Distributed Domination on Graph Classes of Bounded Expansion

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We provide a new constant factor approximation algorithm for the (connected) distance-r dominating set problem on graph classes of bounded expansion. Classes of bounded expansion include many familiar classes of sparse graphs such as planar graphs and graphs with excluded (topological) minors, and notably, these classes form the most general subgraph closed classes of graphs for which a sequential constant factor approximation algorithm for the distance-r dominating set problem is currently known. Our algorithm can be implemented in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model of distributed computing and uses $\mathcal{O}(r^2 \log n)$ communication rounds.

Our techniques, which may be of independent interest, are based on a distributed computation of sparse neighborhood covers of small radius on bounded expansion classes. We show how to compute an r-neighborhood cover of radius $2r$ and overlap $f(r)$ on every class of bounded expansion in $\mathcal{O}(r^2 \log n)$ communication rounds.

Finally, we show how to use the greater power of the $\mathcal{LOCAL}$ model to turn any distance-r dominating set into a constantly larger connected distance-r dominating set in $3r + 1$ rounds on any class of bounded expansion. Combining this algorithm, e.g., with the constant factor approximation algorithm for dominating sets on planar graphs of Lenzen et al. gives a constant factor approximation algorithm for connected dominating sets on planar graphs in a constant number of rounds in the $\mathcal{LOCAL}$ model, where the approximation ratio is only 6 times larger than that of Lenzen et al.'s algorithm.

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1 Introduction and contributions

The DOMINATING SET problem is one of the most fundamental problems in algorithmic graph theory and complexity theory. Given a graph G, the problem is to find a minimum size subset $D \subseteq V(G)$ such that every vertex $w \in V(G)$ has distance at most one to a vertex in D.

A number of generalisations and variations of the DOMINATING SET problem have been studied. In the DISTANCE-r DOMINATING SET problem, we are given a graph G and an integer parameter r and we are asked to find a minimum size set D such that every vertex in G has distance at most r to a vertex in D. By setting $r = 1$ we obtain the original DOMINATING SET problem. The problem is also known as the (k, r)-*center* problem and has been extensively studied in the literature, especially in the distributed community, where it finds applications in distributed databases, routing and distributed data structures, see, e.g., [12, 14, 26, 31, 39, 45, 46, 55, 54, 56].

Another problem is the CONNECTED DOMINATING SET problem, and its distance-r variant, where we ask for a (distance-r) dominating set which is connected. Also this problem has received considerable interest in the literature, see, e.g., [18, 19, 67, 70]. We also refer to the recent survey on distributed computing and on more applications [63].

Finding a minimum size dominating set in a graph G is NP-complete in general [35], and even so in very restricted settings, e.g., on planar graphs of maximum degree 3 (cf. [GT2] in [25]). Therefore it is believed that there is no efficient algorithm that finds a smallest dominating set for a given graph in general, and consequently, much effort has been made to find good approximations.

Of course, the DISTANCE-r DOMINATING SET problem trivially reduces to the DOMINATING SET problem by connecting in the input graph all vertices that are at distance at most r from each other. However, this may lead to the introduction of many edges even if the input graph is very sparse, which may be problematic, e.g., in large databases where linear time algorithms are crucial. Furthermore, much structural information available on the input graph, e.g., planarity, may be lost when introducing these edges. Therefore, a direct solution to the more general problem is desirable.

There is a simple greedy algorithm to approximate dominating sets: at each step add a vertex to the dominating set which dominates the largest number of non-dominated vertices. This algorithm on an n-vertex graph achieves an approximation ratio of $\ln n - \ln \ln n + \Theta(1)$ [16, 33, 43, 60, 61]. Note that these results are for the SET COVER problem, which however reduces to the DOMINATING SET problem by an approximation preserving reduction and, in fact, the two problems achieve exactly the same approximation ratio [34]. In general, no better approximation ratio can be achieved under standard complexity theoretic assumptions [4, 6, 13, 21, 24, 44, 58].

The DOMINATING SET problem is more manageable when restricted to some special classes of sparse graphs. For example, there exists a PTAS for DOMINATING SET in planar graphs [10], minor closed classes of graphs with local bounded tree-width [23] and graphs with excluded minors [27]. Most recently, it was shown that DISTANCE-r DOMINATING SET admits a PTAS on every graph class with polynomial expansion [30].

All these classes are examples of *classes with bounded expansion*, a very general model of uniform sparseness in graphs introduced by Nešetřil and Ossona de Mendez [48, 49, 47] (see also [52] for other examples of classes with bounded expansion). A motivating observation to the theory of classes with bounded expansion is that the classical notion of sparseness (the ratio of edges to vertices) is not necessarily sufficient for algorithmic tractability. Similarly, the notion of degeneracy and a closely related notion of arboricity, which are stronger notions of uniform sparseness than average edge density, are not satisfactory. A graph G is d-*degenerate* if every subgraph of G contains a vertex of degree at most d. In particular, this means (and, in fact, degeneracy is equivalent to saying) that the average degree of every subgraph is bounded by a constant. Observe that we can make every graph 2-degenerate by subdividing every edge once (by subdividing an edge we mean the operation of replacing an edge $\{u, v\}$ by a path of length 2). Now it is easy to see that the DISTANCE-2 DOMINATING SET problem on the class of 2-degenerate graphs is just as hard as the DOMINATING SET problem on the class of all graphs. Hence, in order to obtain a robust notion of sparseness, in particular, in a locality sensitive setting as in distributed computing, we may want to require our notion of uniform sparseness to be invariant under such local modifications. These requirements lead exactly to the definition of classes of bounded expansion.

DOMINATING SET admits a constant factor approximation on bounded degree graphs, in fact, it is complete for APX on these classes [53]. Hence under the assumption P $\neq$ NP there is no PTAS for the DOMINATING SET problem on bounded degree graphs. All the more so the same holds for the more general DISTANCE-r DOMINATING SET problem. However, for every fixed value of r, there is a linear time computable constant factor approximation algorithm on classes of graphs with bounded expansion [22] and these are the most general subgraph closed classes on which a constant factor approximation algorithm for the DISTANCE-r DOMINATING SET problem is known.

Contribution 1. We present a new approximation algorithm for the DISTANCE-r DOMINATING SET problem on classes of bounded expansion which improves the approximation ratio achieved by the algorithm of [22]. Our algorithm can be implemented in linear time on any class of bounded expansion. A key feature of our algorithm is that it is tailored to be executed in a distributed setting.

There has been lots of effort to approximate the DOMINATING SET problem with distributed algorithms, however, similar hardness results also apply to distributed algorithms. It was shown in [38] that in t communication rounds the DOMINATING SET problem on an n-vertex graphs of maximum degree Δ can only be approximated within factor $\Omega(n^{c/t^2})$ and $\Omega(\Delta^{c'/t})$, where c and c' are constants. This implies that, in general, to achieve a constant approximation ratio, every distributed algorithm requires at least $\Omega(\sqrt{\log n})$ and $\Omega(\log \Delta)$ communication rounds. Kuhn et al. [38] also provide the currently best approximation algorithm on general graphs, which achieves a $(1+\varepsilon)\ln \Delta$-approximation in $\mathcal{O}(\log(n)/\varepsilon)$ rounds for any $\varepsilon > 0$.

