On the Generalised Colouring Numbers of Graphs that Exclude a Fixed Minor

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Abstract

The generalised colouring numbers $\mathrm{col}_r(G)$ and $\mathrm{wcol}_r(G)$ were introduced by Kierstead and Yang as a generalisation of the usual colouring number, and have since then found important theoretical and algorithmic applications.

In this paper, we dramatically improve upon the known upper bounds for generalised colouring numbers for graphs excluding a fixed minor, from the exponential bounds of Grohe *et al.* to a linear bound for the r-colouring number col_r and a polynomial bound for the weak r-colouring number wcol_r. In particular, we show that if G excludes K_t as a minor, for some fixed $t \geq 4$, then $\mathrm{col}_r(G) \leq \binom{t-1}{2}(2r+1)$ and $\mathrm{wcol}_r(G) \leq \binom{r+t-2}{t-2}(t-3)(2r+1) \in O(r^{\,t-1})$.

In the case of graphs G of bounded genus g, we improve the bounds to $\mathrm{col}_r(G) \leq (2g+3)(2r+1)$ (and even $\mathrm{col}_r(G) \leq 5r+1$ if $g=0$, i.e. if G is planar) and $\mathrm{wcol}_r(G) \leq \left(2g+\binom{r+2}{2}\right)(2r+1)$.

Keywords: generalised colouring number, graph minor, graph genus, planar graph, tree-width, tree-depth

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1. Introduction

The *colouring number* $\mathrm{col}(G)$ of a graph G is the minimum integer k such that there is a strict linear order $<_L$ of the vertices of G for which each vertex v has *back-degree* at most $k-1$, i.e. at most $k-1$ neighbours u with $u <_L v$. It is well-known that for any graph G, the chromatic number $\chi(G)$ satisfies $\chi(G) \le \mathrm{col}(G)$.

Some generalisations of the colouring number of a graph have been studied in the literature. These include the *arrangeability* [4] used in the study of Ramsey numbers of graphs, the *admissibility* [15], and the *rank* [14] used in the study of the game chromatic number of graphs. But maybe the most natural generalisation of the colouring number are the two series col_r and wcol_r of *generalised colouring numbers* introduced by Kierstead and Yang [16] in the context of colouring games and marking games on graphs. As proved by Zhu [26], these invariants are strongly related to low tree-depth decompositions [20], and can be used to characterise bounded expansion classes of graphs (introduced in [21]) and nowhere dense classes of graphs (introduced in [22]). For more details on this connection, we refer the interested reader to [18].

The invariants col_r and wcol_r are defined in a way similar to the usual definition of the colouring number: the *r-colouring number* $\mathrm{col}_r(G)$ of a graph G is the minimum integer k such that there is a linear order $<_L$ of the vertices for which each vertex v can reach at most $k-1$ other vertices smaller than v (in the order $<_L$) with a path of length at most r, all internal vertices of which are greater than v. For the *weak r-colouring number* $\mathrm{wcol}_r(G)$, we do not require that the internal vertices are greater than v, but only that they are greater than the final vertex of the path. (Formal definitions will be given in Section 2.) As noticed already in [16], the two types of generalised colouring numbers are related by the inequalities

$$\mathrm{col}_r(G) \le \mathrm{wcol}_r(G) \le (\mathrm{col}_r(G))^r.$$

If we allow paths of any length (but still restrictions on the position of the internal vertices), we get the *∞-colouring number* $\mathrm{col}_\infty(G)$ and the *weak ∞-colouring number* $\mathrm{wcol}_\infty(G)$.

Generalised colouring numbers are an important tool in the context of algorithmic sparse graphs theory. They play a key role for example in the model-checking and enumeration algorithms for first-order logic on bounded expansion and nowhere dense graph classes [8, 13, 11], in Dvořák's linear time approximation algorithm for minimum distance-r dominating sets [7], and in the kernelisation algorithms for distance-r dominating sets [6, 9].

An interesting aspect of generalised colouring numbers is that these invariants can also be seen as gradations between the colouring number $\mathrm{col}(G)$ and two important minor monotone invariants, namely the *tree-width* $\mathrm{tw}(G)$ and the *tree-depth* $\mathrm{td}(G)$ (which is the minimum height of a depth-first search tree for a supergraph of G [20]). More explicitly, for every graph G we have the following relations.

Proposition 1.1.
(a) $\mathrm{col}(G) = \mathrm{col}_1(G) \le \mathrm{col}_2(G) \le \cdots \le \mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1$;
(b) $\mathrm{col}(G) = \mathrm{wcol}_1(G) \le \mathrm{wcol}_2(G) \le \cdots \le \mathrm{wcol}_\infty(G) = \mathrm{td}(G)$.

The equality $\mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1$ was first proved in [10]; for completeness we include the proof in Subsection 2.2. The equality $\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$ is proved in [18, Lemma 6.5].

As tree-width [12] is a fundamental graph invariant with many applications in graph structure theory, most prominently in Robertson and Seymour's theory of graphs with forbidden minors [24], it is no wonder that the study of generalised colouring numbers might be of special interest in the context of proper minor closed classes of graphs. As we shall see, excluding a minor indeed allows us to prove strong upper bounds for the generalised colouring numbers.

Using probabilistic arguments, Zhu [26] was the first to give a non-trivial bound for $\mathrm{col}_r(G)$ in terms of the densities of shallow minors of G. For a graph G excluding a complete graph K_t as a minor, Zhu's bound gives

$$\mathrm{col}_r(G) \le 1 + q_r,$$

where q_1 is the maximum average degree of a minor of G, and q_i is inductively defined by $q_{i+1} = q_1 \cdot q_i^{2i^2}$.

Grohe *et al.* [10] improved Zhu's bounds as follows:

$$\mathrm{col}_r(G) \le (crt)^r,$$

for some (small) constant c depending on t.

Our main results is an improvement of those bounds for the generalised colouring numbers of graphs excluding a minor.

Theorem 1.2.
Let H be a graph and x a vertex of H. Set $h = |E(H-x)|$, and let α be the number of isolated vertices of $H - x$. Then for every graph G that excludes H as a minor, we have

$$\mathrm{col}_r(G) \ \le \ h \cdot (2r+1) + \alpha.$$

For classes of graphs that are defined by excluding a complete graph K_t as a minor, we get the following special result.

Corollary 1.3.
For every graph G that excludes the complete graph K_t as a minor, we have

$$\mathrm{col}_r(G) \ \le \ \binom{t-1}{2} \cdot (2r+1).$$

For the weak r-colouring numbers we obtain the following bound.

Theorem 1.4.
Let $t \ge 4$. For every graph G that excludes K_t as a minor, we have

$$\mathrm{wcol}_r(G) \ \le \ \binom{r+t-2}{t-2} \cdot (t-3)(2r+1) \in O(r^{t-1}).$$

We refrain from stating a bound on the weak r-colouring numbers in the case that a general graph H is excluded as minors for conceptual simplicity. It will be clear from the proof that if a proper subgraph of K_t is excluded, the bounds can be slightly improved. Those improvements, however, will only be linear in t.

The *acyclic chromatic number* $\chi_a(G)$ of a graph G is the smallest number of colours needed for a proper vertex-colouring of G such that every cycle has at least three colours. The best known upper bound for the acyclic chromatic number of graphs without a K_t-minor is $O(t^2 \log^2 t)$, implicit in [19]. Kierstead and Yang [16] gave a short prove that $\chi_a(G) \le \mathrm{col}_2(G)$. Corollary 1.3 shows that for graphs G without a K_t-minor we have $\mathrm{col}_2(G) \in O(t^2)$, which immediately gives an improved $O(t^2)$ upper bound for the acyclic chromatic number of those graphs as well.

In the particular case of graphs with bounded genus, we can improve our bounds further.

Theorem 1.5.
For every graph G with genus g, we have $\mathrm{col}_r(G) \le (4g+5)r + 2g + 1$.
In particular, for every planar graph G, we have $\mathrm{col}_r(G) \le 5r + 1$.

Theorem 1.6.
For every graph G with genus g, we have $\mathrm{wcol}_r(G) \le \left(2g + \binom{r+2}{2}\right) \cdot (2r+1)$.

