On Low Rank-Width Colorings

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Abstract

We introduce the concept of *low rank-width colorings*, generalizing the notion of low tree-depth colorings introduced by Nešetřil and Ossona de Mendez in [25]. We say that a class $\mathcal{C}$ of graphs admits *low rank-width colorings* if there exist functions $N\colon \mathbb{N} \to \mathbb{N}$ and $Q\colon \mathbb{N} \to \mathbb{N}$ such that for all $p \in \mathbb{N}$, every graph $G \in \mathcal{C}$ can be vertex colored with at most $N(p)$ colors such that the union of any $i \leq p$ color classes induces a subgraph of rank-width at most $Q(i)$.

Graph classes admitting low rank-width colorings strictly generalize graph classes admitting low tree-depth colorings and graph classes of bounded rank-width. We prove that for every graph class $\mathcal{C}$ of bounded expansion and every positive integer r, the class $\{G^r : G \in \mathcal{C}\}$ of rth powers of graphs from $\mathcal{C}$, as well as the classes of unit interval graphs and bipartite permutation graphs admit low rank-width colorings. All of these classes have unbounded rank-width and do not admit low tree-depth colorings. We also show that the classes of interval graphs and permutation graphs do not admit low rank-width colorings. As interesting side properties, we prove that every graph class admitting low rank-width colorings has the Erdős-Hajnal property and is χ-bounded.

1 Introduction and main results

We are interested in covering a graph with (overlapping) pieces in such a way that (1) the number of pieces is small, (2) each piece is simple, and (3) every small subgraph is fully contained in at least one piece. Despite the graph theoretic interest in such coverings, it also has nice algorithmic applications. Consider e.g. the subgraph isomorphism problem. Here, we are given two graphs G and H as input, and we are asked to determine whether G contains a subgraph isomorphic to H. In many natural settings the pattern graph H we are looking for is small and in such case a covering as described above is most useful. By the first property, we can then iterate through the small number

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of pieces, by the third property, one of the pieces will contain our pattern graph. By the second property, we can test each piece for containment of H.

We can formulate the covering problem in an equivalent way from the point of view of graph coloring as follows. *How many colors are required to color the vertices of a graph G such that the union of any p color classes induce a simple subgraph* (understanding any p color classes as a piece in the above formulation)? It remains to specify what we mean by *simple* subgraphs.

From an algorithmic point of view, trees, or more generally, graphs of bounded tree-width are very well behaved graphs. Many NP-complete problems, in fact, all problems that can be formulated in monadic second order logic, are solvable in linear time on graphs of bounded tree-width [4, 5]. In particular, the subgraph isomorphism problem for every fixed pattern graph H is solvable in polynomial time on any graph of bounded tree-width.

Taking graphs of small tree-width as our simple building blocks, we can define a *p-tree-width coloring* of a graph G as a vertex coloring of G such that the union of any $i \leq p$ color classes induces a subgraph of tree-width at most $i-1$. Using the structure theorem of Robertson and Seymour [32] for graphs excluding a fixed graph as a minor, DeVos et al. [10] proved that for every graph H and every integer $p \geq 1$, there is an integer $N = N(H,p)$, such that every H-minor-free graph admits a p-tree-width coloring with N colors.

Tree-depth is another important and useful graph invariant. It was introduced under this name in [24], but equivalent notions were known before, including the notion of *rank* [27], *vertex ranking number* and minimum height of an *elimination tree* [1, 9, 33], etc. In [24], Nešetřil and Ossona de Mendez introduced the notion of p-tree-depth colorings as vertex colorings of a graph such that the union of any $i \leq p$ color classes induces a subgraph of tree-depth at most i. Note that the tree-depth of a graph is always larger (at least by 1) than its tree-width, hence a low tree-depth coloring is a stronger requirement than a low tree-width coloring. Also based on the structure theorem, Nešetřil and Ossona de Mendez [24] proved that proper minor closed classes admit even low tree-depth colorings.

Not much later, Nešetřil and Ossona de Mendez [25] proved that proper minor closed classes are unnecessarily restrictive for the existence of low tree-depth colorings. They introduced the notion of *bounded expansion classes of graphs*, a concept that generalizes the concept of classes with excluded minors and with excluded topological minors. While the original definition of bounded expansion is in terms of density of shallow minors, it turns out low tree-depth colorings give an alternative characterisation: a class $\mathcal{C}$ of graphs has bounded expansion if and only if for all $p \in \mathbb{N}$ there exists a number $N = N(\mathcal{C}, p)$ such that every graph $G \in \mathcal{C}$ admits a p-tree-depth coloring with $N(p)$ colors [25]. For the even more general notion of *nowhere dense classes of graphs* [26], it turns out that a class $\mathcal{C}$ of graphs closed under taking subgraphs is nowhere dense if and only if for all $p \in \mathbb{N}$ and all $\varepsilon > 0$ there exists n_0 such that every n-vertex graph $G \in \mathcal{C}$ with $n \geq n_0$ admits a p-tree-depth coloring with n^{ε} colors.

Furthermore, there is a simple algorithm to compute such a decomposition in time $\mathcal{O}(n)$ in case $\mathcal{C}$ has bounded expansion and in time $\mathcal{O}(n^{1+\varepsilon})$ for any $\varepsilon > 0$ in case $\mathcal{C}$ is nowhere dense. As a result, the subgraph isomorphism problem for every fixed pattern H can be solved in linear time on any class of bounded expansion and in almost linear time on any nowhere dense class. More generally, it was shown in [12, 16] that every fixed first order property can be tested in linear time on graphs of bounded expansion, implicitly using the notion of low tree-depth colorings, and in almost linear time on nowhere dense classes [17].

Note that bounded expansion and nowhere dense classes of graphs are uniformly sparse graphs. In fact, bounded expansion classes of graphs can have at most a linear number of edges and nowhere dense classes can have no more than $\mathcal{O}(n^{1+\varepsilon})$ many edges. This motivates our new definition of *low rank-width colorings* which extends the coloring technique to dense classes of graphs which are closed under taking induced subgraphs.

Rank-width was introduced by Oum and Seymour [31] and aims to extend tree-width by allowing well behaved dense graphs to have small rank-width. Also for graphs of bounded rank-width there are many efficient algorithms based on dynamic programming. Here, we have the important meta-theorem of Courcelle, Makowsky, and Rotics [7], stating that for every monadic second-order formula (with set quantifiers ranging over sets of vertices) and every positive integer k, there is an $\mathcal{O}(n^3)$-time algorithm to determine whether an input graph of rank-width at most k satisfies the formula. There are several parameters which are equivalent to rank-width in the sense that one is bounded if and only if the other is bounded. These include *clique-width* [6], *NLC-width* [35], and *Boolean-width* [2].

Low rank-width colorings. We now introduce our main object of study.

Definition 1. A class $\mathcal{C}$ of graphs *admits low rank-width colorings* if there exist functions $N : \mathbb{N} \to \mathbb{N}$ and $Q : \mathbb{N} \to \mathbb{N}$ such that for all $p \in \mathbb{N}$, every graph $G \in \mathcal{C}$ can be vertex colored with at most $N(p)$ colors such that the union of any $i \leq p$ color classes induces a subgraph of rank-width at most $Q(i)$.