For graphs of arboricity a there exists a forest decomposition algorithm achieving a factor $\mathcal{O}(a^2)$-approximation in randomized time $\mathcal{O}(\log n)$, and a deterministic $\mathcal{O}(a \log \Delta)$ approximation algorithm requiring $\mathcal{O}(\log \Delta)$ rounds [42]. Given any $\delta > 0$, $(1+\delta)$-approximations of a maximum independent set, of a maximum matching, and of a minimum dominating set can be computed in $\mathcal{O}(\log^* n)$ rounds in planar graphs [17], which is asymptotically optimal [41]. It is easily seen that the algorithm of [17] extends to minor closed classes of graphs. A constant factor approximation on planar graphs [40, 69] and on graphs of bounded genus [5] can be computed locally in a constant number of communication rounds. In terms of lower bounds, it was shown that there is no deterministic local algorithm (constant-time distributed graph algorithm) that finds a $(7-\varepsilon)$-approximation of a minimum dominating set on planar graphs, for any positive constant ε [32].

Observe that all of the above algorithms cannot be directly employed to obtain good approximations for the DISTANCE-r DOMINATING SET problem, as all structural information which is used in the algorithms may be lost when building the r-transitive closure of the graph. The distributed algorithms of [39, 57] find distance-r dominating sets of size $\mathcal{O}(n/r)$ in time $\mathcal{O}(r \cdot \log^* n)$, without any relation to the size of an optimal distance-r dominating set. In very restrictive settings, e.g., in trees [65] or in star-split graphs [68] better solutions are known.

The DISTANCE-r DOMINATING SET problem is closely related to the problem of covering local neighborhoods in a graph by connected clusters of small radius. The r-neighborhood of a vertex v is the set $N_r[v]$ of vertices w of distance at most r to v. An r-neighborhood cover [9] is a set $\mathcal{X}$ of vertex sets $X \subseteq V(G)$ such that for each vertex $v \in V(G)$ there is a set $X \in \mathcal{X}$ with $N_r[v] \subseteq X$. We are interested in covers of small *radius*, that is, $\mathrm{rad}(G[X])$ shall be small for all $X \in \mathcal{X}$ and small *degree*, that is, every vertex $v \in V(G)$ shall lie in only a few clusters.

Sparse covers have many applications such as distance coordinates, routing with succinct routing tables [2, 9], mobile user tracking [9], resource allocation [7], synchronisation in distributed algorithms [8], and many more. Every graph admits an r-neighborhood cover of radius at most $2r - 1$ and degree at most $2r \cdot n^{1/r}$ [9] and asymptotically, this cannot be improved in general [64]. Better covers are known to exist, e.g., for planar graphs [15] and for classes that exclude a fixed minor [3]. In particular, the construction of [1] provides r-neighborhood covers of radius $\mathcal{O}(t^2 r)$ and degree $2^{\mathcal{O}(t)} \cdot t!$ for graphs that exclude the complete graph K_t as a minor. It follows from a construction in [29] that classes of bounded expansion admit r-neighborhood covers of radius at most $2r$ and degree at most $f(r)$ for some function f depending on the class under consideration.

Contribution 2. We show that the algorithm of [29] for constructing sparse r-neighborhood covers on classes of bounded expansion can be implemented in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model of distributed computing in $\mathcal{O}(r^2 \log n)$ communication rounds. Based on this construction, we show that our newly proposed algorithm for the DISTANCE-r DOMINATING SET problem can be implemented in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model in $\mathcal{O}(r^2 \log n)$ communication rounds on any class of graphs of bounded expansion. Our result is based on a routing scheme presented by Nešetřil and Ossona de Mendez in [51], which in turn is based on an iterative application of an algorithm of Barenboim and Elkin [11].

While in a sequential setting one can trivially connect the vertices of a (distance-r) dominating set along a spanning tree to obtain a connected (distance-r) dominating set of small size, creating such connections is a non-trivial task in the distributed setting. Several algorithms were proposed to compute connected dominating sets in general graphs [19, 20, 59, 62, 70]. We also refer to these papers for applications of connected dominating sets for distributed computing and routing. All lower bounds for the DOMINATING SET problem hold all the more so for the CONNECTED DOMINATING SET problem. In particular, none of the above algorithms computes a constant factor approximation of a minimum connected dominating set in a sub-linear number of communication rounds. To our knowledge, there is no distributed algorithm to compute a constant factor approximation to the CONNECTED (DISTANCE-r) DOMINATING SET problem on restricted graph classes.

Contribution 3. We show how to extend our algorithm for the DISTANCE-r DOMINATING SET problem to compute a constant factor approximation for the CONNECTED DISTANCE-r DOMINATING SET problem. We hence prove that there exists a constant factor approximation algorithm for the CONNECTED DISTANCE-r DOMINATING SET problem which works in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model in $\mathcal{O}(r^2 \log n)$ communication rounds on any class of graphs of bounded expansion.

Finally, we show how to use the greater power of the $\mathcal{LOCAL}$ model to turn any distance-r dominating set D into a connected distance-r dominating set of size at most $c(r) \cdot |D|$, for some small constant $c(r)$ depending only on r and the class under consideration. This new algorithm can be implemented in $3r + 1$ communication rounds in the $\mathcal{LOCAL}$ model. In combination with the algorithm of Lenzen et al. [40] we obtain a constant factor approximation algorithm for the CONNECTED DOMINATING SET problem on planar graphs in a constant number of communication rounds in the $\mathcal{LOCAL}$ model (the constant $c(1)$ which we need here is 6). A similar result follows for graphs of bounded genus by combining our new algorithm with an algorithm of [5].

2 Preliminaries

Graphs. In this paper, we consider *finite, undirected simple graphs*. For a graph G, we write $V(G)$ for the *vertex set* of G and $E(G)$ for its *edge set*. A *path of length* ℓ in G is a subgraph $P \subseteq G$ with vertex set $V(P) = \{v_1, \ldots, v_{\ell+1}\}$ and edge set $E(P) = \{\{v_i, v_{i+1}\} \ : \ 1 \leq i < \ell\}$. The path P *connects its endpoints* v_1 and $v_{\ell+1}$. The *distance* between two vertices $u, v \in V(G)$, denoted $\operatorname{dist}(u, v)$, is the minimum length of a path that connects u and v or ∞ if no such path exists. For $v \in V(G)$, we write $N_r[v]$ for the *closed* r*-neighborhood of* v, that is $N_r[v] = \{u \in V(G) \ : \ \operatorname{dist}(u, v) \leq r\}$. Note that we allow paths of length 0, so $N_r[v]$ always contains v itself. For a set $A \subseteq V(G)$, we write $N_r[A]$ for $\bigcup_{v \in A} N_r[v]$. The radius of a connected graph G is the minimum number $\operatorname{rad}(G)$ such that there is a vertex $v \in V(G)$ with $N_{\operatorname{rad}(G)}[v] = V(G)$.

An *orientation* of a graph G is a directed graph $\vec{G}$ on the same vertex set, which is denoted $V(\vec{G})$, such that for each edge $\{u, v\} \in E(G)$ the set of arcs $E(\vec{G})$ contains exactly one of the arcs (u, v) or (v, u). For $v \in V(\vec{G})$, $\{u : (u, v) \in E(\vec{G})\}$ is the set of the *in-neighbours* of v. The *indegree* $d^-(v)$ of a vertex v is the number in-neighbours of v. We denote the *maximum indegree* of $\vec{G}$ by $\Delta^-(\vec{G})$. For any directed graph $\vec{G}$ we denote the underlying undirected graph by G.

