can ask whether we can go even one step further by showing the existence of a polynomial (or even linear) kernel. A *kernelization algorithm*, or a *kernel*, is a polynomial-time preprocessing algorithm that given an instance (I, k) of a parameterized problem outputs another instance (I', k') which is equivalent to (I, k), and whose total size $|I'| + k'$ is bounded by $f(k)$ for some computable function f, called the *size* of the kernel. If f is a polynomial (respectively, linear) function, then the algorithm is called a *polynomial* (respectively, *linear*) *kernel*. Observe that the existence of a kernel immediately implies that a decidable problem is fixed-parameter tractable: after applying the kernelization, any brute-force algorithm runs in time bounded by a function of k only. In fact, the existence of a kernel is equivalent to fixed-parameter tractability, however, in general the function f can be arbitrarily large.

Kernelization of Dominating Set and Distance-r Dominating Set on classes of sparse graphs has received a lot of attention in the literature [2, 5, 18, 19, 20]. In particular, Distance-r Dominating Set admits a linear kernel on any classe of bounded expansion [14] and an almost linear kernel on any nowhere dense class [16]. The kernelization complexity of Distance-r Independent Set on classes of sparse graphs seems less explored; a linear kernel for the problem is known on any class excluding a fixed apex minor [18].

We prove that for every positive integer r, Distance-r Independent Set admits an almost linear kernel on every nowhere dense class of graphs. In fact, we prove the statement for a slightly more general, annotated variant of the problem, which finds an application e.g. in the model-checking result of Grohe et al. [23].

Theorem 6. *Let $\mathscr{C}$ be a fixed nowhere dense class of graphs, let r be a fixed positive integer, and let $\varepsilon > 0$ be a fixed real. Then there exists a polynomial-time algorithm with the following properties. Given a graph $G \in \mathscr{C}$, a vertex subset $A \subseteq V(G)$, and a positive integer k, the algorithm either correctly concludes that $\alpha_r(G, A) < k$, or finds a subset $Y \subseteq V(G)$ of size at most $f_{\mathrm{ker}}(r, \varepsilon) \cdot k^{1+\varepsilon}$, for some function f_{ker} depending only on $\mathscr{C}$, and a subset $B \subseteq Y \cap A$ such that $\alpha_r(G, A) \geqslant k \iff \alpha_r(G[Y], B) \geqslant k$.*

We remark that in case $\mathscr{C}$ is effectively nowhere dense, it is easy to see that the function f_{ker} above is computable and the algorithm can be made uniform w.r.t. r and ε: there is one algorithm that takes r and ε also on input, instead of a different algorithm for each choice of r and ε. Furthermore, as in [16], it is easy to follow the lines of the proof to obtain a linear kernel in case $\mathscr{C}$ is a class of bounded expansion. That is, the size of the obtained set Y is bounded

by $\mathcal{O}(k)$, where the constant hidden in the $\mathcal{O}(\cdot)$-notation depends on $\mathscr{C}$ and r.

It is not difficult to see that for classes closed under taking subgraphs, this result cannot be extended further. More precisely, similarly as in [14] for the case of Distance-r Dominating Set, we provide the following lower bound for completeness.

Theorem 7. *Let $\mathscr{C}$ be a class of graphs that is closed under taking subgraphs and which is not nowhere dense. Then there exists an integer r such that* Distance-r Independent Set *is* W[1]*-hard on $\mathscr{C}$.*

Our proof of Theorem 6 uses a similar approach as [14, 16] for the kernelization of Distance-r Dominating Set. We aim to iteratively remove vertices from A which are irrelevant for distance-r independent sets in the following sense. A vertex $v \in A$ is irrelevant if the following assertion holds: provided A contains a distance-r independent subset of size k, then also $A - \{v\}$ contains a distance-r independent subset of size k.

In order to find such an irrelevant vertex, we start by computing a good approximation of a distance-$\lfloor r/2 \rfloor$ dominating set D of A. If we do not find a sufficiently small such set, we can reject the instance, as by Theorem 5 this implies that there does not exist a large distance-r independent set in A. We now classify the remaining vertices of A with respect to their interaction with the set D and argue that if A is large we may find an irrelevant vertex.

We repeat this construction until A becomes small enough (almost linear in k) and return the resulting set as the set B. It now suffices to add a small set of vertices and edges so that short distances between the elements of B are exactly preserved. The result will be the output of the kernelization algorithm.

Organization. We assume familiarity with graph theory and refer to [12] for undefined notation. We provide basic facts about nowhere dense graph classes in Section 2 and refer to [30] for a broader discussion of the area. We present our results on the VC-dimension of power graphs in Section 3 and the construction of a nowhere dense class witnessing the non-constant gap between distance-r domination and distance-$2r$ independence in Section 4. Finally, we present the kernelization algorithm for Distance-r Independent Set on nowhere dense graph classes in Section 5.

2. Preliminaries

We shall need some basic notions and tools for kernelization in nowhere dense classes used by Eickmeyer et al. [16]. For consistency and completeness of this paper, we have included these preliminaries also here, and they are largely taken verbatim from [16].

Algorithmic aspects. Whenever we say that the running time of some algorithm on a graph G is *polynomial*, we mean that it is of the form $\mathcal{O}((|V(G)|+|E(G)|)^\alpha)$, where α is a universal constant that is independent of $\mathscr{C}$, r, ε, or any other constants defined in the context. However, the constants hidden in the $\mathcal{O}(\cdot)$-notation may depend on $\mathscr{C}$, r, and ε.

Projections and projection profiles. Let G be a graph and let $A \subseteq V(G)$ be a subset of vertices. For vertices $v \in A$ and $u \in V(G) - A$, a path P connecting u and v is called A-*avoiding* if all its vertices apart from v do not belong to A. For a positive integer r, the r-*projection* of any $u \in V(G) - A$ on A, denoted $M_r^G(u, A)$ is the set of all vertices $v \in A$ that can be connected to u by an A-avoiding path of length at most r. The r-*projection profile* of a vertex $u \in V(G) - A$ on A is a function $\rho_r^G[u, A]$ mapping vertices of A to $\{1, \ldots, r, \infty\}$, defined as follows: for every $v \in A$, the value $\rho_r^G[u, A](v)$ is the length of a shortest A-avoiding path connecting u and v, and ∞ in case this length is larger than r. We define

$$\mu_r(G, A) = |\{\rho_r^G[u, A] \colon u \in V(G) - A\}|$$

to be the number of different r-projection profiles realized on A.

One of the main results of [16] was to show that in nowhere dense classes the number of realized projection profiles is small, as stated in the next lemma.

Lemma 8 ([16]). *Let $\mathscr{C}$ be a nowhere dense class of graphs. There is a function $f_{\mathrm{proj}} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$, $\varepsilon > 0$, graph $G \in \mathscr{C}$, and vertex subset $A \subseteq V(G)$, we have $\mu_r(G, A) \leqslant f_{\mathrm{proj}}(r, \varepsilon) \cdot |A|^{1+\varepsilon}$.*

The next lemma states that any vertex subset $X \subseteq V(G)$ can be "closed" to a set X' that is not much larger than X such that all r-projections on X' are small.