In particular, for every planar graph G, we have $\mathrm{wcol}_r(G) \le \binom{r+2}{2} \cdot (2r+1)$.

For planar graphs, the bound on $\mathrm{col}_1(G) = \mathrm{wcol}_1(G) = \mathrm{col}(G)$ is best possible. Also for $t = 2, 3$ and $r = 1$ one can easily give best possible bounds, as expressed in the following observations.

Proposition 1.7.

(a) For every graph G that excludes K_2 as a minor, we have $\mathrm{col}_r(G) = \mathrm{wcol}_r(G) = 1$.

(b) For every graph G that excludes K_3 as a minor, we have $col_r(G) \le 2$ and $\mathrm{wcol}_r(G) \le r + 1$.

(c) For every graph G that excludes K_t as a minor, $t \ge 4$, we have

$$\mathrm{col}_1(G) = \mathrm{wcol}_1(G) \le (0.64 + o(1))\, t\sqrt{\ln t} + 1 \quad (|V(G)| \to \infty).$$

Part (a) in the proposition is a triviality. For part (b), note that excluding K_3 as a minor means that G is acyclic, hence a forest, and that in this case it is obvious that $\mathrm{col}_r(G) \le 2$ and $\mathrm{wcol}_r(G) \le r + 1$. Finally, $\mathrm{col}_1(G) = \mathrm{wcol}_1(G)$ is one more than the degeneracy of G, thus part (c) follows from Thomason's bound for the average degree of graphs with no K_t as a minor [25].

Regarding the sharpness on our upper bounds in the results above, we can make the following remarks.

- Lower bounds for the generalised colouring numbers for minor closed classes are given in [10]. In that paper it is shown that for every k and every r there is a graph $G_{k,r}$ of tree-width k that satisfies $\mathrm{col}_r(G_{k,r}) = k + 1$ and $\mathrm{wcol}_r(G_{k,r}) = \binom{r+k}{k}$. Graphs of tree-width k exclude K_{k+2} as a minor. This shows that our results for classes with excluded minors are optimal up to a factor $(t-1)(2r+1)$.
- Since graphs with tree-width 2 are planar, this also shows that there exist planar graphs G with $\mathrm{wcol}_r(G) = \binom{r+2}{2} \in \Omega(r^2)$. Compare this to the upper bound $\mathrm{wcol}_r(G) \in O(r^3)$ for planar graphs in Theorem 1.6.
- It follows from Proposition 1.1 (a) that a minor closed class of graphs has uniformly bounded colouring number if and only if it has bounded tree-width. For classes with unbounded tree-width, such a uniform bound cannot be expected. By analysing the shape of admissible paths, it is possible to prove that the planar $r \times r$ grid $G_{r\times r}$ satisfies $\mathrm{col}_r(G_{r\times r}) \in \Omega(r)$. This shows that for planar graphs G, a best possible bound for $\mathrm{col}_r(G)$ will be linear in r.
- It follows from [26, Lemma 3.3] that for 3-regular graphs of high girth the weak r-colouring numbers grow exponentially with r. Hence the polynomial bound for $\mathrm{wcol}_r(G)$ in Theorem 1.2

for classes with excluded minors cannot be extended to classes with bounded degree, or even to classes with excluded topological minors.

The structure of this paper is as follows. In the next section we give necessary definitions, and prove the connections between the generalised colouring numbers and tree-width. In Section 3 we introduce *flat decompositions*, which is our main tool in proving our results, and give an upper bound for the minimum width of a flat decomposition of a graph excluding a complete minor. In Section 4 we prove Theorem 1.4 and in Section 5 we prove Theorem 1.2. Our proofs will rely on the notion of the *elimination-width* of a vertex-order $<_L$, and its connection to weak colouring, stated as Theorem 2.1, which was proved in [10]. In Section 6 we prove Theorems 1.5 and 1.6, which have a detailed analysis of the generalised colouring numbers of planar graphs at their base.

2. Preliminaries

All graphs in this paper are finite, undirected and simple, that is, they do not have loops or multiple edges between the same pair of vertices. For a graph G, we denote by $V(G)$ the vertex set of G and by $E(G)$ its edge set.

The *distance* between a vertex v and a vertex w is the length (that is, the number of edges) of a shortest path between v and w. For a vertex v of G, we write $N^G(v)$ for the set of all neighbours of v, $N^G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$, and for $r \in \mathbb{N}$ we denote by $N_r^G[v]$ the *closed r-neighbourhood of* v, that is, the set of vertices of G at distance at most r from v. Note that we always have $v \in N_r^G[v]$. When no confusion can arise regarding the graph G we are considering, we usually omit the superscript G.

Let M be a graph with vertices $h_1, \ldots, h_n$. The graph M is a *minor* of a graph G if in G there are disjoint connected subgraphs $H_1, \ldots, H_n$ such that if $\{h_i, h_j\}$ is an edge of M, then H_i is connected to H_j (in G). We call the subgraphs $H_1, \ldots, H_n$ of G a *model* of M in G.

2.1. Generalised Colouring Numbers

Let $\Pi(G)$ be the set of all linear orders of the vertices of the graph G, and let $L \in \Pi(G)$. For readability, we write $u <_L v$ if u is smaller than v with respect to L, and $u \leq_L v$ if $u <_L v$ or $u = v$.

Let $u, v \in V(G)$. For a positive integer r, we say that u is *weakly r-reachable* from v with respect to L, if there exists a path P of length ℓ, $0 \leq \ell \leq r$, between u and v such that u is minimum among the vertices of P (with respect to L). Let $\mathrm{WReach}_r[G, L, v]$ be the set of vertices that are weakly r-reachable from v with respect to L. Note that $v \in \mathrm{WReach}_r[G, L, v]$.

If we allow paths of any length, then we call u *weakly reachable* from v with respect to L, and the set of such vertices is denoted by $\mathrm{WReach}_\infty[G, L, v]$

Next, u is *strongly r-reachable* from v with respect to L, if there is a path P of length ℓ, $0 \leq \ell \leq r$, connecting u and v such that $u \leq_L v$ and such that all inner vertices w of P satisfy $v <_L w$. Let $\mathrm{SReach}_r[G, L, v]$ be the set of vertices that are strongly r-reachable from v with respect to L. Note that again we have $v \in \mathrm{SReach}_r[G, L, v]$.

Again, if we allow paths of any length, then we say that u is *strongly reachable* from v, and the collection of all such vertices is denoted $\mathrm{SReach}_\infty[G, L, v]$.

For $r \in \mathbb{N} \cup \{\infty\}$, the *weak r-colouring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} \left|\mathrm{WReach}_r[G, L, v]\right|,$$

and the *r-colouring number* $\mathrm{col}_r(G)$ of G is defined as

$$\mathrm{col}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} \left|\mathrm{SReach}_r[G, L, v]\right|.$$

2.2. Tree-width and elimination-width

The concept of tree-width has shown itself to be very useful for the design of efficient graph algorithms. Many NP-hard problems are fixed-parameter tractable when parametrised by the tree-width of the input graph. A very general theorem due to Courcelle [5] states that every problem definable in monadic second-order logic can be solved in linear time on a class of graphs of bounded tree-width.

The most common definition of tree-width is in terms of tree-decompositions. A *tree-decomposition* of a graph G is a pair $(T,(X_t)_{t\in V(T)})$, where T is a tree and $X_t \subseteq V(G)$ for each $t \in V(T)$, such that

(1) $\bigcup_{t\in V(T)} = V(G)$;
(2) for every edge $\{u,v\} \in E(G)$, there is a $t \in V(T)$ such that $u,v \in X_t$; and
(3) if $v \in X_t \cap X_{t'}$ for some $t,t' \in V(T)$, then $v \in X_{t''}$ for all t'' that lie on the unique path between t and t' in T.

The width of a tree-decomposition is $\max_{t\in V(T)} |X_t| - 1$, and the *tree-width of* G is equal to the smallest width of any tree-decomposition of G.