The *arboricity* of a graph is the minimum number of spanning forests that partition its edge set. The arboricity of a graph is within factor 2 of its degeneracy. For a set $X \subseteq V(G)$ we write $G[X]$ for the subgraph of G *induced* by X. For $k \in \mathbb{N}$, G is k*-degenerate* if for each $X \subseteq V(G)$ the graph $G[X]$ contains a vertex of degree at most k. If an n-vertex graph G is k-degenerate, then G contains at most $k \cdot n$ edges.

We assume that all graphs are represented by adjacency lists so that the total size of a graph representation is linear in the number of edges and vertices.

If G is k-degenerate, then an orientation $\vec{G}$ of G with $\Delta^-(G) \leq k$ can be computed in time $\mathcal{O}(k \cdot n)$ by a simple (sequential) greedy algorithm. Equivalently, we can order the vertices as $v_1, \ldots, v_n$ such that every vertex v_i has at most k smaller neighbours $v_{j_1}, \ldots, v_{j_k}$, $j_\ell < i$ for $\ell \in \{1, \ldots, k\}$. In the same time complexity we can order all adjacency lists consistently with the order. For the sequential model in Section 3 we assume that the identifiers of the vertices occupy constant space while in the distributed model we assume $\log n$-bit identifiers.

Distance -*r* dominating sets. For $r \in \mathbb{N}$, a *distance*-r *dominating set* in a graph G is a set $M \subseteq V(G)$ such that $N_r[M] = V(G)$. A distance-1 dominating set is simply called a *dominating set*.

Distributed system model. The clients of a network are modelled as the vertices $V(G)$ of a graph G, its communication links are represented by the edges $E(G)$ of the graph. Each client has a unique identifier *(id)* of size $\log n$ where $n := |V(G)|$ is the order of the graph known to every vertex. Communication is synchronous and reliable. In each round, each vertex $v \in V(G)$ may send a (different) message to each of its neighbors $w \in N_1[v]$ (the vertex specifies which message is sent to which neighbor) and receives all messages from its neighbors. In the $\mathcal{LOCAL}$ model, messages may have arbitrary size, in the $\mathcal{CONGEST}$ model, messages may have size $\mathcal{O}(\log n)$. In the $\mathcal{CONGEST}_{\mathrm{BC}}$ model, every vertex may only broadcast the same message of size $\mathcal{O}(\log n)$ to all its neighbors. After sending and receiving messages, every client may perform arbitrary finite computations. The complexity of a distributed algorithm is its number of communication rounds. The network graph also represents the graph problem that we are trying to solve, e.g., the DISTANCE-r DOMINATING SET instance. At termination, each vertex must output whether it is part of the distance-r dominating set or not, and these outputs must define a valid solution of the problem. Note that every distributed algorithm that runs in t rounds in We refer to [55] for more background.

Bounded expansion classes. A graph H with vertex set $V(H) = \{v_1, \dots, v_n\}$ is a *minor* of G, written $H \preccurlyeq G$, if there are pairwise disjoint connected subgraphs $H_1, \dots, H_n$ of G, called *branch sets*, such that if $\{v_i, v_j\} \in E(H)$, then there are $u_i \in V(H_i)$ and $u_j \in V(H_j)$ with $\{u_i, u_j\} \in E(G)$. We call $(H_1, \dots, H_n)$ a *minor model* of H in G. For $r \in \mathbb{N}$, the graph H is a *depth-r minor* of G, denoted $H \preccurlyeq_r G$, if there is a minor model $(H_1, \dots, H_n)$ of H in G such that each H_i has radius at most r. We write $d(H)$ for the *average degree* of H, that is, for the number $2|E(H)|/|V(H)|$. A class $\mathcal{C}$ of graphs has *bounded expansion* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$ and all graphs H and $G \in \mathcal{C}$, if $H \preccurlyeq_r G$, then $d(H) \leq f(r)$. The concept of bounded expansion as a model for uniform sparseness in graphs was introduced by Nešetřil and Ossona de Mendez in [47, 48, 49]. Note that many familiar sparse graph classes, such as planar graphs, bounded degree graphs, and graphs with excluded (topological) minors have bounded expansion. For an extensive study of classes with bounded expansion we refer the reader to [50]. Observe that every n-vertex graph from a bounded expansion class is $f(0)$-degenerate and hence has at most $f(0) \cdot n$ many edges (depth-0 minors of G are its subgraphs).

Generalized colouring numbers. Let G be a graph. A *linear order* L of $V(G)$ is a reflexive, anti-symmetric, transitive total binary relation $L \subseteq V(G) \times V(G)$. In the following, we will write $u \leq_L v$ instead of $(u, v) \in L$. We write $\Pi(G)$ for the set of all linear orders on $V(G)$. Let $r \in \mathbb{N}$ and let $u, v \in V(G)$. Vertex u is *weakly r-reachable* from vertex v with respect to a linear order $L \in \Pi(G)$ if there exists a path P of length at most r between u and v such that u is minimum among the vertices of P (with respect to L). Let $\mathrm{WReach}_r[G, L, v]$ be the set of vertices that are weakly r-reachable from v with respect to L. Note that $v \in \mathrm{WReach}_r[G, L, v]$. The *weak r-colouring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} |\mathrm{WReach}_r[G, L, v]|.$$

The generalized colouring numbers were introduced by Kierstead and Yang in the context of colouring games and marking games on graphs [36], and received much attention as a measure for uniform sparseness in graphs, in particular, they can be used to characterize classes of bounded expansion.

Theorem 1 (Zhu [71]). *A class $\mathcal{C}$ of graphs has bounded expansion if and only if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{wcol}_r(G) \leq f(r)$ for all $r \in \mathbb{N}$.*

Bounds for several restricted classes such as graphs of bounded tree-width, planar graphs or graphs with excluded (topological) minors, were provided in [28, 37, 66].

Transitive fraternal augmentations. Let G be an undirected graph. For $r \in \mathbb{N}$ and an undirected graph G, a *depth-r fraternal augmentation of G* is a directed graph $\vec{G}_r$ with edge set $E(\vec{G}_r)$ partitioned as $E_1 \cup \ldots \cup E_r$, such that

- the graph $(V(\vec{G}_r), E_1)$ is an orientation of G;
- $(u, v) \in E(\vec{G}_r)$ implies $(v, u) \notin E(\vec{G}_r)$ for all $u, v \in V(\vec{G}_r)$;
- for all $1 \leq i \leq j \leq r$ with $i + j \leq r$, and for all $u, v, w \in V(G)$ we have that $(u, w) \in E_i$ and $(v, w) \in E_j$ implies that (u, v) or (v, u) belongs to $\bigcup_{k=1}^{i+j} E_k$.

For a directed graph $\vec{G}$, a *depth-r transitive augmentation* of $\vec{G}$ is a graph $\vec{G}_r^t$ obtained from $\vec{G}$ by adding all edges (u, v) such that there is a directed path of length at most r between u and v in $\vec{G}$. A *depth-r transitive fraternal augmentation* of $\vec{G}$ is a depth-r transitive augmentation of a depth-r fraternal augmentation of $\vec{G}$.

Transitive fraternal augmentations were introduced in [47][1]. We refer also to Section 7.4 of the textbook [50] for more background. We use low in-degree transitive fraternal augmentations to approximate the weak coloring numbers. This approximation is based on the following lemma, which is a simple variant of Lemma 5.7 of [29].