As proved by Oum [28], every graph G with tree-width k has rank-width at most $k+1$, hence every graph class which admits low tree-depth colorings also admits low rank-width colorings. On the other hand, graphs admitting a low rank-width coloring can be very dense. We also remark that graph classes admitting low rank-width colorings are monotone under taking induced subgraphs, as rank-width does not increase by removing vertices.

Let us remark that due to the model-checking algorithm of Courcelle et al. [7], the (induced) subgraph isomorphism problem is solvable in cubic time for every fixed pattern H whenever the input graph is given together with a low rank-width coloring for $p = |V(H)|$, using $N(p)$ colors. Indeed, it suffices to iterate through all p-tuples of color classes and look for the pattern H in the subgraph induced by these color classes; this can be done efficiently since this subgraph has rank-width at most $Q(p)$. The caveat is that the graph has to be supplied with an appropriate coloring. In this work we do not investigate the algorithmic aspects of low rank-width colorings, and rather concentrate on the combinatorial question of which classes admit such colorings, and which do not.

Our contribution. We prove that for every class $\mathcal{C}$ of bounded expansion and every integer $r \geq 2$, the class $\{G^r : G \in \mathcal{C}\}$ of rth powers of graphs from $\mathcal{C}$ admits low rank-width colorings. It is easy to see that there are classes of bounded expansion such that $\{G^r : G \in \mathcal{C}\}$ has both unbounded rank-width and does not admit low tree-depth colorings. We furthermore prove that the class of unit interval graphs and the class of bipartite permutation graphs admit low rank-width colorings. On the negative side, we show that the classes of interval graphs and of permutation graphs do not admit low rank-width colorings. Finally, we also prove that every graph class admitting low rank-width colorings has the Erdős-Hajnal property [14] and is χ-bounded [19].

2 Preliminaries

All graphs in this paper are finite, undirected and simple, that is, they do not have loops or parallel edges. Our notation is standard, we refer to [11] for more background on graph theory. We write $V(G)$ for the vertex set of a graph G and $E(G)$ for its edge set. A *vertex coloring* of a graph G with colors from S is a mapping $c\colon V(G) \to S$. For each $v \in V(G)$, we call $c(v)$ the color of v. The *distance* between vertices u and v in G, denoted $\mathrm{dist}_G(u,v)$, is the length of a shortest path between u and v in G. The *rth power of a graph* G is the graph G^r with vertex set $V(G)$, where there is an edge between two vertices u and v if and only if their distance in G is at most r.

Rank-width was introduced by Oum and Seymour [31]. We refer to the surveys [20, 29] for more background. For a graph G, we denote the adjacency matrix of G by A_G, where for $x, y \in V(G)$, $A_G[x,y] = 1$ if and only if x is adjacent to y. Let G be a graph. We define the *cut-rank* function $\mathrm{cutrk}_G\colon 2^V \to \mathbb{N}$ such that $\mathrm{cutrk}_G(X)$ is the rank of the matrix $A_G[X, V(G) \setminus X]$ over the binary field (if $X = \emptyset$ or $X = V(G)$, then we let $\mathrm{cutrk}_G(X) = 0$).

A *rank-decomposition* of G is a pair (T, L), where T is a subcubic tree (i.e. a tree where every node has degree 1 or 3) with at least 2 nodes and L is a bijection from $V(G)$ to the set of leaves of T. The *width* of e is define as $\mathrm{cutrk}_G(A_1^e)$ where (A_1^e, A_2^e) is the vertex bipartition of G each A_i^e is the set of all vertices in G mapped to leaves contained in one of components of $T - e$. The *width* of (T, L) is the maximum width over all edges in T, and the *rank-width* of G, denoted by $\mathrm{rw}(G)$, is the minimum width over all rank-decompositions of G. If $|V(G)| \le 1$, then G has no rank-decompositions, and the rank-width of G is defined to be 0.

A graph is an *interval graph* if it is the intersection graph of a family $\mathcal{I}$ of intervals on the real line, an interval graph is a *unit interval graph* if all intervals in $\mathcal{I}$ have the same length. A graph is a *permutation graph* if it is the intersection graph of line segments whose endpoints lie on two parallel lines.

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{B})$ consisting of a tree T and a family $\mathcal{B} = \{B_t\}_{t \in V(T)}$ of sets $B_t \subseteq V(G)$, satisfying the following three conditions:

(T1) $V(G) = \bigcup_{t \in V(T)} B_t$;

(T2) for every $uv \in E(G)$, there exists a node t of T such that $\{u, v\} \subseteq B_t$;

(T3) for $t_1, t_2, t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever t_2 is on the path from t_1 to t_3 in T.

The *width* of a tree-decomposition $(T, \mathcal{B})$ is $\max\{|B_t| - 1 : t \in V(T)\}$. The *tree-width* of G is the minimum width over all tree-decompositions of G.

Let G be a graph and let $G_1, \ldots, G_s$ be its connected components. Then the *tree-depth* of G is recursively defined as

$$\mathrm{td}(G) = \begin{cases} 1 & \text{if } |V(G)| = 1 \\ 1 + \min_{v \in V(G)}\ \mathrm{td}(G - v) & \text{if } |V(G)| > 1 \text{ and } s = 1 \\ \max_{1 \le i \le s}\ \mathrm{td}(G_i) & \text{otherwise.} \end{cases}$$

3 Powers of sparse graphs

In this section we show that the class of rth powers of graphs from a bounded expansion class admit low rank-width colorings. The original definition of bounded expansion classes by Nešetřil and

Ossona de Mendez [25] is in terms of bounds on the density of bounded depth minors. We will work with the characterisation by the existence of low tree-depth colorings as well as by a characterisation in terms of bounds on generalized coloring numbers.

Theorem 1 (Nešetřil and Ossona de Mendez [25]). *A class $\mathcal{C}$ of graphs has bounded expansion if and only if for all $p \in \mathbb{N}$ there exists a number $N = N(\mathcal{C}, p)$ such that every graph $G \in \mathcal{C}$ admits a p-tree-depth coloring with N colors.*

Our main result in this section is the following.

Theorem 2. *Let $\mathcal{C}$ be a class of bounded expansion and $r \geq 2$ be an integer. Then the class $\{G^r : G \in \mathcal{C}\}$ of rth powers of graphs from $\mathcal{C}$ admits low rank-width colorings.*

For a graph G, we denote by $\Pi(G)$ the set of all linear orders of $V(G)$. For $u, v \in V(G)$ and a non-negative integer r, we say that u is *weakly r-reachable* from v with respect to L, if there is a path P of length at most r between u and v such that u is the smallest among the vertices of P with respect to L. We denote by $\mathrm{WReach}_r[G, L, v]$ the set of vertices that are weakly r-reachable from v with respect to L. The *weak r-coloring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) \quad := \quad \min_{L \in \Pi(G)} \max_{v \in V(G)} \big|\mathrm{WReach}_r[G, L, v]\big|.$$

The weak coloring number was introduced by Kierstead and Yang [21] in the context of coloring and marking games on graphs. As shown by Zhu [36], classes of bounded expansion can be characterised by the weak coloring numbers.

Theorem 3 (Zhu [36]). *A class $\mathcal{C}$ has bounded expansion if and only if for all $r \geq 1$ there is a number $f(r)$ such that for all $G \in \mathcal{C}$ it holds that $\mathrm{wcol}_r(G) \leq f(r)$.*

In order to prove Theorem 2, we will first compute a low tree-depth coloring. We would like to apply the following theorem, relating the tree-width (and hence in particular the tree-depth) of a graph and the rank-width of its rth power.