Lemma 9 ([14, 16]). *Let $\mathscr{C}$ be a nowhere dense class of graphs. There is a function $f_{\mathrm{cl}} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ and a polynomial-time algorithm that, given $G \in \mathscr{C}$, $X \subseteq V(G)$, $r \in \mathbb{N}$, and $\varepsilon > 0$, computes a superset $X' \supseteq X$ of vertices with the following properties:*

- *$|X'| \leqslant f_{\mathrm{cl}}(r, \varepsilon) \cdot |X|^{1+\varepsilon}$; and*
- *$|M_r^G(u, X')| \leqslant f_{\mathrm{cl}}(r, \varepsilon) \cdot |X|^{\varepsilon}$ for each $u \in V(G) - X'$.*

The next lemma shows that a set can be closed without increasing its size too much, so that short distances between its elements are preserved in the subgraph induced by the closure.

Lemma 10 ([14, 16]). *There is a function $f_{\mathrm{pth}} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ and a polynomial-time algorithm which on input $G \in \mathscr{C}$, $X \subseteq V(G)$, $r \in \mathbb{N}$, and $\varepsilon > 0$, computes a superset $X' \supseteq X$ of vertices with the following properties:*

- *whenever $\mathrm{dist}_G(u, v) \leqslant r$ for $u, v \in X$, then $\mathrm{dist}_{G[X']}(u, v) = \mathrm{dist}_G(u, v)$; and*
- *$|X'| \leqslant f_{\mathrm{pth}}(r, \varepsilon) \cdot |X|^{1+\varepsilon}$.*

Uniform quasi-wideness. We will use the characterization of nowhere denseness via the notion of *uniform quasi-wideness*, explained next.

Definition 1. A class $\mathscr{C}$ is *uniformly quasi-wide* if for every $r \in \mathbb{N}$ there is a function $N_r \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and constant $s_r \in \mathbb{N}$ such that for all $r, m \in \mathbb{N}$ and all subsets $A \subseteq V(G)$ for $G \in \mathscr{C}$ of size $|A| \geqslant N_r(m)$ there is a set $S \subseteq V(G)$ of size $|S| \leqslant s_r$ and a set $B \subseteq A - S$ of size $|B| \geqslant m$ which is r-independent in $G - S$.

It was shown by Nešetřil and Ossona de Mendez [29] that a class $\mathscr{C}$ of graphs is nowhere dense if and only if it is uniformly quasi-wide. Recently, Kreutzer et al. [27] proved that in this case the function $N_r(\cdot)$ can be bounded by a polynomial, with the exponent depending on $\mathscr{C}$ and r. More precisely, we will use the following fact proved later in [31]; here, the $\mathcal{O}_{r,t}(\cdot)$ notation hides factors depending on r and t.

Theorem 11 ([31]). *For all $r, t \in \mathbb{N}$ there is a polynomial N with $N(m) = \mathcal{O}_{r,t}(m^{(4t+1)^{2rt}})$, such that the following holds. Let G be a graph such that $K_t \not\preccurlyeq_{\lfloor 9r/2 \rfloor} G$, and let $A \subseteq V(G)$ be a vertex subset of size at least $N(m)$, for a given m. Then there exists a set $S \subseteq V(G)$ of size $|S| < t$ and a set $B \subseteq A - S$ of size $|B| \geqslant m$ which is r-independent in $G - S$. Moreover, given G and A, such sets S and B can be computed in time $\mathcal{O}_{r,t}(|A| \cdot |E(G)|)$.*

3. 2VC-Dimension of nowhere dense classes

In this section we prove Theorem 2, which we repeat for convenience. We remark again that our proof follows the lines of the work of Bousquet and Thomassé [7], who proved that the set system of all balls (of all radii) in a graph that excluded K_t as a minor is bounded by $t - 1$. As noted in [7], this result was in turn based on the case of planar graphs considered by Chepoi et al. [9].

Theorem 2. *Let $r \in \mathbb{N}$ and let G be a graph. If $K_t \not\preccurlyeq_r G$, then the 2VC-dimension of the set system $\mathrm{Balls}_r(G)$ is at most $t - 1$.*

Proof. Assume there is a set $A = \{a_1, \ldots, a_t\}$ of size t such that for all subsets $\{i, j\} \subseteq \{1, \ldots, t\}$ of size 2 there is a vertex v_{ij} with $\{u \colon \mathrm{dist}(u, v_{ij}) \leqslant r\} \cap A = \{a_i, a_j\}$. For each subset $\{i, j\} \subseteq \{1, \ldots, t\}$ of size 2, choose a vertex u_{ij} so that:

1. $\mathrm{dist}(v_{ij}, u_{ij}) + \mathrm{dist}(u_{ij}, a_i) \leqslant r$;
2. $\mathrm{dist}(v_{ij}, u_{ij}) + \mathrm{dist}(u_{ij}, a_j) \leqslant r$; and
3. subject to Conditions 1 and 2, $\max(\mathrm{dist}(u_{ij}, a_i), \mathrm{dist}(u_{ij}, a_j))$ is minimized.

Observe that such u_{ij} must exist since setting $u_{ij} = v_{ij}$ satisfies the first two conditions.

Let P^i_{ij} and P^j_{ij} be arbitrarily chosen shortest paths between u_{ij} and a_i, and between u_{ij} and a_j, respectively. We now establish some basic properties of paths P^i_{ij} and P^j_{ij} following from the choice of u_{ij}.

Claim 1. *For each vertex x on P^i_{ij} we have $\mathrm{dist}(v_{ij}, x) + \mathrm{dist}(x, a_i) \leqslant r$, and for each vertex y on P^j_{ij} we have $\mathrm{dist}(v_{ij}, y) + \mathrm{dist}(y, a_j) \leqslant r$.*

Proof. We prove only the first statement for the second is symmetric. We have

$$\begin{aligned}\mathrm{dist}(v_{ij}, x) + \mathrm{dist}(x, a_i) &\leqslant \mathrm{dist}(v_{ij}, u_{ij}) + \mathrm{dist}(u_{ij}, x) + \mathrm{dist}(x, a_i)\\ &= \mathrm{dist}(v_{ij}, u_{ij}) + \mathrm{dist}(u_{ij}, a_i) \leqslant r,\end{aligned}$$

where the last equality is due to x lying on a shortest path between u_{ij} and a_i, and the last inequality is by Condition 1. ⌟

Claim 2. *Let x be vertex on P^i_{ij} that is different from u_{ij}. Then* $\mathrm{dist}(x, a_i) < \mathrm{dist}(x, a_j)$. *Symmetrically, if y lies on P^j_{ij} and is different from u_{ij}, then* $\mathrm{dist}(y, a_i) > \mathrm{dist}(y, a_j)$. *Consequently, paths P^i_{ij} and P^j_{ij} share only one vertex, being the endpoint u_{ij}.*

Proof. We prove only the first claim, for the second is symmetric and the third directly follows from the first two. Suppose for contradiction that $\mathrm{dist}(x, a_i) \geqslant \mathrm{dist}(x, a_j)$. By Claim 1 we have

$$\mathrm{dist}(v_{ij}, x) + \mathrm{dist}(x, a_i) \leqslant r.$$

On the other hand, since $\mathrm{dist}(x, a_i) \geqslant \mathrm{dist}(x, a_j)$, we have

$$\mathrm{dist}(v_{ij}, x) + \mathrm{dist}(x, a_j) \leqslant \mathrm{dist}(v_{ij}, x) + \mathrm{dist}(x, a_i) \leqslant r.$$