Lemma 2. *Let G be a graph and let $r \in \mathbb{N}$. Let $\vec{G}_r^t$ be a depth-r transitive fraternal augmentation of G with $\Delta^-(\vec{G}_r^t) \leq d$. Let G_r^t be the underlying undirected graph of $\vec{G}_r^t$. Let L be an order of $V(G_r^t)$ such that every vertex has at most c smaller neighbors with respect to L. Then $|\mathrm{WReach}_r[G, L, v]| \leq (d+1)c + 1$ for all $v \in V(G)$.*

PROOF. Use Lemma 7.9 of [50] to show that if $v_1 \ldots v_k$ is a path of length at most r in G, then either $v_1 = v_k$ or $(v_1, v_k) \in E(\vec{G}_r^t)$ or $(v_k, v_1) \in E(\vec{G}_r^t)$ or there is some v_i, $1 < i < k$ such that $(v_i, v_1), (v_i, v_k) \in E(\vec{G}_r^t)$.

For each vertex $v \in V(G)$ we count the number of end-vertices of paths of length at most r from v such that the end-vertex is the smallest vertex of the path. This number is exactly $|\mathrm{WReach}_r[G, L, v)]|$.

By our above observation, for each such path with end-vertex $w \neq v$, we either have an edge (v, w) or an edge (w, v) or there is u on the path and we have edges (u, v), (u, w) in $E(\vec{G}_r^t)$. By assumption on L there are at most c edges (v, w) or (w, v) such that $w <_L v$. Furthermore, we have at most d edges (u, v), as v has in-degree at most d and for each such u there are at most c edges (u, w) such that $w <_L u$ by assumption on L. These are exactly the pairs of edges we have to consider, as no vertex on the path from v to w may be smaller than w. Hence in total we have $|\mathrm{WReach}_r[G, L, v]| \leq c + d \cdot c + 1 = (d+1)c + 1$. □

As shown in [47], for every class $\mathcal{C}$ of bounded expansion there is a function $d : \mathbb{N} \to \mathbb{N}$ such that we can compute in linear time a depth-r fraternal augmentation $\vec{G}$ of G and a depth-r transitive augmentation $\vec{G}_r^t$ of $\vec{G}_r$ such that $\Delta^-(\vec{G}_r^t) \leq d(r)$. Note that if $\Delta^-(\vec{G}_r^t) \leq d$, then the underlying undirected graph G_r^t is $2d$-degenerate and hence we can compute an order such that every vertex has at most $2d$ smaller neighbors in linear time.

Corollary 3. *Let $\mathcal{C}$ be a class of bounded expansion. There is a linear time algorithm and a function $d : \mathbb{N} \to \mathbb{N}$ which on input $G \in \mathcal{C}$ and $r \in \mathbb{N}$ computes in linear time a linear order of $V(G)$ witnessing that $\mathrm{wcol}_r(G) \leq d(r)$.*

Another possibility to approximate the weak coloring numbers in linear time is presented in [22]. We remark that implicitly in that algorithm one also has to compute depth-r transitive fraternal augmentations.

The next theorem shows that we can compute transitive fraternal augmentations, and in fact also the weak colouring numbers, in the distributed setting in logarithmically many rounds on graphs of bounded expansion.

Theorem 4 (Nešetřil and Ossona de Mendez in [51]). *Let $\mathcal{C}$ be a class of bounded expansion and let $r \in \mathbb{N}$. There are constants $f(r)$ and $F(r)$ such that one can compute for every $G \in \mathcal{C}$ in $\mathcal{O}(r^2 \log n)$ communication rounds in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model a depth-r transitive fraternal augmentation $\vec{G}_r^t$ of G with $\Delta^-(\vec{G}_r^t) \leq f(r)$ and an order L of $V(G_r^t)$ such that every vertex has at most $F(r)$ smaller neighbors in G_r^t with respect to L.*

[1] Note that in [47] transitive fraternal augmentations were defined by taking a depth-1 fraternal augmentation and then a depth-1 transitive augmentation etc. for r times. Here, as in [51], the definition is changed to work in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model

As the graph topology cannot be changed in the distributed setting, the algorithm simulates the introduced fraternal edges by storing for each vertex v and each in-neighbor w in E_i a path of length i such that routing along the paths is possible with small congestion. (Congestion is the maximum number of paths containing the same vertex.) Routing along transitive edges is realized via a broadcasting protocol. [2]

The procedure described in Theorem 4 uses an algorithm by Barenboim and Elkin [11] which computes an orientation of degenerate graphs. For this, we must assume that all vertices know the order n of the input graph. The order is represented by assigning every vertex a *class-id*, which together with the unique vertex-id induces a total order of $V(G)$. We remark that (though not explicitly stated) this order can be obtained as a by-product of the procedure $\mathrm{orient}(z, C)$ described in [51, Section 4.4].

Corollary 5. *Let $\mathcal{C}$ be a class of bounded expansion and let $r \in \mathbb{N}$. There is a constant $d(r)$ such that one can compute for every $G \in \mathcal{C}$ in $\mathcal{O}(r^2 \log n)$ communication rounds in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model an order L of $V(G)$ witnessing that $\mathrm{wcol}_r(G) \leq d(r)$.*

Sparse neighborhood covers. Let G be a graph. For $r \in \mathbb{N}$, an *r-neighborhood cover* of G is a set $\mathcal{X}$ of subsets $X \subseteq V(G)$, called the *clusters* of $\mathcal{X}$, such that for each $v \in V(G)$ there is some $X \in \mathcal{X}$ with $N_r(v) \subseteq X$. The *radius* of $\mathcal{X}$ is the maximum radius of the graph induced by a cluster $X \in \mathcal{X}$. Note that in every r-neighborhood cover of bounded radius each of the clusters induces a connected subgraph of G. The *degree* $d_{\mathcal{X}}(v)$ of a vertex $v \in V(G)$ with respect to $\mathcal{X}$ is the number of clusters that contain v. The degree of $\mathcal{X}$ is the maximum degree $d_{\mathcal{X}}(v)$ over all vertices $v \in V(G)$. The generalized colouring numbers can be used to construct sparse neighborhood covers.

We fix a number $r \in \mathbb{N}$ for the remainder of the paper. For $v \in V(G)$, let X_v be the set of the vertices w such that $v \in \mathrm{WReach}_{2r}[G, L, w]$.

Theorem 6 (Grohe et al. [29]). *Let G be a graph and let $c, r \in \mathbb{N}$. Let L be an order witnessing that $\mathrm{wcol}_{2r}(G) \leq c$. Then the collection $\mathcal{X} = \{X_v : v \in V(G)\}$ is an r-neighborhood cover of G of radius $2r$ and degree c.*

Hence, by combining Theorem 1 and Theorem 6, we obtain r-neighborhood covers of radius at most r and degree at most $f(r)$ for every class of bounded expansion. Classes that exclude a fixed minor are the most general classes known to admit r-neighborhood covers of constant degree (independent of r). Abraham et al. [3] have shown that if G excludes the complete graph K_t as a minor, then there exists an r-neighborhood cover of radius $\mathcal{O}(t^2 r)$ such that each vertex belongs to at most $2^{\mathcal{O}(t)} t!$ clusters. The currently best known bounds for $\mathrm{wcol}_r(G)$ if $K_t \not\preccurlyeq G$ are from [66] and show that in this case $\mathrm{wcol}_r(G) \in \mathcal{O}(r^{t-1})$. Hence, using Theorem 6, we obtain r-neighborhood covers of radius $2r$ and degree $\mathcal{O}((2r)^{t-1})$ for these graphs. Note that the algorithms of [3, 66] to compute the neighborhood covers are inherently sequential. We will show in Section 4 how to compute the r-neighborhood covers presented in Theorem 6 in a distributed setting.