For a linear order $L \in \Pi(G)$, the *fill-in of* G *with respect to* L is the graph G_L obtained by inductively adding for each vertex v (starting with the largest vertex of the order) an edge $\{u,w\}$ for all $u,w \in N(v)$, $u \neq w$, with $u <_L v$ and $w <_L v$. An equivalent definition of G_L would be the graph obtained by making each vertex v adjacent to all the vertices smaller than v (with respect to L) than can be reached from v in G by a path whose internal vertices are greater than v. The *elimination-width* of an order L is the size of the largest clique in G_L minus 1 (i.e. equal to $\omega(G_L) - 1$, where $\omega(G)$ is the clique number of a graph G).

It is not so hard to prove (see, e.g., [3, Theorem 3.1]) that the tree-width of G is equal to the minimum elimination-width over all orders of $V(G)$:

$$\mathrm{tw}(G) = \min_{L\in\Pi(G)} \omega(G_L) - 1.$$

On the other hand, $\omega(G_L) - 1$ obviously is equal to the maximum over all vertices v in G of the number of vertices smaller than v that can be reached from v by a path whose internal vertices are greater than v. (The largest clique in G_L also includes v itself, which is counted for $\mathrm{col}_\infty(G)$, but not for $\mathrm{tw}(G)$.) This shows that $\mathrm{col}_\infty(G) = \mathrm{tw}(G) + 1$, as was claimed earlier.

We also have that elimination-width is related to weak reachability, as the next result shows.

Theorem 2.1 (Grohe *et al.* [10])**.**
Let G be a graph and let $L \in \Pi(G)$ be a linear order of $V(G)$ with elimination-width at most k. For all $r \in \mathbb{N}$ and all $v \in V(G)$, we have

$$\left|\mathrm{WReach}_r[G,L,v]\right| \leq \binom{r+k}{k}.$$

3. Flat decompositions

Our main tool in proving our results will be flat decompositions, which we introduce now.

Let G be a graph, let $H \subseteq G$ be a subgraph of G, and let $f\colon \mathbb{N} \to \mathbb{N}$ be a function. We say that H *f-spreads on* G if, for every $r \in \mathbb{N}$ and $v \in V(G)$, we have

$$|N_r^G[v] \cap V(H)| \leq f(r).$$

Let H, H' be vertex-disjoint subgraphs of G. We say that H is *connected* to H' if some vertex in H has a neighbour in H', i.e. if there is an edge $\{u, v\} \in E(G)$ such $u \in V(H)$ and $v \in V(H')$.

Definition 3.1.
A *decomposition* of a graph G is a sequence $\mathcal{H} = (H_1, \dots, H_\ell)$ of non-empty subgraphs of G such that the vertex sets $V(H_1), \dots, V(H_\ell)$ partition $V(G)$. The decomposition $\mathcal{H}$ is *connected* if each H_i is connected.

For a decomposition $(H_1, \dots, H_\ell)$ of a graph G and $1 \le i \le \ell$, we denote by $G[H_{\ge i}]$ the subgraph of G induced by $\bigcup_{i \le j \le \ell} V(H_j)$.

Definition 3.2.
We call the decomposition $\mathcal{H}$ *f-flat* if each H_i f-spreads on $G[H_{\ge i}]$.

A *flat decomposition* is a decomposition that is f-flat for some function $f\colon \mathbb{N} \to \mathbb{N}$.

Definition 3.3.
Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a decomposition of a graph G, let $1 \le i < \ell$, and let C be a component of $G[H_{\ge(i+1)}]$. The *separating number of* C is the maximal number s of (distinct) graphs $Q_1, \dots, Q_s \in \{H_1, \dots, H_i\}$ such that all the Q_j's are connected to C.

Note that the separating number of a component C is independent of the value i such that C is a component of $G[H_{\ge(i+1)}]$. Indeed, let i be minimal such that C is a component of $G[H_{\ge(i+1)}]$. Then for all $t > i$ we have that either H_t is not connected to C, or H_t is a subgraph that contains vertices from C.

Definition 3.4.
Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a decomposition of a graph G. The *width* of $\mathcal{H}$ is the maximum separating number of a component C of $G[H_{\ge i}]$, maximised over all i, $1 \le i < \ell$.

We call a path P in G an *isometric path* if P is a shortest path between its endpoints. Isometric paths will play an important role in the analysis of flat decompositions and the generalised colouring numbers. We call a flat decomposition $\mathcal{H} = (H_1, \dots, H_\ell)$ an *isometric paths decomposition* if each H_i is an isometric path in $G[H_{\ge i}]$.

A definition similar to isometric paths decompositions is given in [1], where they are called *cop-decompositions*. The name *cop-decomposition* in [1] is inspired by a result of [2], which shows that such decompositions of small width exist for classes of graphs that exclude a fixed minor, and which uses a cops-and-robber game argument. The difference between a cop-decomposition and a connected decomposition is that in a connected decomposition we allow arbitrary connected subgraphs rather than just paths as in a cop-decomposition

The property of having a partition into connected subgraphs with the above width properties is extremely useful, as it allows us to contract the subgraphs to find a minor of G with bounded tree-width, as expressed in the following lemma.

Lemma 3.1.
Let G be a graph, and let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a connected decomposition of G of width k. By contracting each connected subgraph H_i to a single vertex, we obtain a graph $H = G/\mathcal{H}$ with ℓ vertices and tree-width at most k.

Proof. We identify the vertices of H with the connected subgraphs $\{H_1, \dots, H_\ell\}$. By the contracting operation, two subgraphs H_i, H_j are adjacent in H if there is an edge in G between a vertex of H_i and a vertex of H_j, and there is a path $H_i, H_{i+1}, \ldots, H_j$ in H if and only if there is a path between some vertex of H_i and some vertex of H_j that uses only vertices of $H_i, H_{i+1}, \ldots, H_j$, in that order.

Let L be the order of $V(H)$ given by the order of the subgraphs in the connected decomposition. Consider the graph H_L, the fill-in of H with respect to L. For any vertex H_i of H, the set of neighbours of H_i in H_L that are smaller than H_i (with respect to L) is the set of subgraphs among $H_1, \ldots, H_{i-1}$ that are reachable via a path (in H) with internal vertices larger than H_i. As each such path corresponds to a path in G as described above, this is exactly the set of subgraphs in $\{H_1, \ldots, H_{i-1}\}$ that are reachable in G from the component C of $G[H_{\geq i}]$ that contains H_i. The number of such subgraphs is the separating number of C, which by definition of the width of $\mathcal{H}$ is as most k. Since H_i is also strongly reachable from itself, we see that $\left|\mathrm{SReach}_\infty[H, L, H_i]\right| \leq k+1$ for all $H_i \in V(H)$. This shows that $\mathrm{tw}(H) + 1 = \mathrm{col}_\infty(H) \leq k+1$, as required. □

A fundamental property of isometric paths is that from any vertex v, not many vertices of an isometric path can be reached from v in r steps.

Lemma 3.2.
Let v be a vertex of a graph G, and let P be an isometric path in G. Then P contains at most $2r+1$ vertices of the closed r-neighbourhood of v: $|N_r[v] \cap V(P)| \leq \min\{\,|V(P)|,\, 2r+1\,\}$.

Proof. Assume $P = v_0, \ldots, v_n$ and $\left|N_r[v] \cap V(P)\right| > 2r+1$. Let i be minimal such that $v_i \in N_r[v]$ and let j be maximal such that $v_j \in N_r[v]$. As P is a shortest path, the distance in G between v_i and v_j is $j - i \geq \left|N_r[v] \cap V(P)\right| - 1 > 2r$, which contradicts the hypothesis that both v_i and v_j are at distance at most r from v, thus at distance at most $2r$ from each other. □

From a decomposition $(H_1, \ldots, H_\ell)$ of a graph G, we define a linear order L on $V(G)$ as follows. First choose an arbitrary linear order on the vertices of each subgraph H_i. Now let L be the linear extension of that order where for $v \in V(H_i)$ and $w \in V(H_j)$ with $i < j$ we define $L(v) < L(w)$.