We conclude that x satisfies Conditions 1 and 2 from the definition of u_{ij}. However, since $x \neq u_{ij}$ and x lies on a shortest path between u_{ij} and a_i, we have $\mathrm{dist}(x, a_i) < \mathrm{dist}(u_{ij}, a_i)$. Therefore,

$$\mathrm{dist}(x, a_j) \leqslant \mathrm{dist}(x, a_i) < \mathrm{dist}(u_{ij}, a_i) \leqslant \max(\mathrm{dist}(u_{ij}, a_i), \mathrm{dist}(u_{ij}, a_j)).$$

Thus, the existence of x contradicts Condition 3 from the definition of u_{ij}. ⌟

Now define paths Q^i_{ij} and Q^j_{ij} as follows:

- if $\mathrm{dist}(u_{ij}, a_i) < \mathrm{dist}(u_{ij}, a_j)$, then $Q^i_{ij} = P^i_{ij}$ and $Q^j_{ij} = P^j_{ij} - u_{ij}$;
- if $\mathrm{dist}(u_{ij}, a_i) > \mathrm{dist}(u_{ij}, a_j)$, then $Q^i_{ij} = P^i_{ij} - u_{ij}$ and $Q^j_{ij} = P^j_{ij}$;
- if $\mathrm{dist}(u_{ij}, a_i) = \mathrm{dist}(u_{ij}, a_j)$, then define Q^i_{ij} and Q^j_{ij} using any of the above.

Thus, by Claim 2 we have that paths Q^i_{ij} and Q^j_{ij} are disjoint. Moreover, for each vertex x on Q^i_{ij} we have $\mathrm{dist}(x, a_i) \leqslant \mathrm{dist}(x, a_j)$, and for each vertex y on Q^j_{ij} we have $\mathrm{dist}(y, a_i) \geqslant \mathrm{dist}(y, a_j)$.

Claim 3. *Let $\{i, j\}$ and $\{i', j'\}$ be two different subsets of size 2 of $\{1, \ldots, t\}$. Suppose that paths Q^i_{ij} and $Q^{i'}_{i'j'}$ intersect. Then $i = i'$.*

Proof. Let x be a vertex lying both on Q^i_{ij} and $Q^{i'}_{i'j'}$. We first consider the corner case when $x = u_{ij}$. Suppose first that $\mathrm{dist}(v_{ij}, x) \geqslant \mathrm{dist}(v_{i'j'}, x)$. Then by Claim 1 we have

$$\mathrm{dist}(v_{i'j'}, a_i) \leqslant \mathrm{dist}(v_{i'j'}, x) + \mathrm{dist}(x, a_i) \leqslant \mathrm{dist}(v_{ij}, x) + \mathrm{dist}(x, a_i) \leqslant r,$$

and analogously $\mathrm{dist}(v_{i'j'}, a_j) \leqslant r$. However, we assumed that $a_{i'}$ and $a_{j'}$ are the only vertices of A that are at distance at most r from $v_{i'j'}$, hence $\{i, j\} = \{i', j'\}$, a contradiction. Suppose then that $\mathrm{dist}(v_{ij}, x) < \mathrm{dist}(v_{i'j'}, x)$. Then we have

$$\mathrm{dist}(v_{ij}, a_{i'}) \leqslant \mathrm{dist}(v_{ij}, x) + \mathrm{dist}(x, a_{i'}) < \mathrm{dist}(v_{i'j'}, x) + \mathrm{dist}(x, a_{i'}) \leqslant r,$$

where the last equality follows from Claim 1. Since a_i and a_j are the only vertices of A that are at distance at most r from v_{ij}, we infer that $i' \in \{i, j\}$. If $i' = i$ then we would be done, so suppose $i' = j$. Since $x = u_{ij}$ and x lies on Q^i_{ij}, by the definition of Q^i_{ij} we have that $\mathrm{dist}(x, a_i) \leqslant \mathrm{dist}(x, a_j) = \mathrm{dist}(x, a_{i'})$. Therefore,

$$\mathrm{dist}(v_{i'j'}, a_i) \leqslant \mathrm{dist}(v_{i'j'}, x) + \mathrm{dist}(x, a_i) \leqslant \mathrm{dist}(v_{i'j'}, x) + \mathrm{dist}(x, a_{i'}) \leqslant r.$$

where the last inequality follows from Claim 1. Again, we assumed that $a_{i'}$ and $a_{j'}$ are the only vertices of A that are at distance at most r from $v_{i'j'}$, so $i \in \{i', j'\}$. If $i = i'$ then we are done, and otherwise we have $i = j'$. Together with $i' = j$ this implies $\{i, j\} = \{i', j'\}$, a contradiction.

The second corner case when $x = u_{i'j'}$ leads to a contradiction in a symmetric manner.

We now move to the main case when $x \neq u_{ij}$ and $x \neq u_{i'j'}$. Then by Claim 2 we have $\mathrm{dist}(x, a_i) < \mathrm{dist}(x, a_j)$ and $\mathrm{dist}(x, a_{i'}) < \mathrm{dist}(x, a_{j'})$. By symmetry, without loss of generality assume that $\mathrm{dist}(x, a_i) \leqslant \mathrm{dist}(x, a_{i'})$. Observe now that

$$\mathrm{dist}(v_{i'j'}, a_i) \leqslant \mathrm{dist}(v_{i'j'}, x) + \mathrm{dist}(x, a_i) \leqslant \mathrm{dist}(v_{i'j'}, x) + \mathrm{dist}(x, a_{i'}) \leqslant r,$$

where the last inequality follows from Claim 1. As we assumed that $a_{i'}$ and $a_{j'}$ are the only vertices of A that are at distance at most r from $v_{i'j'}$, we have $i \in \{i', j'\}$. However, it cannot happen that $i = j'$, because $\mathrm{dist}(x, a_{i'}) < \mathrm{dist}(x, a_{j'})$ and $\mathrm{dist}(x, a_{i'}) \geqslant \mathrm{dist}(x, a_i)$. We conclude that $i = i'$. ⌟

For each $i \in \{1, 2, \ldots, t\}$ we define X_i to be the union of vertex sets of paths Q^i_{ij} for $j \neq i$. Each of these paths has length at most r and has a_i as an endpoint, hence the subgraph induced by X_i is connected and has radius at most r. By Claim 3, sets X_i are pairwise disjoint. Finally, observe that for each $\{i, j\} \subseteq \{1, \ldots, t\}$ with $i \neq j$, there is an edge between a vertex of Q^i_{ij} and a vertex of Q^j_{ij}. We conclude that $(X_i)_{i=1,\ldots,t}$ is a depth-r minor model of K_t in G, a contradiction. □

4. Domination and independence duality in nowhere dense classes

Gap between γ_r and $\gamma^\star_r$. We first prove Theorem 4, which we repeat for convenience.

Theorem 4. *There exists a nowhere dense class $\mathscr{C}$ of graphs with the property that for every $r \in \mathbb{N}$ we have*

$$\sup_{G \in \mathscr{C}} \frac{\gamma_r(G)}{\gamma^\star_r(G)} = +\infty.$$

We will first prove the following auxiliary lemma, which essentially encompasses the statement for $r = 1$. Here $\Delta(G)$ denotes the maximum degree of a vertex in G, whereas $\mathrm{girth}(G)$ is the minimum length of a cycle in G.