3 Approximating distance-r dominating sets

Our first result shows how to (sequentially) compute good distance-r dominating sets in any fixed class of bounded expansion. Our result improves the following result of Dvořák [22].

[2] We remark that in [51] one actually computes 2^r fraternal augmentations and r transitive augmentations, to finally compute an r-centered coloring, which is not necessary in our case.

Theorem 7 (Dvořák [22]). *For every class $\mathcal{C}$ of bounded expansion there is a function $c\colon \mathbb{N} \to \mathbb{N}$ and a linear time algorithm which on input $G \in \mathcal{C}$ and $r \in \mathbb{N}$ computes an order $L \in \Pi(G)$ witnessing $\mathrm{wcol}_{2r}(G) \le c(r)$ and a $c(r)^2$-approximation of a minimum distance-r dominating set of G.*

The remainder of this section is devoted to the proof of the following theorem.

Theorem 8. *For every class $\mathcal{C}$ of bounded expansion there is a function $c\colon \mathbb{N} \to \mathbb{N}$ and a linear time algorithm which on input $G \in \mathcal{C}$ and $r \in \mathbb{N}$ computes an order $L \in \Pi(G)$ witnessing $\mathrm{wcol}_{2r}(G) \le c(r)$ and a $c(r)$-approximation of a minimum distance-r dominating set of G.*

The constant $c(r)$ in Theorem 8 is the same as in Theorem 7, hence we improve the currently best known approximation constant from c^2 to c.

Recall the definition of X_v (see Section 2):

$$X_v = \{w \in V(G) \;:\; v \in \mathrm{WReach}_{2r}[G, L, w]\}\,. \tag{1}$$

We define for a fixed vertex $v \in V(G)$,

$$R_v := \{w \in X_v \;:\; v = \min \mathrm{WReach}_r[G, L, w]\}\,.$$

Lemma 9. *For all $v \in V(G)$ and for all $w \in R_v$ we have $N_r[w] \subseteq X_v$.*

PROOF. Let $w \in R_v$. Observe first that v is the minimum element in $N_r[w]$. Now let $u \in N_r[w]$. Then there is a path of length at most r from u to w with all vertices in $N_r[w]$ and a path of length at most r from w to v again with all vertices in $N_r[w]$. Hence there is a path of length at most $2r$ from u to v with all vertices in $N_r[w]$. As v is minimal in $N_r[w]$, this path witnesses that $v \in \mathrm{WReach}_{2r}[G, L, u]$, and hence by definition of X_v it holds that $u \in X_v$. □

PROOF. (OF THEOREM 8) We claim that the set

$$D := \{\min \mathrm{WReach}_r[G, L, w] \;:\; w \in V(G)\} = \{v \in V(G) \;:\; R_v \neq \emptyset\} \tag{2}$$

is a c-approximation of a minimum distance-r dominating set. Obviously, D is a distance-r dominating set of $V(G)$, as every vertex w is dominated by $\min \mathrm{WReach}_r[G, L, w]$. It remains to show that we achieve the claimed approximation ratio.

For $v \in V(G)$, let X_v and R_v be as above. Let $\mathcal{X}$ be the collection $\{X_v \;:\; v \in V(G)\}$ as in Theorem 6. Then $\mathcal{X}$ is an r-neighborhood cover of degree c and by Lemma 9, for $w \in R_v$, we have $N_r[w] \subseteq X_v$.

Let M be a minimum distance-r dominating set in G. As every $w \in V(G)$ can be distance-r dominated only from $N_r[w] \subseteq X_v$, it follows that if $w \in R_v$, then $M \cap X_v \neq \emptyset$.

Hence, as every vertex appears in at most c clusters, it holds that

$$|D| = |\bigcup_{\substack{v \in V(G)\\ R_v \neq \emptyset}} \{v\}| \le \sum_{\substack{v \in V(G)\\ R_v \neq \emptyset}} |M \cap X_v| \le c \cdot |M|.$$

We finally show how to compute D in linear time. We assume that G is stored in memory by n adjacency lists. We first use the linear time approximation algorithm for the weak coloring numbers from Corollary 3 to compute an order L, which will be represented in a way such that one can iterate through the vertices along L in $\mathcal{O}(|V(G)|)$ time and such that a comparison $u <_L w$ for every

Algorithm 1 DomSet(G, L)

Input: Graph $V(G)$; $V(G) = \{v_i : 1 \leq i \leq n\}$; $v_i <_L v_j$ for $i < j$; $A(v_i)$ is the adjacency list of v_i

Output: A c-approximation of a dominating set of G

1: SortLists(L) ▷ Algorithm 2
2: $D \leftarrow \emptyset$
3: $\mathit{Dominated} \leftarrow \emptyset$
4: **for** $i \leftarrow 1, \ldots, n$ **do**
5: $\quad N_i \leftarrow$BFS(v_i, L) ▷ Algorithm 3
6: $\quad$**if** $N_i \setminus \mathit{Dominated} \neq \emptyset$ **then**
7: $\quad\quad D \leftarrow D \cup \{v_i\}$
8: $\quad\quad \mathit{Dominated} \leftarrow \mathit{Dominated} \cup N_i$
9: **return** D

pair $u, w \in V(G)$ takes $\mathcal{O}(1)$ time. Here we assume that vertices are equipped with constant size identifiers representing the order. The algorithm is given as Algorithm 1.

In the first step (Line 1 of Algorithm 1 and Algorithm 2), we ensure in linear time that every adjacency list is sorted increasingly with respect to L. Algorithm 2 iterates through the vertices of G in order L starting from the least vertex (such that the next vertex can always be found in constant time) and thus has running time $\mathcal{O}(m)$. As graphs of bounded expansion are degenerate, we have $m \in \mathcal{O}(n)$.

Algorithm 2 SortLists(L)

Input: Graph $V(G)$; $V(G) = \{v_i : 1 \leq i \leq n\}$; $v_i <_L v_j$ for $i < j$; $A(v_i)$ is the adjacency list of v_i

Output: $A(v_i)$ is increasingly sorted with respect to L

for $i \leftarrow 1, \ldots, n$ **do** $B(v_i) \leftarrow A(v_i)$ and $A(v_i) \leftarrow ()$
for $i \leftarrow 1, \ldots, n$ **do**
$\quad$**for** $v_j \in B(v_i)$ **do**
$\quad\quad$add v_i at the end of $A(v_j)$

Now Algorithm 1 iterates through the vertices of G starting with the least element v_1 along L. For every $v_i \in V(G)$, it uses Algorithm 3 to compute the set of vertices that are bigger than v_i and are dominated by v_i. If such a vertex is not dominated by a vertex smaller than v_i, it serves as a vertex w in the definition of D (see (2)). Indeed, $v_i \in \mathrm{WReach}_r[G, L, w]$ because w was found by a breadth-first search from v restricted to vertices greater than v with respect to L and to distances at most r. On the other hand, if not v was the minimum vertex in $\mathrm{WReach}_r[G, L, w]$ but, say, u, then w would be dominated by u and added to the set $\mathit{Dominated}$ in the earlier iteration j for $u = v_j$. Thus $v_i = \min \mathrm{WReach}_r[G, L, w]$ and v_i is added to D.