Lemma 3.3.
Let $\mathcal{H} = (H_1, \ldots, H_\ell)$ be a decomposition of a graph G, and let L be an order defined from the decomposition. For an integer i, $1 \leq i \leq \ell$, let $G' = G[H_{\geq i}]$. Then we have for every $r \in \mathbb{N}$ and every $v \in V(G)$:

$$\begin{aligned}\mathrm{SReach}_r[G, L, v] \cap V(H_i) &\subseteq N_r^{G'}[v] \cap V(H_i),\\ \mathrm{WReach}_r[G, L, v] \cap V(H_i) &\subseteq N_r^{G'}[v] \cap V(H_i).\end{aligned}$$

Proof. If a path P with one endpoint v visits a vertex that is smaller than a vertex of H_i, then the path cannot be continued to weakly or strongly visit a vertex of H_i. □

Now we are in a position to give upper bounds of $\mathrm{col}_r(G)$ and $\mathrm{wcol}_r(G)$ in terms of the width of a flat decomposition.

Lemma 3.4.
Let $f\colon \mathbb{N} \to \mathbb{N}$ and let $r, k \in \mathbb{N}$. Let G be a graph that admits an f-flat decomposition of width k. Then we have

$$\mathrm{col}_r(G) \;\leq\; (k+1) \cdot f(r).$$

Proof. Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be an f-flat decomposition of G of width k, and let L be a linear order defined from the decomposition. Let $v \in V(G)$ be an arbitrary vertex and choose q such that $v \in V(H_{q+1})$. Let C be the component of $G[H_{\geq(q+1)}]$ that contains v, and let $Q_1, \dots, Q_m$, $1 \leq m \leq q$, be the subgraphs among $H_1, \dots, H_q$ that have a connection to C. Since $\mathcal{H}$ has width k, we have $m \leq k$. By definition of L, the vertices in $\mathrm{SReach}_r[G, L, v]$ can only lie on $Q_1, \dots, Q_m$ and on H_{q+1}, hence on at most $k+1$ subgraphs. For $j = 1, \dots, m$, assume that $Q_j = H_{i_j}$ and let $G'_j = G[H_{\geq i_j}]$. Then by Lemma 3.3 we have $\mathrm{SReach}_r[G, L, v] \cap Q_j \subseteq N_r^{G'_j}[v] \cap Q_j$. Since $H_{i_j} = Q_j$ f-spreads on G'_j, we have $\left|N_r^{G'_j}[v] \cap Q_j\right| \leq f(r)$. The result follows. □

Lemma 3.5.
Let $f\colon \mathbb{N} \to \mathbb{N}$ and let $r, k \in \mathbb{N}$. Let G be a graph that admits a connected f-flat decomposition of width k. Then we have

$$\mathrm{wcol}_r(G) \;\leq\; \binom{r+k}{k} \cdot f(r).$$

Proof. Let $\mathcal{H} = (H_1, \dots, H_\ell)$ be a connected f-flat decomposition of width k, and let L be a linear order defined from it. We contract the subgraphs $H_1, \dots, H_\ell$ to obtain a graph H of tree-width at most k (see Lemma 3.1). We identify the vertices of H with the subgraphs H_i. For a vertex $v \in V(G)$, consider the subgraph H_i with $v \in V(H_i)$. By Theorem 2.1, the vertex H_i weakly r-reaches at most $\binom{r+k}{k}$ vertices in H that are smaller than or equal to H_i in the order on $V(H)$ induced by L. These vertices H_j that are weakly r-reachable from H_i in H are the only subgraphs in G that may contain vertices that are weakly r-reachable from v in G. We conclude that there are at most $\binom{r+k}{k}$ subgraphs among $H_1, \dots, H_\ell$ in G that contain vertices that are weakly r-reachable from v. As in the previous proof we can argue that there are at most $f(r)$ weakly r-reachable vertices on each subgraph, which completes the proof. □

4. The weak r-colouring numbers of graphs excluding a fixed complete minor

In this section we prove Theorem 1.4. We will provide a more detailed analysis for the r-colouring numbers in the next section.

Theorem (Theorem 1.4)
Let $t \geq 4$. For every graph G that excludes K_t as a minor, we have

$$\mathrm{wcol}_r(G) \;\leq\; \binom{r+t-2}{t-2} \cdot (t-3)(2r+1) \in O(r^{t-1}).$$

Theorem 1.4 is a direct consequence of Lemma 3.5 and of Lemma 4.1 below. This lemma states that connected flat decompositions of small width exist for graphs that exclude a fixed complete graph K_t as a minor. This result is inspired by the result on cop-decompositions presented in [2].

Lemma 4.1.
Let $t \geq 4$ and let $f\colon \mathbb{N} \to \mathbb{N}$ be the function $f(r) = (t-3)(2r+1)$. Let G be a graph that excludes K_t as a minor. Then there exists a connected f-flat decomposition of G of width at most $t-2$.

Proof. Without loss of generality we may assume that G is connected. We will iteratively construct a connected f-flat decomposition $H_1, \dots, H_\ell$ of G. For all q, $1 \leq q < \ell$, we will maintain the following invariant. Let C be a component of $G[H_{\geq(q+1)}]$. Then the subgraphs $Q_1, \ldots, Q_s \in \{H_1, \ldots, H_q\}$ that are connected to C form a minor model of the complete graph K_s, for some $s \leq t-2$. This will immediately imply our claim on the width of the decomposition.

To start, we choose an arbitrary vertex $v \in V(G)$ and let H_1 be the connected subgraph $G[v]$. Clearly, H_1 f-spreads on G, and the above invariant holds (with $s = 1$).

Now assume that for some q, $1 \leq q \leq \ell - 1$, the sequence $H_1, \ldots, H_q$ has already been constructed. Fix some component C of $G[H_{\geq(q+1)}]$ and assume that the subgraphs $Q_1, \ldots, Q_s \in \{H_1, \ldots, H_q\}$ that have a connection to C form a minor model of K_s, for some $s \leq t-2$. Because G is connected, we have $s \geq 1$. Let v be a vertex of C that is adjacent to a vertex of Q_1. Let T be a breadth-first search tree in $G[C]$ with root v. We choose H_{q+1} to be a minimal connected subgraph of T that contains v and that contains for each i, $1 \leq i \leq s$, at least one neighbour of Q_i.

It is easy to see that for every component C' of $G[H_{\geq(q+2)}]$, the subgraphs $Q_1, \ldots, Q_{s'} \in \{H_1, \ldots, H_{q+1}\}$ that are connected to C' form a minor model of a complete graph $K_{s'}$, for some $s' \leq t-1$. Let us show that in fact we have $s' \leq t-2$. Towards a contradiction, assume that there are $Q_1, \ldots, Q_{t-1} \in \{H_1, \ldots, H_{q+1}\}$ that have a connection to C' and such that the Q_i form a minor model of K_{t-1}. As each Q_i has a connection to C', we can contract the whole component C' to find K_t as a minor, a contradiction.

Let us finally show that the decomposition is f-flat. We show that the newly added subgraph H_{q+1} f-spreads on $G[H_{\geq(q+1)}]$. By construction, H_{q+1} is a subtree of T that consists of at most $t-3$ isometric paths in $G[H_{\geq(q+1)}]$ (possibly not disjoint), since T is a breadth-first search tree and v is already a neighbour of Q_1. Now the claim follows immediately from Lemma 3.2. □

5. The r-colouring numbers of graphs excluding a fixed minor

For graphs that exclude a complete graph as a minor, we already get a good bound on the strong r-colouring numbers. However, if a sparse graph is excluded, we can do much better. In this case we will construct an isometric paths decomposition, where only few paths are separating (in general, each connected subgraph in our proof may subsume many isometric paths).

The proof idea is essentially the same as that for Lemma 4.1. We will iteratively construct an isometric paths decomposition $(P_1, \ldots, P_\ell)$ of G such that the components C of $G[P_{\geq(q+1)}]$ are separated by a minor model of a proper subgraph M of $H - x$. To optimise the bounds on the width of the decomposition, we will first try to maximise the number of edges in the subgraph M, before we add more vertices to the model. During the construction we will have to *re-interpret* the separating minor model, as otherwise connections of a vertex model (the subgraph representing a vertex of M) to the component may be lost.