Lemma 12. *For every sufficiently large $d \in \mathbb{N}$ there exists a graph G_d, say with n vertices, satisfying the following properties.*

1. $\Delta(G_d) \leqslant d$;
2. $\mathrm{girth}(G_d) \geqslant d$;
3. $\gamma_1(G_d) \geqslant \frac{\ln d}{2d} \cdot n$; *and*
4. $\gamma_1^\star(G_d) \leqslant \frac{2}{d} \cdot n$.

Proof. Let n be a large even integer, to be fixed depending on d later. We choose the graph G at random using the following random procedure. Consider a set of dn vertices, divided into n buckets, each containing d vertices. Choose a matching M on those dn vertices uniformly at random. Collapse each bucket into a single vertex, thus creating a multigraph G_0 with $dn/2$ edges: every edge of M gives rise to one edge in G_0, which connects the vertices corresponding to the buckets containing the endpoints of the original edge. Note that G_0 is d-regular and may contain multiple edges and loops; the latter ones may arise in case some edge of M has both endpoints in the same bucket. This procedure of choosing a random d-regular multigraph G_0 shall be called the *bucket model*. Finally, we obtain G from G_0 as follows: for every cycle C of length at most d in G_0, pick an arbitrary edge of C and remove it. Note that, in particular, in this manner we remove all the loops and reduce the multiplicity of every edge to at most 1; hence G is a simple graph.

Since G_0 is d-regular and G is a subgraph of G_0, it is clear that $\Delta(G) \leqslant d$; hence Property 1 is always satisfied. Property 2 follows directly from the construction. For Properties 3 and 4, we will need that for large enough n, we actually remove only a sublinear (in n) number of edges when constructing G from G_0. This follows from well-known estimates on the number of short cycles in a random regular graph (see e.g. Bollobás [6] and Wormald [33, 34]), but we give a direct proof for completeness.

Claim 4. *Let X be a random variable counting the number of different cycles of length at most d in G_0. Then for large enough even n depending on d, we have*

$$\mathbb{E}X \leqslant \sqrt{n}.$$

Proof. For an even integer k, let $M(k)$ be the number of matchings on k vertices, which is

$$M(k) = (k-1) \cdot (k-3) \cdot \ldots \cdot 3 \cdot 1 = \frac{k!}{2^{k/2} \cdot (k/2)!}.$$

By Stirling's approximation, for large enough k we have

$$(2/e)^{k/2} \cdot (k/2)^{k/2} \leqslant M(k) \leqslant 2 \cdot (2/e)^{k/2} \cdot (k/2)^{k/2}. \tag{1}$$

For $\ell \in \{1, \ldots, d\}$, let X_ℓ be a random variable counting the number of cycles of length ℓ in G_0. Then $X = X_1 + \ldots + X_d$. We may estimate the expected

value of X_ℓ as follows:

$$\mathbb{E}X_\ell \leqslant n^\ell d^{2\ell} \cdot \frac{M(nd-2\ell)}{M(nd)}. \tag{2}$$

Indeed, $M(nd)$ is the cardinality of the probabilistic space in the bucket model, the factor $n^\ell d^{2\ell}$ is an upper bound on the number of ways vertices and edges of a cycle of length ℓ may be chosen, while $M(nd-2\ell)$ is the number of ways that the other edges, outside of this cycle, are chosen. Combining (2) with (1), for n large enough we have

$$\begin{aligned}\mathbb{E}X_\ell &\leqslant 2 \cdot n^\ell \cdot d^{2\ell} \cdot (2/e)^{-\ell} \cdot \frac{(nd/2-\ell)^{nd/2-\ell}}{(nd/2)^{nd/2}} \\ &= 2 \cdot n^\ell \cdot d^{2\ell} \cdot (2/e)^{-\ell} \cdot \left(1 - \frac{\ell}{nd/2}\right)^{nd/2} \cdot (nd/2-\ell)^{-\ell} \\ &\leqslant 2 \cdot n^\ell \cdot d^{2\ell} \cdot (2/e)^{-\ell} \cdot e^{-\ell} \cdot (nd/4)^{-\ell} = 2 \cdot (2d)^\ell.\end{aligned}$$

Thus, we have

$$\mathbb{E}X = \sum_{\ell=1}^{d} \mathbb{E}X_\ell \leqslant \sum_{\ell=1}^{d} 2 \cdot (2d)^\ell \leqslant (2d)^{d+1}.$$

Hence, for $n \geqslant (2d)^{2(d+1)}$ it holds that $\mathbb{E}X \leqslant \sqrt{n}$. ⌟

By Markov's inequality, we conclude that with probability at least $\frac{1}{10}$, G_0 contains at most $10\sqrt{n}$ cycles of length at most d, which in particular implies that in this case G contains at most $10\sqrt{n}$ fewer edges than G_0. Removal of a single edge can increase the domination number and the fractional domination number of a multigraph by at most 1; here, when defining the fractional domination number of a multigraph we determine domination of a vertex u by verifying whether the sum of weights of the second endpoints over all edges incident to u is at least 1. Hence we have the following.

Claim 5. *With probability at least $\frac{9}{10}$ we have*

$$\gamma_1(G_0) \leqslant \gamma_1(G) \leqslant \gamma_1(G_0) + 10\sqrt{n} \quad \text{and} \quad \gamma_1^\star(G_0) \leqslant \gamma_1^\star(G) \leqslant \gamma_1^\star(G_0) + 10\sqrt{n}.$$

Claim 5 essentially reduces checking Properties 3 and 4 for G to checking them for G_0, as the (fractional) domination numbers are almost the same. On one hand, we have

$$\gamma_1^\star(G_0) \leqslant \frac{n}{d},$$

since putting the weight $\frac{1}{d}$ on every vertex of G_0 yields a fractional dominating set of size $\frac{n}{d}$. Since $10\sqrt{n} < \frac{n}{d}$ for large enough n, by Claim 5 we conclude that Property 4 holds with probability at least $\frac{1}{10}$.

To verify that Property 3 also holds with high probability, we use the results of Alon and Wormald [3], who studied the expected size of a minimum dominating set in a random d-regular graph.

Claim 6 (stated in the proof of Thm 1.2 of Alon and Wormald [3]). *Fix large enough d and consider even n tending to infinity. Then for any $c < 1$, the expected number of dominating sets of size at most $\frac{c \ln d}{d} \cdot n$ in a d-regular multigraph on n vertices chosen randomly according to the bucket model tends to* 0.

In particular, by Markov's inequality, for large enough n the probability that G_0 contains a dominating set of size at most $\frac{\ln d}{4d} \cdot n$ is at most $\frac{1}{10}$. By Claim 5 we infer that G contains a dominating set of size at most $\frac{\ln d}{4d} \cdot n + 10\sqrt{n}$ with probability at most $\frac{2}{10}$, and this value is upper bounded by $\frac{\ln d}{2d} \cdot n$ for large enough n.

We conclude that having fixed d large enough, Properties 3 and 4 hold simultaneously for large enough n with probability at least $\frac{7}{10}$. This concludes the proof. □

For a graph G and $r \in \mathbb{N}$, by $G^{(r)}$ we denote the *exact r-subdivision* of G, which is a graph obtained from G by subdviding every edge $r-1$ times, i.e., replacing it with a path of length r. We now lift the statement of Lemma 12 to larger radii by considering the following construction; see Figure 1.