Theorem 4 (Gurski and Wanke [18]). *Let $r \geq 2$ be an integer. If a graph H has tree-width at most p, then H^r has rank-width at most $2(r+1)^{p+1} - 2$.*

We remark that Gurski and Wanke [18] proved this bound for clique-width instead of rank-width, but clique-width is never smaller than the rank-width [31].

The natural idea would be just to combine the bound of Theorem 4 with low tree-depth coloring given by Theorem 1. Note however, that when we consider any subgraph H induced by $i \leq p$ color classes, the graph H^r may be completely different from the graph $G^r[V(H)]$, due to paths that are present in G but disappear in H. Hence we cannot directly apply Theorem 4. Instead, we will prove the existence of a refined coloring of G such that for any subgraph H induced by $i \leq p$ color classes, in the refined coloring there is a subgraph H' such that $G^r[V(H)] \subseteq H'^r$ and such that H' gets only $g(i)$ colors in the original coloring, for some fixed function g. We can now apply Theorem 4 to H' and use fact that rank-width is monotone under taking induced subgraphs.

In the following, we will say that a vertex subset X *receives* a color i under a coloring c if $i \in c^{-1}(X)$. We first need the following definitions.

Definition 2. Let G be a graph, $X \subseteq V(G)$ and $r \geq 2$. A superset $X' \supseteq X$ is called *an r-shortest path hitter* for X if for all $u, v \in X$ with $1 < \text{dist}_G(u,v) \leq r$, X' contains an internal vertex of some shortest path between u and v.

Definition 3. Let G be a graph, let c be a coloring of G, and $r \geq 2$ and $d \geq 1$. A coloring c' is a (d,r)-*good refinement* of c if for every vertex set X that receives at most p colors under c', there exists an r-shortest path hitter X' of X that receives at most $d \cdot p$ colors under c.

We use the weak coloring numbers to prove the existence of a good refinement.

Lemma 5. *Let G be a graph and $r \geq 2$ be an integer. Then every coloring c of G using k colors has a $(2\text{wcol}_r(G), r)$-good refinement using $k^{2\text{wcol}_r(G)}$ colors.*

Proof. Let Γ be the set of colors used by c, and let $d := 2\text{wcol}_r(G)$. The (d,r)-good refinement c' that we are going to construct will use subsets of Γ of size at most d as the color set; the number of such subsets is at most $k^{2\text{wcol}_r(G)}$. Let L be a linear order of $V(G)$ with $\max_{v \in V(G)} \left|\text{WReach}_r[G,L,v]\right| = \text{wcol}_r(G)$. We construct a new coloring c' as follows:

(1) Start by setting $c'(v) := \emptyset$ for each $v \in V(G)$.

(2) For each pair of vertices u and v such that $u \in \text{WReach}_r[G,L,v]$, we add the color $c(u)$ to $c'(v)$.

(3) For each pair u, v of non-adjacent vertices such that $u <_L v$ and $u \in \text{WReach}_r[G,L,v]$, we do the following. Check whether there is a path P of length at most r connecting u and v such that all the internal vertices of P are larger than both u and v in L. If there is no such path, we do nothing for the pair u, v. Otherwise, fix one such path P, chosen to be the shortest possible, and let z be the vertex traversed by P that is the largest in L. Then add the color $c(z)$ to $c'(v)$.

Thus, every vertex v receives in total at most $2\text{wcol}_r(G)$ colors of Γ to its final color $c'(v)$: at most $\text{wcol}_r(G)$ in step (2), and at most $\text{wcol}_r(G)$ in step (3), because we add at most one color per each $u \in \text{WReach}_r[G,L,v]$. It follows that each final color $c'(v)$ is a subset of Γ of size at most $2\text{wcol}_r(G)$.

We claim that c' is a (d,r)-good refinement of c. Let $X \subseteq V(G)$ be a set that receives at most p colors under c', say colors $A_1, \ldots, A_p \subseteq \Gamma$. Let X' be the set of vertices of G that are colored by colors in $A_1 \cup \cdots \cup A_p$ under c. Since $|A_i| \leq d$ for each $i \in \{1, \ldots, p\}$, we have that X' receives at most $d \cdot p$ colors under c.

To show that X' is an r-shortest path hitter of X, let us choose any two vertices u and v in X with $u <_L v$ and $1 < \text{dist}_G(u,v) \leq r$. If there is a shortest path from u to v whose all internal vertices are larger than u and v in L, by step (3), X' contains a vertex that is contained in one such path. Otherwise, a shortest path from u to v contains a vertex z with $L(z) < L(v)$ other than u and v. This implies that there exists $z' \in \text{WReach}_r[G,L,v] \setminus \{u\}$ on the path such that $c(z') \in c'(v)$, and hence $z' \in X'$ by step (2). Therefore, X' is an r-shortest path hitter of X, as required. □

Definition 4. Let G be a graph, let $X \subseteq V(G)$, and let $r \geq 1$ be an integer. A superset $X' \supseteq X$ is called *an r-shortest path closure of* X if for each $u, v \in X$ with $\text{dist}_G(u,v) = \ell \leq r$, $G[X']$ contains a path of length ℓ between u and v.

Definition 5. Let G be a graph, let c be a coloring of G, and let $r \geq 2$ and $d \geq 1$. A coloring c' is a (d,r)-*excellent refinement* of c if for every vertex set $X \subseteq V(G)$ there exists an r-shortest path closure X' of X such that if X receives p colors in c', then X' receives at most $d \cdot p$ colors in c.

We inductively define excellent refinements from good refinements.

Lemma 6. *Let G be a graph, $r \geq 2$ an integer, and let $d_r := \prod_{2 \leq \ell \leq r} 2\mathrm{wcol}_\ell(G)$. Then every coloring c of G using at most k colors has a (d_r, r)-excellent refinement using at most k^{d_r} colors.*

Proof. The proof is by induction on r. For $r = 2$, an r-shortest path hitter of a set X is an r-shortest path closure, and vice versa. Hence, the statement immediately follows from Lemma 5. Now assume $r \geq 3$. By induction hypothesis, there is a $(d_{r-1}, r-1)$-excellent refinement c_1 of c with at most $k^{d_{r-1}}$ colors. By applying Lemma 5 to c_1, we obtain a $(2\mathrm{wcol}_r(G), r)$-good refinement c' of c_1 with at most $(k^{d_{r-1}})^{2\mathrm{wcol}_r(G)} = k^{d_r}$ colors. We claim that c' is a (d_r, r)-excellent refinement of c. Any set X which gets at most p colors from c' can be first extended to an r-shortest path hitter X' for X which receives at most $2\mathrm{wcol}_r(G) \cdot p$ colors. Then X' can be extended by induction hypothesis to an $(r-1)$-shortest path closure X'' of X' that receives at most $d_{r-1} \cdot 2\mathrm{wcol}_r(G) \cdot p = d_r \cdot p$ colors.