Let us estimate the running time of Algorithm 1. Recall that Line 1 has linear running time. Note that every set N_i computed in Line 5 for a vertex v_i is a subset of X_{v_i} because Algorithm 3 restricts its search to vertices bigger than v_i and to distances at most r. That is, if $w \in N_{v_i}$, then $v_i \in \mathrm{WReach}_r[G, L, w] \subseteq \mathrm{WReach}_{2r}[G, L, w]$ and thus $w \in X_{v_i}$. As every graph $G \in \mathcal{C}$ is c-degenerate, every induced subgraph $H \subseteq G$ has at most $c \cdot |V(H)|$ many edges, also the graph induced by N_i. When constructing N_i in Algorithm 3 we will only visit vertices of N_i and, for every $w \in N_i$ at most one vertex in its adjacency list that is not in N_i. This can be achieved if Line 6 of Algorithm 3 is implemented as an iteration through $A(w)$ starting from the biggest vertex and stopping if a vertex $u \in A(w)$ with $u <_L w$ is reached (recall that $A(w)$ sorted). Hence this search requires time at most $\mathcal{O}((c + 1) \cdot |N_{v_i}|) = \mathcal{O}(c \cdot |X_{v_i}|)$. As every vertex w appears in at most c

Algorithm 3 $\mathrm{BFS}(v, L, r)$

▷ As usual BFS, but restricted to vertices $w >_L v$ and only for r steps.

Input: Graph G; for $w \in V(G)$, the adjacency list $A(w)$ is increasingly sorted w.r.t. to L

```
Q ← empty queue
Enqueue (v, 0) in Q
while Q ≠ ∅ do
    (w, dist) ← dequeue from Q
    if dist < r then
        for u ∈ A(w) and u >_L v do
            if u not marked as visited then
                mark u as visited
                enqueue (u, dist + 1) in Q
return marked vertices
```
□

clusters X_v, we obtain a running time of $\sum_{v \in V(G)} \mathcal{O}(c \cdot |X_v|) = \mathcal{O}(c^2 \cdot n)$.

Note that by Theorem 1 the constant $c(r)$ in the theorem exists for every class of bounded expansion. Besides the improved approximation ratio, our algorithm is simpler than that of [22]. In particular, given an order $L \in \Pi(G)$, it can straightforwardly be implemented in a distributed way.

Remark 10. Algorithm 3 with parameters v, L and $2r$ computes X_v in time $\mathcal{O}(c \cdot |X_v|)$.

4 Distributed *r*-neighborhood covers and *r*-dominating sets

In this section we will show how to compute sparse r-neighborhood covers as described in Theorem 6 and the DISTANCE-r DOMINATING SET of Theorem 8 in a distributed setting.

In order to compute r-neighborhood covers according to Theorem 6, we want to compute an order L of $V(G)$ which witnesses that $\mathrm{wcol}_{2r}(G) \leq c$. In the distributed setting, that means that every vertex w learns its weak reachability set $\mathrm{WReach}_{2r}[G, L, w]$ and, for each $v \in \mathrm{WReach}_{2r}[G, L, w]$, a path within X_v of length at most $2r$ from w to v.

In order to find the distance-r dominating set described in Theorem 8, every vertex w will choose $\min \mathrm{WReach}_r[G, L, w]$ as its dominator and send a message to that vertex along the stored path. (Note that we computed the order L for the parameter $2r$, but are using it for r.) Even if all vertices send their messages at once, no vertex will have to forward more than c messages.

First, using Corollary 5, we compute for a given graph G an order L witnessing that $\mathrm{wcol}_{2r}(G) \leq c(2r)$ in $\mathcal{O}(r^2 \log n)$ communication rounds. We show that every vertex can learn its weak reachability set as well as a routing scheme which preserves short distances. Recall from (1) on Page 9 that X_v is defined as $X_v = \{w \in V(G) : v \in \mathrm{WReach}_{2r}[G, L, w]\}$. The proof can be found in .

Lemma 11. *Let $\mathcal{C}$ be a class of bounded expansion and let $r \in \mathbb{N}$. There is a constant $c = c(2r)$ such that for every $G \in \mathcal{C}$ there is a linear order L on $V(G)$ such that $|\mathrm{WReach}_{2r}[G, L, w]| \leq c$ for all $w \in V(G)$ and in $\mathcal{O}(r^2 \cdot \log n)$ communication rounds every vertex w can learn $\mathrm{WReach}_{2r}[G, L, w]$ and for each $v \in \mathrm{WReach}_{2r}[G, L, w]$ a path $P_{v,w}$ of length at most $2r$ from w to v, which is a shortest path between v and w in the graph induced by X_v. In particular, if $v = \min \mathrm{WReach}_r[G, L, w]$, then the path $P_{v,w}$ is a shortest path between v and w in G.*

PROOF. **Lemma 11.** *Let $\mathcal{C}$ be a class of bounded expansion and let $r \in \mathbb{N}$. There is a constant $c = c(2r)$ such that for every $G \in \mathcal{C}$ there is a linear order L on $V(G)$ such that $|\mathrm{WReach}_{2r}[G, L, w]| \leq c$ for all $w \in V(G)$ and in $\mathcal{O}(r^2 \cdot \log n)$ communication rounds (in $\mathcal{CONGEST}_{\mathrm{BC}}$) every vertex w can learn $\mathrm{WReach}_{2r}[G, L, w]$ and for each $v \in \mathrm{WReach}_{2r}[G, L, w]$ a path $P_{v,w}$ of length at most $2r$ from w to v, which is a shortest path between v and w in the graph induced by X_v. In particular, if $v = \min \mathrm{WReach}_r[G, L, w]$, then the path $P_{v,w}$ is a shortest path between v and w in G.*

PROOF. The pseudocode is given in Algorithm 4. First, using Corollary 5, we compute for a given graph G an order L witnessing that $\mathrm{wcol}_{2r}(G) \leq c(2r)$ in $\mathcal{O}(r^2 \log n)$ communication rounds.

The procedure implicitly uses an algorithm of Barenboim and Elkin [11], which assigns to each vertex v a *class id* $cl(v)$, which together with the unique vertex identifier induces the linear order L. For ease of presentation, we write v_i for the vertex at position i in the order L and call i the *super-id* of the vertex.

Algorithm 4 $\mathrm{WReachDist}(r)$

Input (for a vertex w): $n = |V(G)|$, adjacency list of w, id of w
Output: $\mathrm{WReach}_{2r}[G, L, w]$, $\min \mathrm{WReach}_r[G, L, w]$ for a particular linear order L (see text)

```
in parallel, compute L                                              ▷ by Corollary 5
                                      ▷ when done, every vertex w knows its super-id sid(w)
in parallel, P_w = {{sid(w)}}
for i = 1, ..., r do
    in parallel, broadcast P_w, receive new paths in P
    in parallel (for vertex w):
    toSend ← ∅
    for u_1 first vertex in a path from P do
        if sid(u_1) < sid(w) then
            P ← shortest path from P ∪ P_w that starts in u_1, break ties using super-ids
        if exists P' ∈ P_w with P' = u_1, ... then
            remove P' from P_w
        P_w ← P_w ∪ {P}
        toSend ← toSend ∪ {u_1, ..., u_j, w}                     ▷ where P = u_1, ..., u_j.
    broadcast the set toSend
```

The remaining part of the computation has $2r$ rounds which correspond to $2r$ rounds of a breadth-first search as in Algorithm 3. This time, the search is performed in parallel and we have to make sure that only a logarithmic amount of information is sent by every vertex for the $\mathcal{CONGEST}_{\mathrm{BC}}$ model. The idea is that every vertex w forwards only information about paths that start in a vertex $v \in \mathrm{WReach}_{2r}[G, L, w]$.