Definition 2. Let G be a graph and let $r \in \mathbb{N}$. Construct a graph $G^{\langle r \rangle}$ from the exact-r subdivision $G^{(r)}$ of G by adding two new vertices x and y, connecting x to every subdivision vertex using a path of length r, and connecting y to x using a path of length r.

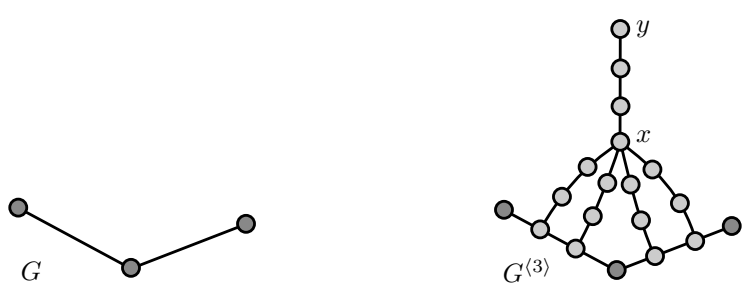

Figure 1: Construction of $G^{\langle 3 \rangle}$ from G, where G is a path on 3 vertices.

We note the following properties of the above construction.

Lemma 13. *The following assertions hold:*

1. *If $\mathscr{C}$ is a nowhere dense graph class, then $\{G^{\langle r \rangle} : G \in \mathscr{C}, r \in \mathbb{N}\}$ is nowhere dense as well.*
2. *For every graph G and every $r \in \mathbb{N}$ we have $\gamma_r(G^{\langle r \rangle}) = \gamma_1(G) + 1$.*
3. *For every graph G and every $r \in \mathbb{N}$ we have $\alpha^\star_{2r}(G^{\langle r \rangle}) = \alpha^\star_2(G) + 1$.*

Proof. Assertion 1 is trivial: if $K_t \not\preccurlyeq_s G$, then $K_{t+1} \not\preccurlyeq_s G^{\langle r \rangle}$, for every $r \in \mathbb{N}$.

For Assertion 2, observe first that if $D \subseteq V(G)$ is a dominating set in G, then $D \cup \{x\}$ (as a subset of $V(G^{\langle r \rangle})$) is a distance-r dominating set of $G^{\langle r \rangle}$. For the reverse inequality, let $O = V(G)$ be the set of vertices of $G^{\langle r \rangle}$ that were originally in G, and define a function $f \colon V(G^{\langle r \rangle}) \to O \cup \{x\}$ as follows: every vertex of O is mapped to itself, all vertices of the paths from y to x are mapped to x, and all vertices on any length-r path from x (exclusive) to some subdivision vertex w (inclusive), where w lies on the length-r path between some $u, v \in O$ in $G^{(r)}$, are mapped to either u or v (chosen arbitrarily). It is not hard to see that if D is a distance-r dominating set in $G^{\langle r \rangle}$, then so does $f(D)$: the set D needs to contain some vertex from the x-to-y path, due to the necessity of dominating y, hence $x \in f(D)$ and x already r-dominates all the vertices of $G^{\langle r \rangle}$ apart from O, and f maps every vertex to a vertex that can only dominate more vertices from O. Since $\{x\} \subseteq f(D) \subseteq O \cup \{x\}$ and x does not r-dominate any vertex of O, it follows that $f(D) - \{x\}$ is a dominating set in G.

Finally, for Assertion 3, let $I^\star$ be a fractional distance-2 independent set of G. Then we obtain a fractional distance-$2r$ independent set of $G^{\langle r \rangle}$ of value one larger by adding weight 1 to the vertex y. Vice versa, let $I^\star$ be a fractional distance-$2r$ independent set of $G^{\langle r \rangle}$. Observe that all vertices of $V(G^{\langle r \rangle}) - O$ can collect a total weight of at most 1, as they all lie in the r-neighborhood of the vertex x. It follows that $I^\star$ restricted to O is a fractional distance-$2r$ independent set in $G^{(r)}$, which means that it is also a fractional distance-2 independent set in G. □

We are now ready to prove Theorem 4.

Proof (Proof of Theorem 4). The class $\mathscr{D} = \{G \colon \mathrm{girth}(G) \geqslant \Delta(G)\}$ is known to be nowhere dense [30]. By Lemma 12, for every large enough $d \in \mathbb{N}$ there is a graph $G_d \in \mathscr{D}$ with $\frac{\gamma_1(Gd)}{\gamma^\star(G_d)} \geqslant \frac{\ln d}{4}$. It follows that

$$\sup_{G \in \mathscr{D}} \frac{\gamma_1(G)}{\gamma_1^\star(G)} = +\infty.$$

By Lemma 13, Assertion 1, the class $\mathscr{C} := \{G^{\langle r \rangle} \colon G \in \mathscr{D}, r \in \mathbb{N}\}$ is nowhere dense. By Lemma 13, Assertions 2 and 3, and the fact that $\gamma_1^\star(G) = \alpha_2^\star(G)$ for all graphs G, for every $r \in \mathbb{N}$ we have

$$\sup_{G^{\langle r \rangle} \in \mathscr{C}} \frac{\gamma_r(G^{\langle r \rangle})}{\gamma_r^\star(G^{\langle r \rangle})} = \sup_{G \in \mathscr{D}} \frac{\gamma_1(G) + 1}{\gamma_1^\star(G) + 1} \geqslant \sup_{G \in \mathscr{D}} \frac{\gamma_1(G)}{2\gamma_1^\star(G)} = \frac{1}{2} \sup_{G \in \mathscr{D}} \frac{\gamma_1(G)}{\gamma_1^\star(G)} = +\infty.$$

This concludes the proof. □

Gap between α_{2r} and γ_r. We now prove Theorem 5, which we repeat for convenience.

Theorem 5. *Let $\mathscr{C}$ be a nowhere dense class of graphs. There exists a function $f_{\mathrm{dual}} \colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ such that for all $G \in \mathscr{C}$, $A \subseteq V(G)$, $r \in \mathbb{N}$, and $\varepsilon > 0$, we have*

$$\gamma_r(G, A) \leqslant f_{\mathrm{dual}}(r, \varepsilon) \cdot \alpha_{2r+1}(G, A)^{1+\varepsilon} \leqslant f_{\mathrm{dual}}(r, \varepsilon) \cdot \alpha_{2r}(G, A)^{1+\varepsilon}.$$

Furthermore, there is a polynomial-time algorithm that given G, A, r, ε as above, computes a distance-r dominating set of A in G of size bounded as above.

We are going to make use of the following kernelization result of [16] for distance-r dominating sets.

Lemma 14 ([16]). *Let $\mathscr{C}$ be a nowhere dense class of graphs. There exists a function $f_{\mathrm{ker}}\colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ and a polynomial-time algorithm with the following properties. Given a graph $G \in \mathscr{C}$, a vertex subset $A \subseteq V(G)$, positive integers k, r and a real number $\varepsilon > 0$, the algorithm either correctly concludes that $\gamma_r(G, A) > k$, or finds a subset $Y \subseteq V(G)$ of size at most $f_{\mathrm{ker}}(r, \varepsilon) \cdot k^{1+\varepsilon}$ and a subset $B \subseteq Y \cap A$ such that*

$$\min(\gamma_r(G, A), k) = \min(\gamma_r(G[Y], B), k).$$

Moreover, every subset of Y that distance-r dominates B in $G[Y]$ also distance-r dominates A in G.