To implement the above mentioned re-interpretation of the minor model it will be more convenient to work with a slightly different (and non-standard) definition of a minor model. Let M be a graph with vertices $h_1, \ldots, h_n$. The graph M is a minor of G if there are pairwise in G disjoint connected subgraphs $H_1, \ldots, H_n$ and pairwise internally disjoint paths E_{ij} for $\{h_i, h_j\} \in E(M)$ that are also internally disjoint from the $H_1, \ldots, H_n$, such that if $e_{ij} = \{h_i, h_j\}$ is an edge of M, then E_{ij} connects a vertex of H_i with a vertex of H_j. We call the subgraph H_i of G the *model* of h_i in G and the path E_{ij} the *model* of e_{ij} in G.

One can easily see that a graph H is a minor of a graph G according to the definition in Section 2 if and only if H is a minor of G according to the definition given above. The reason to introduce paths E_{ij} (rather than edges e_{ij}) is that we want to control the number of vertices in vertex models connected to a component. This is impossible for the connecting paths E_{ij}, so it would be impossible if we let the vertex models grow to encompass the E_{ij}.

Lemma 5.1 (following [2]).
Let H be a graph and x a vertex of H. Set $h = |E(H - x)|$, and let α be the number of isolated vertices of $H - x$. Then every graph G that excludes H as a minor admits an isometric paths decomposition of width at most $3h + \alpha$.

Proof. Without loss of generality we may assume that G is connected. Assume $H - x$ has vertices $h_1, \dots, h_k$, $k = |V(H)| - 1$. For $1 \le i \le k$, denote by d_i the degree of h_i in $H - x$.

We will iteratively construct an isometric paths decomposition $(P_1, \dots, P_\ell)$ of G. For all q, $1 \le q < \ell$, we will maintain the four invariants given below. With each component C of $G[P_{\ge(q+1)}]$ we associate a minor model of a proper subgraph M of $H - x$.

1. For $h_i \in V(M)$, the models H_i of h_i in G use vertices of $P_1, \dots, P_q$ only.
2. For each H_i with $h_i \in V(M)$ such that h_i is an isolated vertex in $H - x$, H_i will consist of a single vertex only.
   For each H_i with $h_i \in V(M)$ such that h_i is not an isolated vertex in $H - x$, it is possible to place a set of d_i pebbles $\{\, p_{ij} \mid \{h_i, h_j\} \in E(H - x) \,\}$ on the vertices of H_i (with possibly several pebbles on a vertex), in such a way that the pebbles occupy exactly the set of vertices of H_i with a neighbour in C. In particular, each H_i has between 1 and d_i vertices with a neighbour in C.
3. For each edge $e_{ij} = \{h_i, h_j\} \in E(M)$, the model E_{ij} of e_{ij} in G has the following properties.
   (a) The endpoints of E_{ij} are the vertices with pebbles p_{ij} in H_i and p_{ji} in H_j.
   (b) The internal vertices of E_{ij} belong to a single path P_p, where $p \le q$.
   (c) Assume E_{ij} has internal vertices in P_p. Let D be the component of $G[P_{\ge p}]$ that contains P_p. Let v_{ij} and v_{ji} be the vertices of H_i and H_j, respectively that are pebbled with p_{ij} and p_{ji} (at the time P_p was defined). Then E_{ij} is an isometric path in $G[D \cup \{v_{ij}, v_{ji}\}] - e_{ij}$. (This condition is not necessary for the proof of the lemma; it will be used in the proof of Theorem 1.2, though.)
4. All vertices on a path of $P_1, \dots, P_q$ that have a connection to C are part of the minor model.

Let us first see that maintaining these invariants implies that the isometric paths decomposition has the desired width. By Condition 4, the separating number of the component C is determined by the number of isometric paths that are part of the minor model of M and have a connection to C. To count this number of paths, we count the number m_1 of paths that lie in any vertex model H_i for $h_i \in V(M)$ and have a connection to C, and we count the number m_2 of paths that correspond to the edges e_{ij} of M. By Condition 2, m_1 is at most the number of pebbles in H plus the number of isolated vertices of $H - x$. Since the number of pebbles of each model H_i is at most d_i, the number of pebbles is at most the sum of the vertex degrees, and therefore $m_1 \le 2|E(H - x)| + \alpha$. By Condition 3(b), m_2 is at most $|E(H - x)|$. Finally, since M is a proper subgraph of $H - x$, either $m_1 < 2|E(H - x)| + \alpha$ or $m_2 < |E(H - x)|$ and hence we have $m_1 + m_2 < 3|E(H - x)| + \alpha$.

We show how to construct an isometric paths decomposition with the desired properties. To start, we choose an arbitrary vertex $v \in V(G)$ and let P_1 be the path of length 0 consisting of v only. For every connected component of $G - V(P_1)$, we define M as the single vertex graph K_1 and the model H_1 of this vertex as P_1. All pebbles are placed on v. As G is connected, we see that Condition 4 is satisfied; all other invariants are clearly satisfied.

Now assume that for some q, $1 \le q \le \ell - 1$, the sequence $P_1, \dots, P_q$ has already been constructed. Fix some component C of $G[P_{\ge(q+1)}]$ and assume that the pebbled minor model of a proper subgraph $M \subsetneq H - x$ with the above properties for C is given. We first find an isometric path P_{q+1} that lies completely inside C and add it to the isometric paths decomposition. The exact choice of P_{q+1} depends on which of the following two cases we are in.

Case 1: There is a pair h_i, h_j of non-adjacent vertices in M such that $\{h_i, h_j\} \in E(H - x)$.
By Condition 2, the pebbles p_{ij} and p_{ji} lie on some vertices v_{ij} of H_i and v_{ji} of H_j, respectively, that have a neighbour in C. Let v_i and v_j be vertices of C with $\{v_{ij}, v_i\}, \{v_{ji}, v_j\} \in E(G)$ (possibly $v_i = v_j$) such that the distance between v_i and v_j in C is minimum among all possible neighbours of v_{ij} and v_{ji} in C. We choose P_{q+1} as an arbitrary shortest path in C with endpoints v_i and v_j. We add the edge $\{h_i, h_j\}$ to M and the path $E_{ij} = \{v_{ij}, v_i\} + P_{q+1} + \{v_j, v_{ji}\}$ to the model of M.

Case 2: M is an induced subgraph of $H - x$.
We choose an arbitrary vertex $v \in V(C)$ and define P_{q+1} as the path of length 0 consisting of v only. We add an isolated vertex h_a to M, for some a with $1 \leq a \leq k$, such that h_a was not already a vertex of M and define $H_a = P_{q+1}$, with any pebbles on v.

Because in both cases the new path P_{q+1} lies completely in C, every other component of $G[P_{\geq(q+1)}]$ (and its respective minor model) is not affected by this path. Therefore, it suffices to show how to find a pebbled minor model with the above properties for every component of $C - V(P_{q+1})$. Let C' be such a component and let M be the proper subgraph of $H - x$ associated with C. We show how to construct from M a graph M' and a corresponding minor model with the appropriate properties for C'. Note that the vertex model H_a added in Case 2 automatically satisfies Conditions 1 and 2.

We iteratively re-establish the properties for the vertex models H_i with $h_i \in V(M)$, in any order. Fix some i with $h_i \in M$ and consider a path E_{ij} such that the vertex $v_{ij} \in V(H_i)$ that is pebbled by p_{ij} has no connection to C'. Let $E_{ij} = w_1, \ldots, w_s$, where $w_1 = v_{ij}$. Let a be minimal such that w_a has a connection to C', or let $a = s - 1$ if no such vertex exists on E_{ij}. We add all vertices $w_1, \ldots, w_a$ to H_i. If w_a has a connection to C', we redefine E_{ij} as the path $w_a, \ldots, w_s$ and place the pebble p_{ij} on w_a. If w_a has no connection to C', we delete the edge $\{h_i, h_j\}$ from M'. If after fixing every path E_{ij} for H_i in the above way, H_i has no connections to C', we delete h_i from M'. Otherwise, if there are pebbles that do not lie on a vertex with a connection to C', we place these pebbles on arbitrary vertices that are occupied by another pebble, that is, that have a connection to C'.