It remains to show that X'' is an r-shortest path closure of X. Take any $u, v \in X$ with $\mathrm{dist}_G(u, v) = \ell \leq r$. If $\ell \leq 1$, then u, v are already adjacent in $G[X]$. Otherwise, since X' is an r-shortest path hitter for X, there is a vertex $z \in X'$ that lies on some shortest path connecting u and v in G. In particular, $\mathrm{dist}_G(u, z) = \ell_1$ and $\mathrm{dist}_G(z, v) = \ell_2$ for ℓ_1, ℓ_2 satisfying $\ell_1, \ell_2 < \ell$ and $\ell_1 + \ell_2 = \ell$. Since X'' is an $(r-1)$-shortest path closure of X', we infer that $\mathrm{dist}_{G[X'']}(u, z) = \ell_1$ and $\mathrm{dist}_{G[X'']}(z, v) = \ell_2$. Hence $\mathrm{dist}_{G[X'']}(u, v) = \ell$ by the triangle inequality. □

Proof (of Theorem 2). Let G be a graph in $\mathcal{C}$ and let $d_r := \prod_{2 \leq \ell \leq r} 2\mathrm{wcol}_\ell(G)$. Since $\mathcal{C}$ has bounded expansion, by Theorem 3, for each r, $\mathrm{wcol}_r(G)$ is bounded by a constant only depending on $\mathcal{C}$. We start by taking c to be a $(d_r \cdot p)$-tree-depth coloring with $N(d_r \cdot p)$ colors, where N is the function from Theorem 1. Then its (d_r, r)-excellent refinement c' has the property that c' uses at most $N(d_r \cdot p)^{d_r}$ colors, and every subset X which receives at most p colors in c' has an r-shortest path closure X' that receives at most $d_r \cdot p$ colors in c. Thus, the graph induced on X in the rth power G^r is the same at the graph induced on X in the rth power $G[X']^r$. Since $G[X']$ has tree-depth at most $d_r \cdot p$, by Theorem 4, $G[X']^r$ has rank-width at most $2(r+1)^{d_r \cdot p + 1} - 2$. Therefore, $G^r[X]$ has rank-width at most $2(r+1)^{d_r \cdot p+1} - 2$ as well. □

We now give two example applications of Theorem 2. A *map graph* is a graph that can be obtained from a plane graph by making a vertex for each face, and adding an edge between two vertices, if the corresponding faces share a vertex. One can observe that every map graph is an induced subgraph of the second power of another planar graph, namely the *radial graph* of the original graph.

Lemma 7. *Every map graph is an induced subgraph of the second power of a planar graph.*

Proof. Let G be a map graph, and let H be a planar graph defining the map for G. Consider the *radial graph* R of H: the vertex set of R consists of vertices and faces of H, and a vertex u is adjacent to a face f iff u is incident to f in H. Obviously R is planar. It follows that G is the subgraph induced in R^2 by the vertices corresponding to faces of H. □

Thus, map graphs have low rank-width colorings. A similar reasoning can be performed for line graphs of graphs from any bounded expansion graph class. Thus, both map graphs and line graphs of graphs from any fixed bounded expansion graph class admit low rank-width colorings.

Lemma 8. *If $\mathcal{C}$ is a graph class of bounded expansion, then there is a graph class of bounded expansion $\mathcal{C}_1$ such that all line graphs of graphs from $\mathcal{C}$ are induced subgraphs of graphs from $\mathcal{C}_1^2$.*

Proof. Observe that for any graph G, if by G_1 we denote G with every edge subdivided once, then the line graph of G is a subgraph of G_1^2 induced by the subdividing vertices. It follows that we may take $\mathcal{C}_1$ to be the class of all 1-subdivisions of graphs from $\mathcal{C}$. This class also has bounded expansion. □

4 Other positive results

We now prove that unit interval graphs and bipartite permutation graphs admit low rank-width colorings.

Theorem 9. *The class of unit interval graphs and the class of bipartite permutation graphs admit low rank-width colorings.*

Our results follow from characterizations of these classes obtained by Lozin [22]. Let $n, m \geq 1$. We denote by $H_{n,m}$ the graph with $n \cdot m$ vertices which can be partitioned into n independents sets $V_1 = \{v_{1,1}, \ldots, v_{1,m}\}, \ldots, V_n = \{v_{n,1}, \ldots, v_{n,m}\}$ so that for each $i \in \{1, \ldots, n-1\}$ and for each $j, j' \in \{1, \ldots, m\}$, vertex $v_{i,j}$ is adjacent to $v_{i+1,j'}$ if and only if $j' \in \{1, \ldots, j\}$, and there are no edges between V_i and V_j if $|i - j| \geq 2$. The graph $\widetilde{H}_{n,m}$ is the graph obtained from $H_{n,m}$ by replacing each independent set V_i by a clique.

Lemma 10. *The following statements hold:*

1. *(Lozin [22]) The rank-width of $H_{n,m}$ and of $\widetilde{H}_{n,m}$ is at most $3n$.*
2. *(Lozin [22]) Every bipartite permutation graph on n vertices is isomorphic to an induced subgraph of $H_{n,n}$.*
3. *(Lozin [23]) Every unit interval graph on n vertices is isomorphic to an induced subgraph of $\widetilde{H}_{n,n}$.*

Hence, in order to prove Theorem 9, it suffices to prove that the graphs $H_{n,m}$ and $\widetilde{H}_{n,m}$ admit low rank-width colorings.

Proof (of Theorem 9). For every positive integer p, let $N(p) := p + 1$ and $Q(i) := 3i$ for each $i \in \{1, \ldots, p\}$. We prove that for all $n, m \geq 1$, the graphs $H_{n,m}$ and $\widetilde{H}_{n,m}$ can be vertex colored using $N(p)$ colors so that each of the connected components of the subgraph induced by any $i \leq p$ color classes has rank-width at most $R(i)$. As rank-with and rank-width colorings are monotone under taking induced subgraphs, the statement of the theorem follows from Lemma 10.

Assume that the vertices of $H_{n,m}$ (and $\widetilde{H}_{n,m}$, respectively) are, as in the definition, named $v_{1,1}, \ldots, v_{1,m}, \ldots, v_{n,1}, \ldots, v_{n,m}$. We color the vertices in the ith row, $v_{i,1}, \ldots, v_{i,m}$, with color $j+1$ where $j \in \{0, 1, \ldots, p\}$ and $i \equiv j \pmod{p+1}$. Then any connected component H of a subgraph induced by $i \leq p$ colors is isomorphic to $H_{i',m}$ ($\widetilde{H}_{i',m}$, respectively) for some $i' \leq i$. Hence, according to Lemma 10, H has rank-width at most $3i = Q(i)$, as claimed. □

5 Negative results

In contrast to the result in Section 4, we prove that interval graphs and permutation graphs do not admit low rank-width colorings. For this, we introduce twisted chain graphs. Briefly, a twisted chain graph G consists of three vertex sets A, B, C where each of $G[A \cup C]$ and $G[B \cup C]$ is a chain graph, but the ordering of C with respect to the chain graphs $G[A \cup C]$ and $G[B \cup C]$ are distinct.

Definition 6. For a positive integer n, a graph on the set of $3n^2$ vertices $A \cup B \cup C$, where $A = \{v_1, \ldots, v_{n^2}\}$, $B = \{w_1, \ldots, w_{n^2}\}$, and $C = \{z_{(i,j)} : 1 \leq i, j \leq n\}$, is called a *twisted chain graph* of order n if

- for integers $x, y, i, j \in \{1, \ldots, n\}$ and $k = n(x-1) + y$, v_k is adjacent to $z_{(i,j)}$ if and only if either $(x < i)$ or $(x = i$ and $y \leq j)$;
- for integers $x, y, i, j \in \{1, \ldots, n\}$ and $k = n(x-1) + y$, w_k is adjacent to $z_{(i,j)}$ if and only if either $(x < j)$ or $(x = j$ and $y \leq i)$;
- the edge relation within $A \cup B$ and within C is arbitrary.