We remark that the statement of this result given in [16] is somewhat weaker, but the above formulation follows readily from the proof. First, in [16] the result is stated for distance-r dominating sets of the whole graph G, i.e., for $A = V(G)$; however, throughout the whole reasoning, the more general problem of dominating a subset of vertices is considered and it is straightforward to lift the argument to this setting. Second, in [16], only assertion $\gamma_r(G, A) \leqslant k \Leftrightarrow \gamma_r(G[Y], B) \leqslant k$ is stated, whereas the stronger assertion that the distance-r domination numbers are equal in case they are both bounded by k follows immediately from the proof. Finally, the fact that every subset of Y that distance-r dominates B in $G[Y]$ also distance-r dominates A in G is the key property used to prove the above.

Let us remark one more property of the set Y provided by Lemma 14 that follows implicitly from the proof in [16]: adding any vertices of G to Y does not change the asserted properties of Y. We may hence apply Lemma 10 with parameter $2r + 1$ to the set B in the graph G, thus obtaining its superset B', and add all the vertices of B' to the set Y. Thus, we additionally ensure that distances up to value $2r + 1$ between elements of B are preserved in $G[Y]$. We derive the following lemma.

Lemma 15. *Let $\mathscr{C}$ be a nowhere dense class of graphs. There exists a function $f_{\mathrm{ker}}\colon \mathbb{N} \times \mathbb{R} \to \mathbb{N}$ and a polynomial-time algorithm with the following properties. Given a graph $G \in \mathscr{C}$, a vertex subset $A \subseteq V(G)$, positive integers k, r and a real number $\varepsilon > 0$, the algorithm either correctly concludes that $\gamma_r(G, A) > k$, or finds a subset $Y \subseteq V(G)$ of size at most $f_{\mathrm{ker}}(r, \varepsilon) \cdot k^{1+\varepsilon}$ and a subset $B \subseteq Y \cap A$ such that $\alpha_{2r+1}(G, B) = \alpha_{2r+1}(G[Y], B)$ and*

$$\min(\gamma_r(G, A), k) = \min(\gamma_r(G[Y], B), k).$$

Moreover, every subset of Y that r-dominates B in $G[Y]$ also r-dominates A in G.

We now apply Dvořák's duality theorem [15] for bounded expansion classes, which is in fact stated in terms of a parameter called the *weak-r coloring number*,

denoted $\mathrm{wcol}_r(\cdot)$. We refer to [15] for a formal definition of this parameter and only note the following lemma, which characterizes nowhere dense classes in terms of weak coloring numbers.

Lemma 16 (Zhu [35]). *A class of graphs $\mathscr{C}$ is nowhere dense if and only if there exists a function $f_{\mathrm{wcol}}\colon \mathbb{N}\times\mathbb{R}\to\mathbb{N}$ such that for all $r\in\mathbb{N}$ and all $\varepsilon>0$ and all n-vertex graphs $H\subseteq G$ for $G\in\mathscr{C}$ we have $\mathrm{wcol}_r(H)\leqslant f_{\mathrm{wcol}}(r,\varepsilon)\cdot n^{\varepsilon}$.*

The key ingredient of our proof is the following result of Dvořák [15], which relates domination and independence numbers in graphs with bounded weak coloring numbers.

Lemma 17 (Dvořák [15], see also [14]). *For every graph G and vertex subset $A\subseteq V(G)$,*

$$\gamma_r(G,A)\leqslant \mathrm{wcol}_{2r+1}(G)^2\cdot\alpha_{2r+1}(G,A).$$

We are ready to give the proof of Theorem 5.

Proof (Proof of Theorem 5). Given $G\in\mathscr{C}$ and $A\subseteq V(G)$, let $k:=\gamma_r(G,A)$. We define $\delta>0$ as a constant depending on ε which will be determined at the end of the proof. Apply Lemma 15 to find a subset $Y\subseteq V(G)$ of size at most $f_{\mathrm{ker}}(r,\delta)\cdot k^{1+\delta}$ and a subset $B\subseteq Y\cap A$ such that $\gamma_r(G[Y],B)=k$ and $\alpha_{2r+1}(G,B)=\alpha_{2r+1}(G[Y],B)$. As $G[Y]$ is a subgraph of G, according to Lemma 16 we have

$$\begin{aligned}&\mathrm{wcol}_{2r+1}(G[Y])\leqslant\\ &\qquad f_{\mathrm{wcol}}(2r+1,\delta)\cdot(f_{\mathrm{ker}}(r,\delta)\cdot k^{1+\delta})^{\delta}\leqslant f_{\mathrm{wcol}}(2r+1,\delta)\cdot f_{\mathrm{ker}}(r,\delta)^{\delta}\cdot k^{2\delta}.\end{aligned}$$

By Lemma 17, we have

$$\begin{aligned}k=\gamma_r(G,A)&=\gamma_r(G[Y],B)\leqslant \mathrm{wcol}_{2r+1}(G[Y])^2\cdot\alpha_{2r+1}(G[Y],B)\\ &=\mathrm{wcol}_{2r+1}(G[Y])^2\cdot\alpha_{2r+1}(G,B)\leqslant \mathrm{wcol}_{2r+1}(G[Y])^2\cdot\alpha_{2r+1}(G,A)\\ &\leqslant (f_{\mathrm{wcol}}(2r+1,\delta)\cdot f_{\mathrm{ker}}(r,\delta)^{\delta}\cdot k^{2\delta})^2\cdot\alpha_{2r+1}(G,A).\end{aligned}$$

Hence

$$k^{1-4\delta}\leqslant f_{\mathrm{wcol}}(2r+1,\delta)^2\cdot f_{\mathrm{ker}}(r,\delta)^{2\delta}\cdot\alpha_{2r+1}(G,A),$$

which means that

$$\begin{aligned}k&\leqslant \big(f_{\mathrm{wcol}}(2r+1,\delta)^2\cdot f_{\mathrm{ker}}(r,\delta)^{2\delta}\cdot\alpha_{2r+1}(G,A)\big)^{1/(1-4\delta)}\\ &\leqslant \big(f_{\mathrm{wcol}}(2r+1,\delta)^2\cdot f_{\mathrm{ker}}(r,\delta)^{2\delta}\cdot\alpha_{2r+1}(G,A)\big)^{1+8\delta}.\end{aligned}$$

Conclude by setting $\delta=\varepsilon/8$ and $f_{\mathrm{dual}}(r,\varepsilon)=\big(f_{\mathrm{wcol}}(2r+1,\delta)^2\cdot f_{\mathrm{ker}}(r,\delta)^{2\delta}\big)^{1+8\delta}$. Finally, note that sets Y and B obtained using Lemma 15 and the distance-r dominating set of B in $G[Y]$ obtained using Lemma 17 are computable in polynomial time, while the last statement of Lemma 15 asserts that this set also r-dominates A in G. □

5. Kernelization

We now come to the proof of Theorem 6, which we repeat for convenience.