After performing these operations for every H_i, all conditions are satisfied. Condition 2 is re-established for every H_i: if h_i is not removed from M', then every pebble that lies on a vertex that has no connection to C' is pushed along a path until it lies on a vertex that does have a connection to C', or finally, if there is no such connection on the path that it guards, it is placed at an arbitrary vertex that has a connection to C'. The operations on H_i also re-establish Condition 3(a) for one endpoint of E_{ij}. And after the operations are performed on H_j, Condition 3(a) is re-established for E_{ij}. Furthermore, if C' does not have a connection to a vertex model H_i, it may clearly be removed without violating Condition 4. All other conditions are clearly satisfied.

It remains to show that the graph M for a component C is always a proper subgraph of $H - x$. This however is easy to see. Assume that $M = H - x$ and all conditions are satisfied. By Condition 2, every H_i, $1 \leq i \leq k$, has a connection to C. Then, by adding C as a subgraph H_{k+1} to the minor model, we find H as a minor, a contradiction. □

Theorem (Theorem 1.2)
Let H be a graph and x a vertex of H. Set $h = |E(H - x)|$, and let α be the number of isolated vertices of $H - x$. Then for every graph G that excludes H as a minor, we have

$$\mathrm{col}_r(G) \leq h \cdot (2r + 1) + \alpha.$$

Proof. We strengthen the analysis in the proof of Lemma 3.4 by taking into account the special properties of the isometric paths decomposition constructed in the proof of Lemma 5.1.

Let $\mathcal{P} = (P_1, \dots, P_\ell)$ be an isometric paths decomposition of G that was constructed as in the proof of Lemma 5.1, and let L be an order defined from the decomposition. Let $v \in V(G)$ be an arbitrary vertex and choose q such that $v \in V(P_{q+1})$. Let C be the component of $G[P_{\geq(q+1)}]$ that contains v, and let $Q_1, \dots, Q_m$, $1 \leq m \leq q$, be the paths among $P_1, \dots, P_q$ that have a connection to C. By definition of L, the vertices in $\mathrm{SReach}_r[G, L, v]$ can only lie on $Q_1, \dots, Q_m$ and on P_{q+1}.

In the proof of Lemma 5.1, we associated with the component C a pebbled minor model of a proper subgraph M of $H - x$. The paths $Q_1, \dots, Q_m$ were either associated with a vertex model H_i representing a vertex h_i of M, or with a path E_{ij} representing an edge e_{ij} of M. Just as in the proof of Lemma 3.4, we can argue that $\left|\mathrm{SReach}_r[G, L, v] \cap Q_j\right| \leq \min\{\, |V(Q_j)|, 2r + 1 \,\}$ for each path Q_j. However, the paths that lie inside a vertex model H_i can have only as many connections to C as there are pebbles on it, since, by Condition 2 of the proof of Lemma 5.1, every connection of H_i to C must be pebbled. Let q be the number of paths E_{ij} that have vertices connected to C and in the r-neighbourhood of v. By Condition 3(c) from the proof, for every such path E_{ij} with endpoints v_i and v_j, the pebbles p_{ij} and p_{ji} lie on vertices v_{ij} and v_{ji} such that the path $E'_{ij} = \{v_{ij}, v_i\} + E_{ij} + \{v_j, v_{ji}\}$ is isometric. Thus $N_r[v]$ meets only at most h many paths E'_{ij}. It follows form Lemma 3.4 that $\mathrm{col}_r(G) \leq h(2r + 1) + \alpha$. □

6. The generalised colouring numbers of planar graphs

In this section we prove Theorems 1.5 and 1.6, providing upper bounds for $\mathrm{col}_r(G)$ and $\mathrm{wcol}_r(G)$ when G is a graph with bounded genus. Since for every genus g there exists a t such that every graph with genus at most g does not contain K_t as a minor, we could use Theorems 1.2 to obtain upper bounds for the generalised colouring numbers of such graphs. But the bounds obtained in this section are significantly better.

6.1. The weak r-colouring number of planar graphs

By a *maximal planar graph* we mean a (simple) graph that is planar, but where we cannot add any further edges without destroying planarity. It is well know that a maximal planar graph G with $|V(G)| \geq 3$ has a unique plane embedding (up to the choice of the outer face), which is a triangulation of the plane. We will use that implicitly regularly in what follows.

We start by obtaining an upper bound for $\mathrm{wcol}_r(G)$ that is much smaller than the bound given by Theorem 1.2. Our method for doing this again uses isometric paths decompositions. For maximal planar graphs, we will provide isometric paths decompositions of width at most 2. Using Lemma 3.5 and the fact that $\mathrm{wcol}_r(G)$ cannot decrease if edges are added, we conclude that $\mathrm{wcol}_r(G) \leq \binom{r+2}{2} \cdot (2r + 1) \in O(r^3)$. In [10], Grohe *et al.* proved that for every r there is a graph $G_{2,r}$ of tree-width 2 such that $\mathrm{wcol}_r(G_{2,r}) = \binom{r+2}{2} \in \Omega(r^2)$. Since graphs with tree-width 2 are planar, this shows that the maximum of $\mathrm{wcol}_r(G)$ for planar graphs is both in $\Omega(r^2)$ and $O(r^3)$.

Lemma 6.1.
Every maximal planar graph G has an isometric paths decomposition of width at most 2.

Proof. Fix a plane embedding of G. Since the proof is otherwise trivial, we assume $|V(G)| \geq 4$.

We will inductively construct an isometric paths decomposition $P_1, \dots, P_\ell$ such that each component C of $G - \bigcup_{1 \leq j \leq \ell} V(P_j)$ satisfies that the boundary of the region in which C lies is a cycle in G that has its vertices in exactly two paths from $P_1, \dots, P_\ell$.

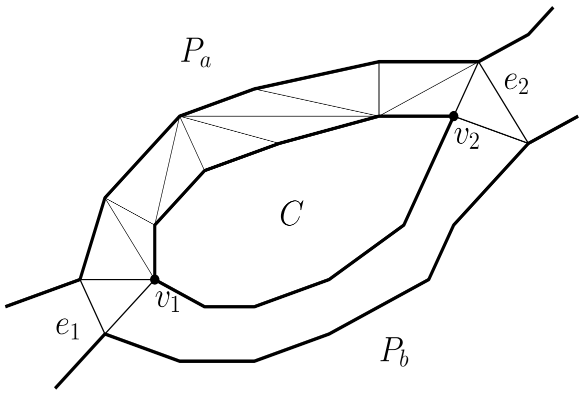

Figure 1: The path P_{i+1} is chosen from the vertices of a connected component C.

As the first path P_1, choose an arbitrary edge of the (triangular) outer face, and as P_2 choose the vertex of that triangle that is not contained in P_1. There is only one connected component in $G - (P_1 \cup P_2)$, and it is in the interior of the cycle which has vertices $V(P_1) \cup V(P_2)$.

Now assume that $P_1, \ldots, P_i$ have been constructed in the desired way, and choose an arbitrary connected component C of $G - \bigcup_{1 \le j \le i} V(P_j)$. Let D be the cycle that forms the boundary of the region in which C lies, and let P_a, P_b, $1 \le a, b \le i$, be the paths that contain the vertices of D. Notice that at least one of these paths must have more than one vertex, and let P_a be such a path.

Since P_a and P_b are disjoint and isometric paths, D must contain exactly two edges e_1, e_2 that do not belong to P_a and P_b. (Of course, more than two edges can connect P_a and P_b. But only two of them are on D.) Each of these edges belongs to a triangle in G which is in the interior of D. By definition of D, the triangle that consists of e_1 and a vertex v_1 in the interior of D has the property that v_1 must lie in C. Similarly, the triangle that consists of e_2 and a vertex v_2 in the interior of D has the property that v_2 must lie in C ($v_1 = v_2$ is possible). See Figure 1 for a sketch of the situation.

Any path P in C that connects v_1 and v_2 has the property that every vertex of C that is adjacent to P_a is either in P or in the region defined by P_a and P that does not contain P_b. Hence, as a next path P_{i+1} we can take any isometric path in C connecting v_1 and v_2.