We first show that a large twisted chain graph has large rank-width. We remark that a proof of this fact seems to follow also from a careful examination and modification of general constructions given by Dabrowski and Paulusma [8]; however, we prefer to give our own direct proof for the sake of completeness.

Lemma 11. *For every integer $n > 0$, every twisted chain graph of order $12n$ has rank-width at least n.*

Before we proceed to the proof of Lemma 11, we need to introduce some basic tools A vertex bipartition (X, Y) of a graph G is *balanced with respect to a set $C \subseteq V(G)$* if $\frac{|C|}{3} \leq |X \cap C|, |Y \cap C|$. We will need the following standard fact.

Lemma 12. *If G is a graph of rank-width at most w and $C \subseteq V(G)$ with $|C| \geq 3$, then G admits a vertex bipartition (X, Y) with $\mathrm{cutrk}_G(A) \leq w$ that is balanced with respect to C.*

Proof. Let (T, L) be a rank-decomposition of G of width at most w. We subdivide an edge of T, and regard the new vertex as a root node. For each node $t \in V(T)$, let $\mu(t)$ be the number of leaves of T that are descendants of t and correspond to vertices of S. Now, we choose a node t that is farthest from the root node and such that $\mu(t) \geq \frac{|C|}{3}$. By the choice of t, either t is a leaf or for each child t' of t we have $\mu(t') < \frac{|C|}{3}$. Therefore, since $|C| \geq 3$, in any case $\frac{|C|}{3} \leq \mu(t) \leq \frac{2|C|}{3}$. Let e be the edge connecting the node t and its parent. By the construction, the vertex bipartition associated with e satisfies the required property. □

We now proceed to the proof of Lemma 11.

Proof (of Lemma 11). Let $m := 12n$ and let G be a twisted chain graph of order m. Adopt the notation from Definition 6 for G. Suppose for the sake of contradiction that the rank-width of G is smaller than n. By Lemma 12, there exists a vertex bipartition (S, T) of G with $\mathrm{cutrk}_G(A) < n$ such that $|C \cap S| \geq \frac{|C|}{3} = m^2/3$ and similarly $|C \cap T| \geq m^2/3$.

Suppose we have vertices $v_{a_1}, \ldots, v_{a_k} \in A \cap S$ and $z_{(b_1,c_1)}, \ldots, z_{(b_k,c_k)} \in C \cap T$ with the following property satisfied:

$$a_1 \le (b_1-1)m + c_1 < a_2 \le (b_2-1)m + c_2 < \cdots < a_k \le (b_k-1)m + c_k.$$

Such a structure will be called an *A-ordered (S,T)-matching* of order k. By the definition of adjacency in G it follows that the submatrix of $A_G[S,T]$ induced by rows corresponding to vertices v_{a_i} and columns corresponding to vertices $z_{(b_i,c_i)}$ has ones in the upper triangle and on the diagonal, and zeroes in the lower triangle. The rank of this submatrix is k, so since $\mathrm{cutrk}_G(A) < n$, there is no A-ordered (S,T)-matching of order n. We similarly define the A-ordered (T,S)-matching of rank n, where the vertices v_{a_i} belong to T and vertices $z_{(b_i,c_i)}$ belong to S. Likewise, there is no A-ordered (T,S)-matching of order n.

Suppose now we have vertices $w_{a_1}, \ldots, w_{a_k} \in B \cap S$ and $z_{(b_1,c_1)}, \ldots, z_{(b_k,c_k)} \in C \cap T$ with the following property satisfied:

$$a_1 \le (c_1-1)m + b_1 < a_2 \le (c_2-1)m + b_2 < \cdots < a_k \le (c_k-1)m + b_k.$$

Such a structure will be called a *B-ordered (S,T)-matching* of order k, and a *B-ordered (T,S)-matching* of order k is defined analogously. Again, the same reasoning as above shows that there is no B-ordered (S,T)-matching of order n, and no B-ordered (T,S)-matching of order n.

Let $\preceq_1$ and $\preceq_2$ be lexicographic orders on $\{1,\ldots,m\} \times \{1,\ldots,m\}$, with the leading coordinate being the first one for $\preceq_1$ and the second for $\preceq_2$.

Claim 1. *If there is a sequence $(x_1,y_1) \prec_1 (x_2,y_2) \prec_1 \ldots \prec_1 (x_{4k},y_{4k})$ such that $z_{(x_j,y_j)} \in S$ for odd j and $z_{(x_j,y_j)} \in T$ for even j, then there is either an A-ordered (S,T)-matching of order k, or an A-ordered (T,S)-matching of order k.*

PROOF. Denote $r_j = (x_j-1)m + y_j$ for $j = 1,2,\ldots,4k$. For $i = 1,2,\ldots,2k$, we define a_i and (b_i,c_i) as follows:

- if $v_{r_{2i-1}} \in S$, then $a_i = r_{2i-1}$ and $(b_i,c_i) = z_{(b_{2i},c_{2i})}$, and
- if $v_{r_{2i-1}} \in T$, then $a_i = r_{2i-1}$ and $(b_i,c_i) = z_{(b_{2i-1},c_{2i-1})}$.

It follows that vertices $v_{a_1}, \ldots, v_{a_{2k}}$ and $z_{(b_1,c_1)}, \ldots, z_{(b_{2k},c_{2k})}$ satisfy

$$a_1 \le (b_1-1)m + c_1 < a_2 \le (b_2-1)m + c_2 < \cdots < a_k \le (b_k-1)m + c_k,$$

and for each i we have that v_{a_i} and $z_{(b_i,c_i)}$ belong to different sides of the bipartition (S,T). Now, if for at least k indices i we have $v_{a_i} \in S$ and $z_{(b_i,c_i)} \in T$, then the subsequence induced by elements with these indices gives an A-ordered (S,T)-matching of order k. Otherwise, there are at least k indices i with $v_{a_i} \in T$ and $z_{(b_i,c_i)} \in S$, and the subsequence induced by them gives an A-ordered (T,S)-matching of order k. ⌟

A symmetric proof gives the following.

Claim 2. *If there is a sequence $(x_1,y_1) \prec_2 (x_2,y_2) \prec_2 \ldots \prec_2 (x_{4k},y_{4k})$ such that $z_{(x_j,y_j)} \in S$ for odd j and $z_{(x_j,y_j)} \in T$ for even j, then there is either a B-ordered (S,T)-matching of order k, or a B-ordered (T,S)-matching of order k.*

From Claim 1 and Claim 2 it follows that the largest possible length of sequences as in the statements is smaller than $4n$. For $i, j \in \{1, \ldots, m\}$, call the sets $\{z_{(i,y)} \colon y \in \{1, \ldots, m\}\}$ and $\{z_{(x,j)} \colon x \in \{1, \ldots, m\}\}$ the *ith row* and the *jth column*, respectively. A row or a column is *mixed* if it contains both elements of S and elements of T. Observe that if there were at least $4n$ mixed rows, then by choosing vertices from S and T alternately from these rows we would obtain a sequence of length $4n$ as in the statement of Claim 1. Then Claim 1 gives us an A-ordered (S, T)-matching of order n or an A-ordered (T, S)-matching of order n, a contradiction. Hence, there are less than $4n$ mixed rows, and symmetrically there are less than $4n$ mixed columns.