Theorem 6. *Let $\mathscr{C}$ be a fixed nowhere dense class of graphs, let r be a fixed positive integer, and let $\varepsilon > 0$ be a fixed real. Then there exists a polynomial-time algorithm with the following properties. Given a graph $G \in \mathscr{C}$, a vertex subset $A \subseteq V(G)$, and a positive integer k, the algorithm either correctly concludes that $\alpha_r(G, A) < k$, or finds a subset $Y \subseteq V(G)$ of size at most $f_{\mathrm{ker}}(r, \varepsilon) \cdot k^{1+\varepsilon}$, for some function f_{ker} depending only on $\mathscr{C}$, and a subset $B \subseteq Y \cap A$ such that $\alpha_r(G, A) \geqslant k \iff \alpha_r(G[Y], B) \geqslant k$.*

For the rest of this section let us fix a nowhere dense class $\mathscr{C}$, $k, r \in \mathbb{N}$ and $\varepsilon > 0$, as well as a graph $G \in \mathscr{C}$ and $A \subseteq V(G)$. We will carry out the construction with a real $\delta > 0$ that will be defined as a function of ε at the end of the proof. Let $d := \lfloor r/2 \rfloor$.

Using the algorithm of Theorem 5, we first compute an approximate distance-d dominating set D for A. We have $|D| \leqslant f_{\mathrm{dual}}(d, \delta) \cdot \alpha_r(G, A)^{1+\delta}$, hence if $|D| > f_{\mathrm{dual}}(d, \delta) \cdot k^{1+\delta}$, then we may immediately conclude that $\alpha_r(G, A) > k$ and terminate the algorithm. Therefore, from now on we may assume that $|D| \leqslant f_{\mathrm{dual}}(d, \delta) \cdot k^{1+\delta}$.

Now, using the algorithm of Lemma 9 (with parameters $2r$ and δ) we compute a vertex subset Z with the following properties:

- $D \subseteq Z$, in particular, Z is a distance-d dominating set of A;
- $|Z| \leqslant f_{\mathrm{cl}}(2r, \delta) \cdot |D|^{1+\delta}$; and
- $|M^G_{2r}(u, Z)| \leqslant f_{\mathrm{cl}}(2r, \delta) \cdot |D|^{\delta}$ for each $u \in V(G) - Z$.

Here, f_{cl} is the function provided by Lemma 9 for the class $\mathscr{C}$.

We now classify the elements of $A - Z$ with respect to their $2r$-projections profiles onto Z. More precisely, we define the following equivalence relation:

$$u \sim_Z v \quad \iff \quad \rho^G_{2r}[u, Z] = \rho^G_{2r}[v, Z].$$

According to Lemma 8 (applied with parameters $2r$ and δ), there exists a function f_{proj} such that this equivalence relation has at most $f_{\mathrm{proj}}(2r, \delta) \cdot |Z|^{1+\delta}$ equivalence classes. Observe that the empty projection profile (i.e., one that maps all of Z to ∞) is not realized, because Z is a distance-d dominating set of A.

Examine the sizes of equivalence classes of $\sim_Z$ and suppose that some class K of $\sim_Z$ has more than

$$N(4r, f_{\mathrm{cl}}(2r, \delta) \cdot |D|^{\delta} + (r+1)^{s(4r)} \cdot (s(4r) + 1) + 1)$$

elements, where N and s are the functions from Definition 1, characterizing $\mathscr{C}$ as uniformly quasi-wide; note here that for fixed r, $s(4r)$ is a fixed constant and $N(4r, \cdot)$ is a fixed polynomial, both hard-codable in the algorithm, so the

above number can be computed. Then we proceed as follows. In the following, we simply write s for $s(4r)$. We apply the algorithm of Theorem 11 to find a subset $S \subseteq V(G)$ of size at most s and a set $L \subseteq K - S$ of size at least $f_{\mathrm{cl}}(2r, \delta) \cdot |D|^\delta + (r+1)^s \cdot (s+1) + 1$ which is $4r$-independent in $G - S$.

Claim 7. *There are at most $f_{\mathrm{cl}}(2r, \delta) \cdot |D|^\delta$ elements in L which are at distance at most $2r$ from Z in the graph $G - S$.*

Proof. No two elements of L can be connected by a path of length at most $2r$ in $G - S$ to the same element of Z, as by assumption, the elements of L are $4r$-independent in $G - S$. However, every element of Z that is at distance at most $2r$ in $G - S$ from some element of L must belong to the common $2r$-projection of vertices of L onto Z. This projection has size at most $f_{\mathrm{cl}}(2r, \delta) \cdot |D|^\delta$, so there can be at most this many vertices in L that are at distance at most $2r$ from Z in $G - S$. ⌟

Hence, at least $(r+1)^s \cdot (s+1) + 1$ elements of L cannot be connected to Z by a path of length at most $2r$ in $G - S$. We classify these elements with respect to their r-projections onto S, that is, we define the following equivalence relation:

$$u \sim_S v \quad \Longleftrightarrow \quad \rho_r^G[u, S] = \rho_r^G[v, S].$$

As S has size at most s and $\rho_r^G[u, S]$ is a function mapping from S to $\{1, \ldots, r, \infty\}$, this equivalence relation has at most $(r+1)^s$ equivalence classes. Hence, there exists at least one equivalence class L' of $\sim_S$ which has at least $s + 2$ elements.

Claim 8. *Every element of L' is irrelevant, i.e., for every $a \in L'$ there exists a distance-r independent subset of A of size k if and only if there exists a distance-r independent subset of $A - \{a\}$ of size k*

Proof. Let $I \subseteq A$ be a distance-r independent set which contains an element $a \in L'$. We show that we can replace a with some other element $a' \in L'$ to obtain a distance-r independent set I' of the same size.

For this, we show that if we cannot find such an element $a' \in L'$, then I was not a distance-r independent set. So assume that the r-neighborhood (in G) of every $a' \in L' - \{a\}$ contains an element $b \in I$, which conflicts choosing a' into the distance-r independent set. For each $a' \in L'$ fix such an element $b(a')$ of I.

First assume that some path of length at most r between a' and $b(a')$ contains an element of S and let t be the one closest to a'. As $a \sim_S a'$, the vertex a is at the same distance from t as a', which implies that a is at distance at most r from $b(a')$. However, by assumption $a \in I$ and $b(a') \in I$, which implies that I is not a distance-r independent set. Hence in the following assume that no path of length at most r between a' and $b(a')$ contains an element of S, for any $a' \in L' - \{a\}$.

We show that this assumption implies that every element $b(a') \in I$, for $a' \in L' - \{a\}$, can be connected in G by a path of length at most d to a vertex of S. To see this, recall that every element of L' can be connected in G by a path of length at most d to some $z \in Z$, as Z is a distance-d dominating set

of A in G. As a' and $b(a')$ are at distance at most r, the distance between a' and z is at most $r + d \leqslant 2r$. However, the elements of L' are at distance more than $2r$ from Z in $G - S$. Hence, as we assumed that no path of length at most r between a' and $b(a')$ contains an element of S, $b(a')$ must be within distance at most d to some $t \in S$.

Now, since $|S| \leqslant s$ and $|L'| \geqslant s + 2$, at least 2 elements $b(a')$ and $b(a'')$ in I associated with a' and a'' in $L' - \{a\}$ must be at distance at most d from the same element of S. Hence, their distance in G is at most $2d \leqslant r$, which implies that I is not a distance-r independent set, a contradiction. This finishes the proof of the claim. ⌟

Hence, if A is large, we can safely remove any element $a' \in L'$ from A. We repeat the whole procedure as long as this is possible, i.e. until we do not find an equivalence class K of size $N(f_{\mathrm{cl}}(2r, \delta) \cdot |D|^{\delta} + (r+1)^{4s} \cdot (s+1) + 1)$ anymore. Then, we return the current set A as the set B whose existence we claimed in the theorem. Obviously, the classification procedure can be carried out in polynomial time and has to be repeated at most $|A|$ times. Hence, we can compute the set B in polynomial time.