It is clear that any component C' of $G - \bigcup_{1 \le j \le i+1} V(P_j)$ that was not already a component of $G - \bigcup_{1 \le j \le i} V(P_j)$ is connected to at most two paths from P_a, P_b, P_{i+1}, and no such component is connected to both P_a and P_b. To finish the construction of the decomposition we must prove that such a component C' is connected to exactly two of these three paths. Let us assume that C' lies in the interior of some cycle D' contained in $V(P_a) \cup V(P_{i+1})$. Suppose for a contradiction that D' only has vertices from one of these paths, say from P_a. But since any cycle contains at least one edge not in P_a and D' has length at least 3, this implies that there is an edge between two non-consecutive vertices of P_a. This contradicts that P_a was chosen as an isometric path. Exactly the same arguments apply when C' lies in the interior of some cycle contained in $V(P_b) \cup V(P_{i+1})$.

The isometric paths decomposition we constructed has width 2, and thus the result follows. $\square$

Theorem (Theorem 1.6)
For every graph G with genus g, we have $\mathrm{wcol}_r(G) \le \left(2g + \binom{r+2}{2}\right) \cdot (2r+1)$.

In particular, for every planar graph G, we have $\mathrm{wcol}_r(G) \le \binom{r+2}{2} \cdot (2r+1)$.

Proof. We first prove the bound for planar graphs. According to Lemma 6.1, maximal planar graphs have isometric paths decompositions of width at most 2. Using Lemma 3.5, we see that any maximal planar graph G satisfies $\mathrm{wcol}_r(G) \le \binom{r+2}{2} \cdot (2r+1)$. Since $\mathrm{wcol}_r(G)$ cannot decrease when edges are added, we conclude that any planar graph satisfies the same inequality.

It is well known (see e.g. [17, Lemma 4.2.4] or [23]) that for a graph of genus $g > 0$, there exists a non-separating cycle C that consists of two isometric paths such that $G - C$ has genus $g - 1$. We construct a linear order of $V(G)$ by starting with the vertices of such a cycle. We repeat this procedure inductively until all we are left to order are the vertices of a planar graph G'. We have seen that we can order the vertices of G' in such a way that they can weakly r-reach at most $\binom{r+2}{2} \cdot (2r+1)$ vertices in G'. By Lemma 3.3 and Lemma 3.2 we see that any vertex in the graph can weakly r-reach at most $2g \cdot (2r+1)$ vertices from the cycles we put first in the linear order. The result follows immediately. □

6.2. The r-colouring number of planar graphs

From Lemma 6.1 and Lemma 3.4, we immediately conclude that $\mathrm{col}_r(G) \le 3(2r+1)$ if G is planar. This is already an improvement of what we would obtain using Theorem 1.2 with the fact that planar graph do not contain K_5 or $K_{3,3}$ as a minor. Yet we can further improve this by showing that $\mathrm{col}_r(G) \le 5r+1$, a bound which is tight for $r = 1$. The method we use to prove this again uses isometric paths, but differs from the techniques we have used before because we will use sequences of separating paths that are not disjoint.

Let G be a maximal planar graph and fix a plane embedding of G. Let v be any vertex of G and let S be a lexicographic breadth-first search tree of G with root v. For each vertex w, let P_w be the unique path in S from the root v to w.

The following tree-decomposition $(T, (X_t)_{t \in V(T)})$ is a well-known construction that has been used to show that the tree-width of a graph is linear in its radius.

1. $V(T)$ is the set of faces of G (recall that all these faces are triangles);
2. $E(T)$ contains all pairs $\{t, t'\}$ where the faces t and t' share an edge in G which is not an edge of S;
3. for each face $t \in V(T)$ with vertices $\{a, b, c\}$, let $X_t = V(P_a) \cup V(P_b) \cup V(P_c)$.

We define a linear order L on the vertices of G as follows. Let t' be the outer face of G, with vertices $\{a, b, c\}$. We pick one of the paths P_a, P_b, P_c, say P_a, arbitrarily as the first path and order its vertices starting from the root v and moving up to a. We pick a second path arbitrarily, say P_b, and order its vertices which have not yet been ordered, starting from the one closest to v and moving up to b. After this, we do the same with the vertices of the third path P_c.

We now pick the outer face as the root of the tree T from the tree-decomposition and perform a depth-first search on T. Each bag X_t contains the union of three paths, but at the moment t is reached by the depth-first search on T, at most one of these paths contains vertices which have not yet been ordered. We order the vertices of such a path starting from the one closest to v and moving up towards the vertex which lies in t.

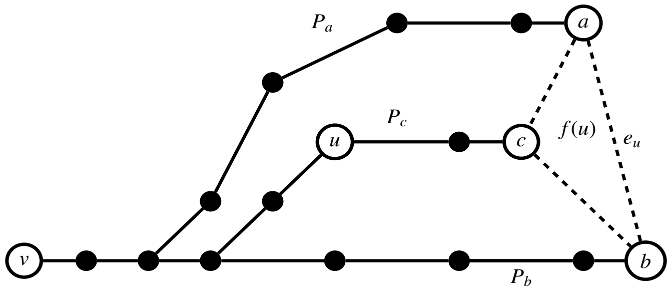

Figure 2: Situation for a vertex u such that $f(u)$ is not the outer face. Solid edges represent those in $G[X_{f(u)}] \cap S$. The cycle $C(u)$ is the one contained in $P_a \cup P_b \cup e_u$. The vertices u and c lie in $O(u)$.

For $u \in V(G)$, let $f(u)$ be the first face (in the depth-first search traversal of T) for which the bag $X_{f(u)}$ contains u. If u is a vertex for which $f(u)$ is the outer face, then let $C(u)$ be the cycle formed by the three edges in the outer face. Otherwise, if $f(u)$ is not the outer face, then let e_u be the unique edge of $f(u)$ not in S such that the other face containing e_u was found by T before $f(u)$, and let $C(u)$ be the cycle formed in $S + e_u$. Finally, let $O(u)$ be the set of vertices lying in the interior of $C(u)$. See Figure 2 for a sketch of the situation.

The following lemma tells us that if $f(u)$ is not the outer face, then the paths in $X_{f(u)}$ separate u from any other smaller vertex in L.

Lemma 6.2.
For all $u \in V(G)$, we have that the vertices of $X_{f(u)}$ are smaller, with respect to L, than all vertices in $O(u) \setminus X_{f(u)}$.

Proof. If $f(u)$ is the outer face, then by the construction of L the vertices of $X_{f(u)}$ are smaller than all other vertices in $V(G)$.

Assume next that $f(u)$ is not the outer face and let $z <_L u$. If $f(z) = f(u)$, then also $X_{f(z)} = X_{f(u)}$ and by the definition of $X_{f(z)}$ we have that z is contained in $X_{f(u)}$. If $f(z) \neq f(u)$, then it must be that $f(z)$ is a face which is encountered before $f(u)$ in the depth-first search of T. We know that $X_{f(z)}$ is the union of three paths. Assume for a contradiction that one of these paths, say P_1, contains a vertex x in $O(u)$. One of the endpoints of P_1 is in $f(z)$ and therefore cannot be in $O(u)$. The fact that P_1 has both vertices in $O(u)$ and vertices not in $O(u)$, means that there must be a vertex $w \neq v$ of P_1 in $C(u)$. Notice that $C(u) - e_u$ is a subset of two of the paths of $X_{f(u)}$. Therefore, w also belongs to a path P_2 contained in $X_{f(u)}$ that does not have any of its vertices in $O(u)$. That means we have two paths between w and the root v, one is a subpath of P_1 containing x, and the other one is a subpath of P_2 that does not contain x. However, any of the paths that form $X_{f(z)}$ and $X_{f(u)}$ are paths of S, and this means we have found a cycle in S, a contradiction. We conclude that no path of $X_{f(z)}$ contains a vertex of $O(u)$ and so z does not lie in $O(u)$. □

We will use the ordering L and Lemma 6.2 to prove that $\mathrm{col}_r(G) \leq 5r + 1$ for any planar graph G. For the purpose of the following proof, it is particularly important that S is a *lexicographic* breadth-first search tree.