Observe that if there were two non-mixed rows such that the first one was contained in S while the second was contained in T, then all the $12n$ columns would be mixed, a contradiction. Hence, either all the non-mixed rows belong to S, or all of them belong to T. However, there are more than $12n - 4n = 8n = m/3$ non-mixed rows, so either S or T contains more than a third of vertices of C. This is a contradiction with the assumption that (S, T) is balanced with respect to C. ☐

We now observe that if a graph class contains arbitrarily large twisted chain graphs, then it does not admit low rank-width colorings.

Theorem 13. *Let $\mathcal{C}$ be a hereditary graph class, and suppose for each positive integer n some twisted chain graph of order n belongs to $\mathcal{C}$. Then $\mathcal{C}$ does not admit low rank-width colorings.*

Proof. We show that for every pair of integers $m \geq 3$ and $n \geq 1$, there is an graph $G \in \mathcal{C}$ such that for every coloring of G with m colors, there is an induced subgraph H that receives at most 3 colors and has rank-width at least n. This implies that $\mathcal{C}$ does not admit low rank-width colorings. We will need the following simple Ramsey-type argument. Claim 3 follows, e.g., from [34, Theorem 11.5], but we give a simple proof for the sake of completeness.

Claim 3. *For all positive integers k, d, there exists an integer $M = M(k, d)$ such that for all sets X, Y with $|X| = |Y| = M$ and all functions $f \colon X \times Y \to \{1, \ldots, d\}$, there exist subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| = |Y'| = k$ such that f sends all elements of $X' \times Y'$ to the same value.*

Proof. We prove the claim for $M(k, d) = k \cdot d^{dk}$. Let X_0 be an arbitrary subset X of size dk. For each $y \in Y$, define the *type* of y as the function $g_y \colon X_0 \to \{1, \ldots, d\}$ defined as $g_y(x) = f(x, y)$. There are at most d^{dk} different types possible, so there is a subset $Y' \subseteq Y$ of size k such that each element of Y' has the same type g. Since $|X_0| = dk$, there is some $i \in \{1, \ldots, d\}$ such that g yields value i for at least k elements of X_0. Then if we take X' to be an arbitrary set of k elements mapped to i by g, then $f(x, y) = i$ for each $(x, y) \in X' \times Y'$, as required. ⌟

Let $M_1 := M(12n, m)$, $M_2 := M(M_1, m)$, and $M_3 := M(M_2, m)$. Let $G \in \mathcal{C}$ be a twisted chain graph of order M_3; adopt the notation from Definition 6 for G. Suppose G is colored by m colors by a coloring c. By Claim 3, there exist $X_1, Y_1 \subseteq \{1, \ldots, M_3\}$ with $|X_1| = |Y_1| = M_2$ such that $\{z_{(x,y)} \colon (x, y) \in X_1 \times Y_1\}$ is monochromatic under c.

Now, for an index $k \in \{1, \ldots, M_3^2\}$, let $(i_1(k), j_1(k)) \in \{1, \ldots, m\} \times \{1, \ldots, m\}$ be the unique pair such that $k = (i_1(k) - 1)M_3 + j_1(k)$, and let $(i_2(k), j_2(k)) \in \{1, \ldots, m\} \times \{1, \ldots, m\}$ be the unique pair such that $k = (j_2(k) - 1)M_3 + i_2(k)$. By reindexing vertices A and C using pairs $(i_1(k), j_1(k))$ and $(i_2(k), j_2(k))$, we may view coloring c on A and C as a coloring on $\{1, \ldots, M_3\} \times \{1, \ldots, M_3\}$. By applying Claim 3 to the vertices from A indexed by $X_1 \times Y_1$, we obtain subsets $X_2 \subseteq X_1$ and $Y_2 \subseteq Y_1$ such that $|X_2| = |Y_2| = M_1$ and the set $\{v_{(x-1)M_3+y} \colon x \in X_2, y \in Y_2\}$ is monochromatic. Finally,

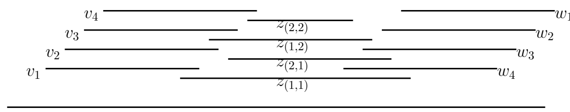

Figure 1: An interval intersection model of a twisted chain graph of order 2.

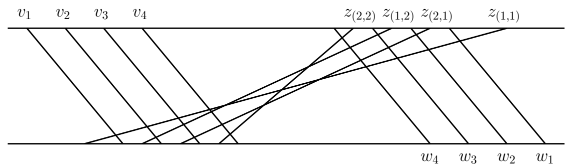

Figure 2: A permutation intersection model of a twisted chain graph of order 2.

by applying Claim 3 to the vertices from B indexed by $X_2 \times Y_2$, we obtain subsets $X_3 \subseteq X_2$ and $Y_3 \subseteq Y_2$ such that $|X_3| = |Y_3| = 12n$ and the set $\{w_{(y-1)M_3+x} \colon (x, y) \in X_3 \times Y_3\}$ is monochromatic. Now observe that the subgraph $G[\{v_{(x-1)M_3+y}, w_{(y-1)M_3+x}, z_{(x,y)} \colon (x, y) \in X_3 \times Y_3\}]$ receives at most 3 colors, and is a twisted chain graph of order $12n$. By Lemma 11 it has rank-width at least n, so this proves the claim. $\square$

We now observe that a twisted chain graph of order n is an interval graph, provided each of A, B, and C is a clique, and there are no edges between A and B. Similarly, for each n there is a twisted chain graph of order n that is a permutation graph. See Figures 1 and 2 for examples of intersection models.

Lemma 14. *Let G be a twisted chain graph of order n, for some positive integer n. If each of A, B, and C is a clique, and there are no edges between A and B, then G is an interval graph.*

Proof. Let $M > 2n^2$. Consider the following interval model:

- For each $i \in \{1, \ldots, n^2\}$, assign interval $[0, i]$ to v_i.
- For each $i \in \{1, \ldots, n^2\}$, assign interval $[M - i, M]$ to w_i.
- For each $x, y \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, assign interval $[(x-1)n + y, M - (y-1)n - x]$ to $z_{(x,y)}$.

It can be easily seen that this is an interval model of the twisted chain graph of order n. $\square$

Lemma 15. *For each positive integer n, there is a twisted chain graph of order n that is a permutation graph.*

Proof. Let $M > 10n^2$. Consider the following permutation model, spanned between horizontal lines ℓ_1 with y-coordinate 0 and ℓ_2 with y-coordinate 1:

- For each $i \in \{1, \ldots, n^2\}$, assign the segment with endpoints $(i, 0)$ and $(i, 1)$ to v_i.
- For each $i \in \{1, \ldots, n^2\}$, assign the segment with endpoints $(M - i, 0)$ and $(M - i, 1)$ to w_i.
- For each $x, y \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, assign the segment with endpoints $((x-1)m + y, 0)$ and $(M - (y-1)m - x, 1)$ to $z_{(x,y)}$

It can be easily seen that this is a permutation model of some twisted chain graph of order n. $\square$

By Theorem 13, we obtain the following.