According to Lemma 10 there exists a function f_{pth} and a polynomial-time algorithm which computes for the set B a superset Y such that

- whenever $\mathrm{dist}_G(u, v) \leqslant r$ for $u, v \in B$, then $\mathrm{dist}_{G[Y]}(u, v) = \mathrm{dist}_G(u, v)$; and
- $|Y| \leqslant f_{\mathrm{pth}}(r, \varepsilon) \cdot |B|^{1+\varepsilon}$.

We compute such a set Y. The following observation is immediate.

Claim 9. *The sets A, B and Y satisfy the following property. There exists an r-independent subset $I \subseteq A$ of size k in G if and only if there exists an r-independent subset $I' \subseteq B$ of size k in $G[Y]$.*

Hence the set B and Y provide us with a kernel, as desired. It remains to choose δ so that we obtain the claimed bounds on the size of the kernel. We have

$$\begin{aligned}|Y| &\leqslant f_{\mathrm{pth}}(r, \varepsilon) \cdot |B|^{1+\delta} \\ &\leqslant f_{\mathrm{pth}}(r, \varepsilon) \cdot \big(f_{\mathrm{proj}}(2r, \delta) \cdot |Z|^{1+\delta} \cdot N(f_{\mathrm{cl}}(2r, \delta) \cdot |D|^{\delta} + (r+1)^{s(4r)} \\ &\qquad \cdot (s(4r)+1)+1)\big)^{1+\delta}.\end{aligned}$$

Let $p \coloneqq (4t(r)+1)^{2r \cdot t(r)}$, where graphs from $\mathscr{C}$ exclude $K_{t(r)}$ as a depth-r minor. Using the bounds of Theorem 11, we can now define a function f_{uqw} so that

$$N(f_{\mathrm{cl}}(2r, \delta) \cdot |D|^{\delta} + (r+1)^{s(4r)} \cdot (s(4r)+1)+1) \leqslant f_{\mathrm{uqw}}(r, \delta) \cdot |D|^{\delta \cdot p}.$$

Hence

$$|Y| \leqslant f_{\mathrm{pth}}(r, \varepsilon) \cdot \big(f_{\mathrm{proj}}(2r, \delta) \cdot |Z|^{1+\delta} \cdot f_{\mathrm{uqw}}(r, \delta) \cdot |D|^{\delta \cdot p}\big)^{1+\delta}.$$

Now using $|Z| \leqslant f_{\mathrm{cl}}(2r,\delta) \cdot |D|^{1+\delta}$, $|D| \leqslant f_{\mathrm{dual}}(r,\delta) \cdot k^{1+\delta}$ and $\delta^2 \leqslant \delta$, we can define f_{ker} so that

$$|Y| \leqslant f_{\mathrm{ker}}(r,\delta) \cdot k^{1+23p\delta}.$$

By defining $\delta := \varepsilon/23p$ we conclude the proof of Theorem 6.

6. Hardness on somewhere dense classes

We finally prove Theorem 7, which we repeat for convenience.

Theorem 7. *Let $\mathscr{C}$ be a class of graphs that is closed under taking subgraphs and which is not nowhere dense. Then there exists an integer r such that* Distance-r Independent Set *is* W[1]*-hard on $\mathscr{C}$.*

We shall use the following well-known characterization of somewhere dense graph classes; recall that $G^{(r)}$ denotes the exact r-subdivision of a graph G.

Lemma 18 ([29]). *Let $\mathscr{C}$ be somewhere dense graph class that is closed under taking subgraphs. Then there exists $r \in \mathbb{N}$ such that $G^{(r)} \in \mathscr{C}$ for all graphs G.*

The hardness proof is based on a very simply reduction. Using Lemma 18 we will give an easy reduction from the classical Independent Set problem, which is knwon to be W[1]-hard.

Lemma 19 ([13]). Independent Set *is* W[1]*-hard on the class of all graphs.*

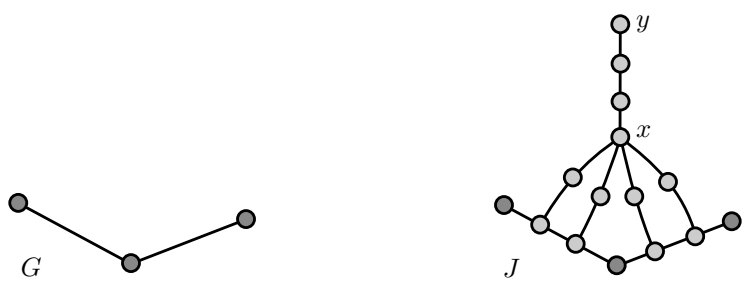

Figure 2: Construction of J from G, where G is a path on 3 vertices. The graph H is the exact r-subdivision of J.

Proof (Proof of Theorem 7). Let $r \in \mathbb{N}$ be the number given by Lemma 18 for the class $\mathscr{C}$, that is, $G^{(r)} \in \mathscr{C}$ for every graph G. Now fix an arbitrary graph G. We construct in polynomial time the following graph H; see Figure 2. We first construct $G^{(3)}$, i.e., we replace in G every edge by a path of length 3. Next, we add two new vertices x and y. We connect x to each of the previously added subdivision vertices using a path of length 2, and we connect y to x using a path of length 3. Denote the resulting graph by J. Now construct $H := J^{(r)}$, which by assumption on r belongs to $\mathscr{C}$. We denote the vertices of H which are also vertices of G by O (for *original*).

The following claims summarize the main distance properties of H.

Claim 1. *For all $u, v \in O$ we have* $\mathrm{dist}_G(u, v) \geqslant 2 \Leftrightarrow \mathrm{dist}_H(u, v) \geqslant 6r$.

Proof. It is straightforward to see that the distance between any two vertices of O is exactly $3r$ times larger in H than in G. The claim follows. ⌟

Claim 2. *For all $u, v \in V(H) - O$ we have* $\mathrm{dist}_H(u, v) < 6r$.

Proof. The distance from any $u \in V(H) - (O \cup \{y\})$ to x is at most $(r-1)+2r = 3r-1$, while the distance from y to x is $3r$. Hence, the distance between any $u, v \in V(H) - O$ is smaller than $6r$. ⌟

We can hence relate the size of a distance-$(6r-1)$ independent set in H to the size of an independent set in G as follows.

Claim 3. *We have* $\alpha_{6r-1}(H) = \alpha_1(G) + 1$.

Proof. Let I be a distance-$(6r-1)$ independent set in H. By Claim 2, I can contain at most one vertex of $V(H) - O$. We may assume that this vertex is the vertex y, because y is at distance at least $6r$ from every vertex of O. Now $I' := I - \{y\} \subseteq O$ is an independent set in G by Claim 1.

Conversely, let I be an independent set in G. Then $I \cup \{y\}$ is a distance-$(6r-1)$ independent set in H. ⌟

Hence, finding an independent set of size k in an arbitrary graph G reduces to finding a distance-$(6r-1)$ independent set of size $k+1$ on a polynomial-time computable graph H belonging to $\mathscr{C}$. This proves that Distance-$(6r-1)$ Independent Set on $\mathscr{C}$ is W[1]-hard. □

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