Theorem (Theorem 1.5)
For every graph G with genus g, we have $\mathrm{col}_r(G) \leq (4g + 5)r + 2g + 1$.
In particular, for every planar graph G, we have $\mathrm{col}_r(G) \leq 5r + 1$.

Proof. Also this time we first prove the bound for planar graphs. Since $\mathrm{col}_r(G)$ cannot decrease when edges are added, we can assume that G is maximal planar. Therefore, we can order its vertices according to a linear order L as defined above.

Fix a vertex $u \in V(G)$ such that $f(u)$ is not the outer face, and let a, b, c be the vertices of $f(u)$. Recall that this means that $X_{f(u)} = V(P_a) \cup V(P_b) \cup V(P_c)$. Choose P_c to be the unique path of $X_{f(u)}$ containing u. Let P_u be the subpath of P_c from u to the root v. Notice that $C(u) - e_u \subseteq P_a \cup P_b$. Then by Lemma 6.2, P_a and P_b separate u from all smaller vertices not in $X_{f(u)}$. Therefore, using the definition of the ordering L, we see that all vertices in $V(P_a) \cup V(P_b) \cup V(P_u) \setminus \{u\}$ are smaller than u in L, and that all the vertices in $O(u) \setminus V(P_u)$ are larger than u in L. Hence, we have

$$\mathrm{SReach}_r[G, L, u] \;\subseteq\; N_r^G[u] \cap (V(P_a) \cup V(P_b) \cup V(P_u)). \tag{1}$$

Since S is a breadth-first search tree, using Lemma 3.2, we see that $|N_r^G[u] \cap V(P_a)| \leq 2r + 1$ and $|N_r^G[u] \cap V(P_b)| \leq 2r + 1$. Also, by the definition of L we have $|N_r^G[u] \cap V(P_u)| \leq r + 1$. These inequalities together with (1) tell us that $\left|\mathrm{SReach}_r[G, L, u]\right| \leq 5r + 3$.

In the remainder of this proof we will show that in fact there are at least 2 fewer vertices in $\mathrm{SReach}_r[G, L, u]$.

We say that the *level* d_u of a vertex u is the distance u has from v, i.e. the height of u in the breadth-first search tree S. For equality to occur in $|N_r^G[u] \cap V(P_a)| \leq 2r + 1$, there must be vertices $z_1, z_2 \in V(P_a)$ in $N_r^G[u]$ such that the level of z_1 in S is $d_u - r$ and the level of z_2 is $d_u + r$. We will show that at most one of z_1 and z_2 can belong to $\mathrm{SReach}_r[G, L, u] \setminus V(P_u)$.

Suppose $z_2 \in \mathrm{SReach}_r[G, L, u]$ and let P_2 be a path from u to z_2 that makes z_2 strongly r-reachable from u. Since z_2 is at level $d_u + r$, P_2 has length r and all of its vertices must be at different levels of S. For any path P with all of its vertices at different levels of S, we will denote by $P(d)$ the vertex of P at level d. By definition of L, we know that $P_a(d_u + i) <_L z_2$ for all $0 \leq i \leq r - 1$. This, together with the definition of P_2, tells us that P_2 cannot share any vertex with P_a other than z_2. Moreover, the edge incident to z_2 in P_2 cannot belong to S, because there already is an edge in $E(P_a) \subseteq E(S)$ joining a vertex at level $d_u + r - 1$ to z_2. This means that the vertex $P_a(d_u + r - 1)$ was found by the lexicographic breadth-first search S before the vertex $P_2(d_u + r - 1)$. This in its turn implies that $P_a(d_u + r - 2)$ was found by S before $P_2(d_u + r - 2)$. Continuing inductively we find that this is true for every level $d_u + i$, $0 \leq i \leq r - 1$. In a similar way, we can check that this implies that S found the vertex $P_a(d_u - i)$ before the vertex $P_u(d_u - i)$, for $1 \leq i \leq r$, whenever these vertices differ. In particular, z_1 was found before $P_u(d_u - r)$ if $z_1 \notin V(P_u)$.

Let us use this last fact to show that if $z_1 \in \mathrm{SReach}_r[G, L, u]$, then z_1 must also belong to P_u. We do this by assuming that $z_1 \notin V(P_u)$. This tells us that the vertices $P_a(d_u - i)$ and $P_u(d_u - i)$ are distinct for all $0 \leq i \leq r$. Therefore, given that z_1 was found by S before $P_u(d_u - r)$, there exists no edge between z_1 and $P_u(d_u - r + 1)$, because if it did exist, then the edge joining $P_u(d_u - r)$ and $P_u(d_u - r + 1)$ would not be in S. It follows that any vertex at level $d_u - r + 1$ belonging to $N(z_1)$ was found by S before $P_u(d_u - r + 1)$. By the same argument there is no edge between $N(z_1)$ and $P_u(d_u - r + 2)$. Inductively, we find that for $0 \leq i \leq r - 1$, any vertex at level $d_u - r + i$ belonging to $N_i^G[z_1]$ was found by S before $P_u(d_u - r + i)$, and so there is no edge between $N_i^G[z_1]$ and $P(d_u - r + i + 1)$. But for $i = r - 1$ this means that $u \notin N_r^G[z_1]$ which implies that $z_1 \notin \mathrm{SReach}_r[G, L, u]$. Hence we can conclude that if $z_2 \in \mathrm{SReach}_r[G, L, u]$, then z_1 can only be strongly r-reachable from u if it also belongs to P_u.

Now suppose $z_1 \in \mathrm{SReach}_r[G, L, u] \setminus V(P_u)$, and let P_1 be a path from u to z_1 that makes z_1 strongly r-reachable from u. Since z_1 is at level $d_u - r$, P_1 has length r and all of its vertices are at different levels of S. Let $d_u - j$ be the minimum level of a vertex in $V(P_1) \cap V(P_u)$. Notice that $j < r$, since $z_1 \notin V(P_u)$. Since $E(P_u) \subseteq E(S)$, it is clear that the vertex $P_u(d_u - j - 1)$ was found by S before $P_1(d_u - j - 1)$. This tells us that $P_u(d_u - j - 2)$ was found before $P_1(d_u - j - 2)$. Using induction, we can check that this will also be true for all levels $d_u - i$, $j + 1 \leq i \leq r$. In particular, this means that $P_u(d_u - r)$ was found before z_1. This implies that the lexicographic search found the vertex $P_u(d_u - i)$ before the vertex $P_a(d_u - i)$, for all $0 \leq i \leq r$. Hence S found u before $P_a(d_u)$. Now suppose for a contradiction that there is a path P_2 that makes z_2 strongly r-reachable from u. The path P_2 can only intersect $V(P_a)$ at z_2 and, since u was found before $P_a(d_u)$, it must be that $P_2(d_u + i)$ was found by S before $P_a(d_u + i)$, for $1 \leq i \leq r - 1$. Then the edge going from level $d_u + r - 1$ to level $d_u + r$ in P_a does not belong to S. This is a contradiction, given the definition of P_a.

By the analysis above, we have that

$$\left|(V(P_a) \setminus V(P_u)) \cap \mathrm{SReach}_r[G, L, u]\right| \leq 2r.$$

In a similar way we can show that $\left|(V(P_b) \setminus V(P_u))) \cap \mathrm{SReach}_r[G, L, u]\right| \leq 2r$. Then by (1) it follows that $\left|\mathrm{SReach}_r[G, L, u]\right| \leq 5r + 1$ for this choice of u.

We still have to do the case that u is a vertex such that $f(u)$ is the outer face. We notice that it might be possible that when u was added to the order L, fewer than two paths reaching $f(u)$ had been ordered. In this case it is clear that $\left|\mathrm{SReach}_r[G, L, u]\right| \leq (2r + 1) + (r + 1) \leq 5r + 1$. If u is on the third chosen path leading from the root to the vertices of $f(u)$, then we can use the arguments above to show that $\left|\mathrm{SReach}_r[G, L, u]\right| \leq 5r + 1$.

Having proved the bound on $\mathrm{col}_r(G)$ for planar graphs, the bound for graphs with genus $g > 0$ can be easily proved following the same procedure as in the proof of Theorem 1.6 in the previous subsection. □

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