Every vertex w maintains a set $\mathcal{P}_w$ of paths of length at most $2r$ from vertices $v \in \mathrm{WReach}_{2r}[G, L, w]$. For every vertex $v \neq w$ there is at most one path P_v in $\mathcal{P}_w$ that starts in v and certifies that $v \in \mathrm{WReach}_{2r}[G, L, w]$. In the first round, every vertex broadcasts its super-id, which we understand as a path of length 0. A vertex w receives super-ids and stores only those which are smaller than its own super-id.

In a later iteration, every vertex w receives some sets of paths from its neighbors and computes their union $\mathcal{P}$. For every vertex u_1, with a super-id greater than the super-id of w, all paths from $\mathcal{P}$ starting in u_1 are discarded. For every vertex u_1 with a smaller super-id, vertex w selects the shortest path starting in u_1 among all paths in $\mathcal{P}$ and $\mathcal{P}_w$. (There is at most one path in $\mathcal{P}_w$ that starts in u_1.) If there are many such shortest paths, w chooses the lexicographically least one (with respect to the super-ids).

Let this path be $P = u_1, \ldots, u_j$ for some $j \leq 2r$. Then P is stored in $\mathcal{P}_w$ (if there is already a path in $\mathcal{P}_w$ that starts in u_1, it is replaced by P). If $j < 2r$, then w broadcasts the path $u_1, \ldots, u_j, w$.

Observe that every vertex w forwards information about a vertex v only if $v \in \mathrm{WReach}_{2r}[G, L, w]$. Hence, w forwards only at most c paths simultaneously and the whole procedure works in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model. Observe also that we perform a breadth-first search through the cluster X_v and break ties according to the order by vertex super-ids. This implies our claims on shortest paths. $\square$

We can now combine Theorem 6 and Lemma 11 to obtain the first main theorem of this section.

Theorem 12. *Let $\mathcal{C}$ be a class of bounded expansion. There is a distributed algorithm which for every graph $G \in \mathcal{C}$ and every $r \in \mathbb{N}$ computes a representation of a sparse r-neighborhood cover in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model in $\mathcal{O}(r^2 \cdot \log n)$ communication rounds. More precisely, the algorithm computes an order L, represented by $\log n$-sized labels and for every vertex v a routing scheme of length at most $2r$ to every vertex in $\mathrm{WReach}_{2r}[G, L, v]$.*

Also Theorem 8 can now be implemented as a distributed algorithm.

Theorem 13. *Let $\mathcal{C}$ be a class of bounded expansion and let $r \in \mathbb{N}$. There is a constant c and a distributed algorithm which for every graph $G \in \mathcal{C}$ computes a c-approximation of a minimum distance-r dominating set in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model in $\mathcal{O}(r^2 \cdot \log n)$ communication rounds.*

PROOF. Recall that we want to compute the distance-r dominating set

$$D := \{v \in V(G) \ : \ v = \min \mathrm{WReach}_r[G, L, w] \text{ for some } w \in V(G)\},$$

that is, every vertex w elects the smallest vertex from its r-neighborhood with respect to L to the distance-r dominating set. As w knows $\mathrm{WReach}_r[G, L, w]$ and a routing scheme to these vertices, all vertices can send to the smallest vertex in the list a short message that it should be included in the dominating set. Observe that if a vertex u has to forward the identifier of a vertex $w \in \mathrm{WReach}_r[G, L, v]$ from some other vertex v, then also $w \in \mathrm{WReach}_r[G, L, u]$. Hence, no vertex has to forward more than c messages of total size at most $\mathcal{O}(c \cdot \log n)$. $\square$

5 Connected Dominating Sets

In this section we study the CONNECTED DISTANCE-r DOMINATING SET problem. Our main result in this section is the following theorem.

Theorem 14. *Let $\mathcal{C}$ be a class of bounded expansion and let $r \in \mathbb{N}$. There is a constant c and a distributed algorithm which for every graph $G \in \mathcal{C}$ computes a c-approximation of a minimum connected distance-r dominating set in the $\mathcal{CONGEST}_{\mathrm{BC}}$ model in $\mathcal{O}(r^2 \cdot \log n)$ communication rounds.*

The following observation is folklore, we provide a proof for completeness.

Lemma 15. *Let G be a connected graph and let D be a distance-r dominating set of G. Let $\mathcal{P}$ be a set of paths in G such that for each pair $u, v \in D$ with $\mathrm{dist}(u, v) \leq 2r + 1$ there is a path $P_{u,v} \in \mathcal{P}$ connecting u and v. Then the subgraph H induced by $D \cup \bigcup_{P \in \mathcal{P}} V(P)$ is connected.*

PROOF. We show by induction on $\mathrm{dist}(u, v)$ that all $u, v \in D$ are connected in H. The claim holds by definition of H if $\mathrm{dist}(u, v) \leq 2r + 1$. Now assume that $\mathrm{dist}(u, v) \geq 2r + 2$ and let $P = (u = v_0, v_1, \ldots, v_t = v)$ with $t \geq 2r + 3$ be a shortest path connecting u and v. As P is a shortest path, neither u nor v dominate v_{r+1}. Hence there is another vertex $w \in D$ which dominates v_{r+1}. As v_1 and v_{r+1} are connected by a path of length r, and w and v_{r+1} are connected by a path $(w = w_0, w_1, \ldots, w_{r'} = v_{r+1})$ of length $r' \leq r$, $\mathrm{dist}(u, w) \leq 2r$, hence w and u are connected in H. Furthermore, the path $P' = (w, w_1, \ldots, w_{r'-1}, v_{r+1}, \ldots, v_t = v)$ is shorter than P, hence, by induction hypothesis, w and v are connected in H. This implies that u, v are connected in H. $\square$

Now, we use the local separation properties of the weak colouring numbers to connect the dominating set we computed in Theorem 13. The proof of the following lemma is immediate by definition of weak reachability.

Lemma 16. *Let G be a graph and let L be a linear order on $V(G)$. Let $u, v \in V(G)$ be such that there exists a path P between u and v of length at most r. Let w be the minimal vertex of P with respect to L. Then $w \in \mathrm{WReach}_r[G, L, u]$ and $w \in \mathrm{WReach}_r[G, L, v]$.*

Corollary 17. *Let G be a connected graph and let L be a linear order on $V(G)$. Let D be an r-dominating set of G. Let D' be a set which is obtained by adding for each $v \in D$ and each $w \in \mathrm{WReach}_{2r+1}[G, L, v]$ the vertex set of a path between v and w. Then D' is a connected distance-r dominating set of G.*

PROOF. Fix a set $\mathcal{P}$ of paths in G such that for each pair $u, v \in D$ with $\mathrm{dist}(u, v) \leq 2r + 1$ there is a path $P_{u,v} \in \mathcal{P}$ connecting u and v. According to Lemma 15, the subgraph H induced by $D \cup \bigcup_{P \in \mathcal{P}} V(P)$ is connected. According to Lemma 16, for each path $P_{u,v}$ between u and v, there is a vertex $w \in V(P_{u,v})$ is weakly $2r + 1$-reachable both from u and from v. As D' contains the vertex set of a path between u and w and of a path between v and w, it follows that D' is a connected distance-r dominating set of G. $\square$

We are now ready to prove the main theorem.