Theorem 16. *The classes of interval graphs and permutation graphs do not admit low rank-width colorings.*

6 Erdős-Hajnal property and χ-boundedness

A graph class $\mathcal{C}$ has the *Erdős-Hajnal property* if there is $\varepsilon > 0$, depending only on $\mathcal{C}$, such that every n-vertex graph in $\mathcal{C}$ has either an independent set or a clique of size n^{ε}. The conjecture of Erdős and Hajnal [14] states that for every fixed graph H, the class of graphs not having H as an induced subgraph has the Erdős-Hajnal property; cf. [3]. We prove that every class admitting low rank-width colorings has the Erdős-Hajnal property. The proof is based on the fact that every class of graphs of bounded rank-width has this property, which was shown by Oum and Seymour [30]. Since this claim is not written in any published work, we include the proof for the completeness.

A graph is *cograph* if it can be recursively constructed from isolated vertices by means of the following two operations: (1) taking disjoint union of two graphs and (2) joining two graphs, i.e., taking their disjoint union and adding all possible edges with one endpoint in the first graphs and the second endpoint in the second. It is well-known that every n-vertex cograph contains either an independent set or a clique of size $n^{1/2}$; this follows from the fact that cographs are perfect.

Lemma 17 (Oum and Seymour [30]). *For every positive integer p, there exists a constant $\delta = \delta(p)$ such that every n-vertex graph of rank-width at most p contains either an independent set or a clique of size at least n^{δ}.*

Proof. Let $\kappa(p) := \frac{1}{\log_2 3+p}$ and $\delta(p) := \frac{\kappa(p)}{2}$. We first prove by induction on n that every n-vertex graph of rank-width at most p contains a cograph of size at least $n^{\kappa(p)}$. Assume $n \geq 3$, for otherwise every graph on at most 2 vertices is a cograph. Let G be an n-vertex graph of rank-width at most p. By Lemma 12, G has a vertex bipartition (A, B) where $\operatorname{cutrk}_G(A) \leq p$ and $|A|, |B| \geq \frac{n}{3}$. Since $\operatorname{cutrk}_G(A) \leq p$, there exist $A' \subseteq A$ and $B' \subseteq B$ with $|A'|, |B'| \geq \frac{n}{3 \cdot 2^p}$, such that either there are no edges between A' and B', or every vertex in A' is adjacent to every vertex in B'. By induction hypothesis, $G[A']$ contains a cograph H_1 and $G[B']$ contains a cograph H_2, both as induced subgraphs, such that $|V(H_1)|, |V(H_2)| \geq (\frac{n}{3 \cdot 2^p})^{\kappa(p)}$. Note that $G[V(H_1) \cup V(H_2)]$ is a cograph. Since $\kappa(p) = \frac{1}{\log_2 3+p}$, we have $|V(H_1)| + |V(H_2)| \geq 2(\frac{n}{3 \cdot 2^p})^{\kappa(p)} = n^{\kappa(p)}$. We conclude that G contains either an independent set or a clique of size at least $(n^{\kappa(p)})^{1/2} = n^{\delta(p)}$. $\square$

Proposition 18. *Every class of graphs that admits low rank-width colorings has the Erdős-Hajnal property.*

Proof. Let $\mathcal{C}$ be the class of graphs in question. Fix any $G \in \mathcal{C}$, say on n vertices. Since $\mathcal{C}$ admits low rank-width colorings, there exist functions $N \colon \mathbb{N} \to \mathbb{N}$ and $R \colon \mathbb{N} \to \mathbb{N}$, depending only on $\mathcal{C}$, such that for all p, G can be colored using $N(p)$ colors so that each induced subgraph of G that receives $i \leq p$ colors has rank-width at most $R(i)$. Let c be such a coloring for $p = 1$; then c uses $N(1)$ colors.

Let $\delta(p)$ be the function defined in Lemma 17. We define

$$\varepsilon := \min\left(\frac{\delta(R(1))}{2}, \frac{1}{2\log_2 N(1)}\right),$$

and claim that G has an independent set or a clique of size at least n^ε.

First, assume $n \geq N(1)^2$. Then there is a color i such that $|c^{-1}(i)| \geq \frac{n}{N(1)}$. Thus, the subgraph H induced by the vertices with color i has at least $\frac{n}{N(1)}$ vertices and rank-width at most $R(1)$. Since $n \geq N(1)^2$, by Lemma 17, H, and thus also G, contains either an independent set or a clique of size at least

$$|V(H)|^{\delta(R(1))} \geq \left(\frac{n}{N(1)}\right)^{\delta(R(1))} \geq n^{\frac{\delta(R(1))}{2}} \geq n^\varepsilon.$$

Second, assume $n < N(1)^2$. Then $n^\varepsilon < N(1)^{2\varepsilon} \leq 2$, so any two-vertex induced subgraph of G is either a clique or an independent set of size at least n^ε. □

A class $\mathcal{C}$ of graphs is *χ-bounded* if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$ and an induced subgraph H of G, we have $\chi(H) \leq f(\omega(H))$, where $\chi(H)$ is the chromatic number of H and $\omega(H)$ is the size of a maximum clique in H. It was proved by Dvořák and Král' [13] that for every p, the class of graphs of rank-width at most p is χ-bounded.

Theorem 19 (Dvořák and Král' [13]). *For each positive integer p, the class of graphs of rank-width at most p is χ-bounded.*

We observe that this fact directly generalizes to classes admitting low rank-width colorings.

Proposition 20. *Every class of graphs that admits low rank-width colorings is χ-bounded.*

Proof. Let $\mathcal{C}$ be the class of graphs in question, let $G \in \mathcal{C}$, and let $q = \omega(G)$. Since $\mathcal{C}$ admits low rank-width colorings, there exist functions $N\colon \mathbb{N} \to \mathbb{N}$ and $R\colon \mathbb{N} \to \mathbb{N}$, depending only on $\mathcal{C}$, such that for all p, G can be colored using $N(p)$ colors so that each induced subgraph of G that receives $i \leq p$ colors has rank-width at most $R(i)$. Let c be such a coloring for $p = 1$, and w.l.o.g. suppose that c uses colors $\{1, \ldots, N(1)\}$. By Theorem 19, there is function $f_{R(1)}$ such that for every graph H of rank-width at most $R(1)$, we have $\chi(H) \leq f_{R(1)}(\omega(H))$.

For $i \in \{1, \ldots, N(1)\}$, let $G_i = G[c^{-1}(i)]$ be the subgraph induced by vertices of color i. Since G_i is an induced subgraph of G, we have that $\omega(G_i) \leq q$, so G_i has a proper coloring c_i using $f_{R(1)}(q)$ colors, say colors $\{1, \ldots, f(q)\}$. Then we can take the product coloring c' of G defined as $c'(u) = (c(u), c_{c(u)}(u))$. Observe that since each coloring c_i is proper, c' is a proper coloring of G. It follows that the chromatic number of G is at most $N(1) \cdot f(q)$, so we can take $f'(q) = N(1) \cdot f_{R(1)}(q)$ as the χ-bounding function for $\mathcal{C}$. □

7 Conclusions

In this work we introduced the concept of low rank-width colorings, and showed that such colorings exist on rth powers of graphs from any bounded expansion class, for any fixed r, as well as on unit interval and bipartite permutation graphs. These classes are non-sparse and have unbounded rank-width. On the negative side, the classes of interval and permutation graphs do not admit low rank-width colorings.