PROOF. (OF THEOREM 14) Instead of computing an order L for $\mathrm{wcol}_{2r}(G)$ as in Theorem 13, we compute an order L for $\mathrm{wcol}_{2r+1}(G)$. Assume $|\mathrm{WReach}_{2r+1}[G, L, v]| \leq c'$ for all $v \in V(G)$. We compute an r-dominating set D based on the order L. Note that in Section 4 we used L computed for parameter $2r$ and now we use L computed for $2r + 1$, but for all orders L and all $v \in V(G)$ we have $|\mathrm{WReach}_{2r}[G, L, v]| \leq |\mathrm{WReach}_{2r+1}[G, L, v]|$.

By Theorem 8, D is at most c' times larger than a minimum distance-r dominating set. As a by-product, see Lemma 11, every vertex v learns a path of length at most $2r + 1$ to each $w \in \mathrm{WReach}_{2r+1}[G, L, v]$. Now, every vertex broadcasts its set of paths to construct the set D'. As in the proof of Theorem 13, observe that if a vertex x has to forward a path from $w \in \mathrm{WReach}_r[G, L, v]$ to v for some other vertex v, then also $w \in \mathrm{WReach}_r[G, L, x]$. Hence, no vertex has to forward more than c' messages of total size at most $\mathcal{O}(c' \cdot r \cdot \log n)$. Clearly, the computed set D' has size at most $c' \cdot (2r + 1) \cdot |D|$ and by Corollary 17 it is a connected distance-r dominating set. We conclude be defining $c := c'^2 \cdot (2r + 1)$. $\square$

We now show how to use the greater power of the $\mathcal{LOCAL}$ model to compute connected dominating sets with much smaller constants involved. Our theorem is based on the simple observation that in the $\mathcal{LOCAL}$ model we can construct for every connected graph from an r-dominating set D a connected depth-r minor with $|D|$ vertices. This minor (by definition of bounded expansion classes) has only a linear number of edges and we can hence choose a set of short paths realizing the corresponding connections to connect the dominating set.

We want to define a partition of $V(G)$ into balls around vertices from an r-dominating set D. For a connected graph G and an injection $id\colon V(G) \to \mathbb{N}$, we define the *lexicographic order* $<_{\mathrm{lex}}$

on the set of paths in $V(G)$ with respect to id as follows. Consider two paths $P_1 = v_1, \ldots, v_k$ and $P_2 = w_1, \ldots, w_\ell$. If $k < \ell$, then $P \leq_{\text{lex}} P_2$. If $k = \ell$, then $P_1 \leq_{\text{lex}} P_2$ if the sequence $id(v_1), \ldots, id(v_k)$ is lexicographically smaller than the sequence $id(w_1), \ldots, id(w_\ell)$ or $P_1 = P_2$. For vertices $v, w \in V(G)$, let $P(v, w)$ be the lexicographically shortest path from v to w.

Let G be a connected graph, let $id(v)$ be the unique identifier of v and let D be a distance-r dominating set of G. For each $v \in D$ let

$$B(v) := \{w \in V(G) \ : \ P(v, w) \leq_{\text{lex}} P(u, w) \text{ for all } u \in D, u \neq v\}\,.$$

The *D-partition* $\mathcal{B}(D)$ of G with respect to id is the set $\{B(v) \ : \ v \in D\}$.

The proofs of the following two lemmas can be found in **??**.

Lemma 18. *Let G be a connected graph and let D be a distance-r dominating set of G. Then $\mathcal{B}(D) = \{B(v) : v \in D\}$ is a partition of $V(G)$ and $G[B(v)]$ has radius at most r for all $v \in D$.*

PROOF. As G is connected and D is a distance-r dominating set, $\mathcal{B}$ is a partition of $V(G)$. Furthermore, for each $w \in V(v)$, there is a lexicographically shortest path P of length at most r from v to w in G. Assume towards a contradiction that P is not also a path in $B(v)$. Then there is $z \in V(P)$ and $u \in D$ such that $z \in B(u)$. By definition of $B(u)$, the lexicographically shortest path Q' from u to z is smaller than the lexicographically shortest path Q between v and z. But then the path P' obtained by replacing the initial part Q of P by Q' is lexicographically smaller than P, a contradiction. □

Lemma 19. *Let G be a connected graph and let D be a distance-r dominating set of G. By contracting the sets $B(v)$ for $v \in D$, we obtain a connected depth-r minor of G.*

PROOF. It is immediate by definition of depth-r minors and Lemma 18 that we construct a depth-r minor $H \preccurlyeq_r G$. Furthermore, as G is connected and as $\mathcal{B}(D)$ is a partition of $V(G)$ by the same lemma, it is easy to see that H is connected. □

Lemma 20. *Let G be a connected graph such that for each depth-r minor $H \preccurlyeq_r G$ we have $|E(H)| \leq d \cdot |V(H)|$. Let D be a distance-r dominating set of G. We can compute a connected dominating set D' of size at most $2r \cdot d \cdot |D|$ in $3r + 1$ communication rounds in the $\mathcal{LOCAL}$ model.*

PROOF. In this proof, we write $H(D)$ for the depth-r minor constructed from a distance-r dominating set D as in Lemma 19.

Every vertex $v \in D$ can find its $2r + 1$-neighborhood in $2r + 1$ communication rounds. With this information, each $v \in D$ can construct $B(v)$, as all possible dominators for $w \in N_r[v]$ must come from $N_{2r}[v]$. Each vertex $v \in D$ (now understood as representing a vertex of $H(D)$) can also learn its neighbors in $H(D)$ (here we need to learn $N_{2r+1}[v]$). Now each vertex v computes the lexicographically shortest path P_{uv} of length at most $2r + 1$ for each neighbor u in $H(D)$ (take the ordering induced by vertex id's). Observe that u and v fix the same path P_{vu}, hence, the two vertices can report to all vertices on P_{vu} in another r communication rounds that they shall be included in the connected dominating set D'.

By Lemma 19, the constructed set D' is a connected distance-r dominating set. Furthermore, by assumption, $H(D)$ has at most $d \cdot |D|$ many edges. Each edge is replaced by at most $2r - 1$ vertices in the above construction. Adding the $|D|$ vertices of the original set D, we obtain the claimed bounds. □

As a corollary from Lemma 18 and Lemma 20 we obtain the following theorem.

Theorem 21. *Let $\mathcal{C}$ be a class of graphs of bounded expansion and assume that for every graph $G \in \mathcal{C}$ we can compute a c-approximation D of a minimum distance-r dominating set of G in t rounds in the $\mathcal{LOCAL}$ model. Let $f : \mathbb{N} \to \mathbb{N}$ denote the edge density function of depth-r minors of $\mathcal{C}$. Then there is a distributed algorithm which finds a $2rcf(r)$-approximation for connected distance-r dominating set of G in $\mathcal{O}(t + r)$ rounds in the $\mathcal{LOCAL}$ model.*

The theorem can be applied, e.g., to extend the algorithm of Lenzen et al. [40] to obtain a connected dominating set on planar graphs in the local model which is only 6 times larger than the dominating set computed for the planar graph (an n-vertex planar graph has at most $3n - 6$ edges). Similarly, it applies to the extension of Lenzen et al.'s algorithm by Amiri et al. [5] for graphs of bounded genus or to the randomized $\mathcal{O}(a^2)$ approximation of Lenzen and Wattenhofer [42] applied to graphs with excluded minors (here, $a \in \mathcal{O}(t \log t)$ if K_t is excluded as a minor).

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