The obvious open problem is to characterise hereditary graph classes which admit low rank-width colorings in the spirit of the characterisation theorem for graph classes admitting low tree-depth colorings. We believe that Theorem 13 may provide some insight into this question, as it shows that containing arbitrarily large twisted chain graphs is an obstacle for admitting low rank-width colorings. Is it true that every hereditary graph class that does not admit low rank-width colorings has to contain arbitrarily large twisted chain graphs?

In this work we did not investigate the question of computing low rank-width colorings, and this question is of course crucial for any algorithmic applications. Our proof for the powers of sparse graphs can be turned into a polynomial-time algorithm that, given a graph G from a graph class of bounded expansion $\mathcal{C}$, first computes a low tree-depth coloring, and then turns it into a low rank-width coloring of G^r, for a fixed constant r. However, we do not know how to efficiently compute a low rank-width coloring given the graph G^r alone, without the knowledge of G. The even more general problem of efficiently constructing an approximate low rank-width coloring of any given graph remains wide open.

Finally, we remark that our proof for the existence of low rank-width colorings on powers of graphs from a class of bounded expansion actually yields a slightly stronger result. Precisely, Ganian et al. [15] introduced a parameter *shrub-depth* (or *SC-depth*), which is a depth analogue of rank-width, in the same way as tree-depth is a depth analogue of tree-width. It can be shown that for constant r, the rth power of a graph of constant tree-depth belongs to a class of constant shrub-depth, and hence our colorings for powers of graphs from a class of bounded expansion are actually low shrub-depth colorings. We omit the details.

References

[1] H. L. Bodlaender, J. S. Deogun, K. Jansen, T. Kloks, D. Kratsch, H. Müller, and Z. Tuza. Rankings of graphs. *SIAM Journal on Discrete Mathematics*, 11(1):168–181, 1998.

[2] B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle. Boolean-width of graphs. *Theoretical Computer Science*, 412(39):5187–5204, 2011.

[3] M. Chudnovsky. The Erdös-Hajnal conjecture—a survey. *Journal of Graph Theory*, 75(2):178–190, 2014.

[4] B. Courcelle. Graph rewriting: An algebraic and logic approach. *Handbook of Theoretical Computer Science, volume B*, pages 193–242, 1990.

[5] B. Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and Computation*, 85(1):12–75, 1990.

[6] B. Courcelle, J. Engelfriet, and G. Rozenberg. Handle-rewriting hypergraph grammars. *Journal of Computer and System Sciences*, 46(2):218–270, 1993.

[7] B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory of Computing Systems*, 33(2):125–150, 2000.

[8] K. Dabrowski and D. Paulusma. Clique-width of graph classes defined by two forbidden induced subgraphs. *Comput. J.*, 59(5):650–666, 2016.

[9] J. S. Deogun, T. Kloks, D. Kratsch, and H. Müller. On vertex ranking for permutation and other graphs. In *STACS 1994*, pages 747–758. Springer, 1994.

[10] M. DeVos, G. Ding, B. Oporowski, D. P. Sanders, B. Reed, P. Seymour, and D. Vertigan. Excluding any graph as a minor allows a low tree-width 2-coloring. *Journal of Combinatorial Theory, Series B*, 91(1):25–41, 2004.

[11] R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate Texts in Mathematics*. Springer, 2012.

[12] Z. Dvořák, D. Král', and R. Thomas. Testing first-order properties for subclasses of sparse graphs. *Journal of the ACM (JACM)*, 60(5):36, 2013.

[13] Z. Dvořák and D. Král'. Classes of graphs with small rank decompositions are χ-bounded. *European Journal of Combinatorics*, 33(4):679–683, 2012.

[14] P. Erdős and A. Hajnal. Ramsey-type theorems. *Discrete Applied Mathematics*, 25(1-2):37–52, 1989.

[15] R. Ganian, P. Hliněný, J. Nešetřil, J. Obdržálek, P. Ossona de Mendez, and R. Ramadurai. When trees grow low: shrubs and fast MSO_1. In *MFCS 2012*, volume 7464 of *Lecture Notes in Computer Science*, pages 419–430. Springer, 2012.

[16] M. Grohe and S. Kreutzer. Methods for algorithmic meta theorems. *Model Theoretic Methods in Finite Combinatorics*, 558:181–206, 2011.

[17] M. Grohe, S. Kreutzer, and S. Siebertz. Deciding first-order properties of nowhere dense graphs. In *STOC 2014*, pages 89–98. ACM, 2014.

[18] F. Gurski and E. Wanke. The NLC-width and clique-width for powers of graphs of bounded tree-width. *Discrete Applied Mathematics*, 157(4):583–595, 2009.

[19] A. Gyárfás. Problems from the world surrounding perfect graphs. *Zastosowania Matematyki (Applicationes Mathematicae)*, 19:413–441, 1987.

[20] P. Hliněný, S. Oum, D. Seese, and G. Gottlob. Width parameters beyond tree-width and their applications. *The Computer Journal*, 51(3):326–362, 2008.

[21] H. A. Kierstead and D. Yang. Orderings on graphs and game coloring number. *Order*, 20(3):255–264, 2003.

[22] V. V. Lozin. Minimal classes of graphs of unbounded clique-width. *Annals of Combinatorics*, 15(4):707–722, 2011.

[23] V. V. Lozin and G. Rudolf. Minimal universal bipartite graphs. *Ars Combinatoria*, 84:345–356, 2007.

[24] J. Nešetřil and P. Ossona de Mendez. Tree-depth, subgraph coloring and homomorphism bounds. *European Journal of Combinatorics*, 27(6):1022–1041, 2006.

[25] J. Nešetřil and P. Ossona de Mendez. Grad and classes with bounded expansion I. Decompositions. *European Journal of Combinatorics*, 29(3):760–776, 2008.

[26] J. Nešetřil and P. Ossona de Mendez. On nowhere dense graphs. *European Journal of Combinatorics*, 32(4):600–617, 2011.

[27] J. Nešetřil and S. Shelah. On the order of countable graphs. *European Journal of Combinatorics*, 24(6):649–663, 2003.

[28] S. Oum. Rank-width is less than or equal to branch-width. *Journal of Graph Theory*, 57(3):239–244, 2008.

[29] S. Oum. Rank-width: Algorithmic and structural results. *Discrete Applied Mathematics*, 2016.

[30] S. Oum and P. Seymour. Personal communication.

[31] S. Oum and P. Seymour. Approximating clique-width and branch-width. *Journal of Combinatorial Theory, Series B*, 96(4):514–528, 2006.

[32] N. Robertson and P. D. Seymour. Graph minors. XVI. Excluding a non-planar graph. *Journal of Combinatorial Theory, Series B*, 89(1):43–76, 2003.

[33] A. A. Schäffer. Optimal node ranking of trees in linear time. *Information Processing Letters*, 33(2):91–96, 1989.

[34] W. T. Trotter. *Combinatorics and partially ordered sets*. Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, 1992.

[35] E. Wanke. k-NLC graphs and polynomial algorithms. *Discrete Applied Mathematics*, 54(2-3):251–266, 1994.

[36] X. Zhu. Colouring graphs with bounded generalized colouring number. *Discrete Mathematics*, 309(18):5562–5568, 2009.