Algorithmic Properties of Sparse Digraphs

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The notions of bounded expansion [49] and nowhere denseness [51], introduced by Nešetřil and Ossona de Mendez as structural measures for undirected graphs, have been applied very successfully in algorithmic graph theory. We study the corresponding notions of directed bounded expansion and nowhere crownfulness on directed graphs, as introduced by Kreutzer and Tazari in [41]. These classes are very general classes of sparse directed graphs, as they include, on one hand, all classes of directed graphs whose underlying undirected class has bounded expansion, such as planar, bounded-genus, and H-minor-free graphs, and on the other hand, they also contain classes whose underlying class is not nowhere dense.

We show that many of the algorithmic tools that were developed for undirected bounded expansion classes can, with some care, also be applied in their directed counterparts, and thereby we highlight a rich algorithmic structure theory of directed bounded expansion classes.

More specifically, we show that the directed Steiner tree problem is fixed-parameter tractable on any class of directed bounded expansion parameterized by the number k of non-terminals plus the maximal diameter s of a strongly connected component in the subgraph induced by the terminals. Our result strongly generalizes a result of Jones et al. [35], who proved that the problem is fixed parameter tractable on digraphs of bounded degeneracy if the set of terminals is required to be acyclic.

We furthermore prove that for every integer $r \geq 1$, the distance-r dominating set problem can be approximated up to a factor $\mathcal{O}(\log k)$ and the connected distance-r dominating set problem can be approximated up to a factor $\mathcal{O}(k \cdot \log k)$ on any class of directed bounded expansion, where k denotes the size of an optimal solution. If furthermore, the class is nowhere crownful, we are able to compute a polynomial kernel for distance-r dominating sets. Polynomial kernels for this problem were not known to exist on any other existing digraph measure for sparse classes.

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1 Introduction

Structural graph theory has made a deep impact on the analysis of complex algorithmic graph problems and the design of graph algorithms for hard problems. It provides a wealth of new and different tools for dealing with the intrinsic complexity of NP-hard problems on graphs and these methods have been applied very successfully in algorithmic graph theory, in approximation, optimization or the design of exact and parameterized algorithms for problems on undirected graphs, see e.g. [10, 12, 13, 14, 15, 16, 24, 25, 58]

Concepts such as *tree width* or *excluded (topological) minors* as well as density based graph parameters such as *bounded expansion* [49] or *nowhere denseness* [51] capture important properties of graphs and make them applicable for algorithmic applications.

Developing a similar structural theory for directed graphs resulting in classes of digraphs with a similarly broad algorithmic impact has so far not been blue as successful as for undirected graphs. The general goal is to identify structural parameters of digraphs which define interesting and general classes of digraphs for which at the same time we have a rich set of algorithmic tools available that can be used in the design of algorithms on these classes. However, essentially all approaches, e.g. in [5, 6, 30, 34, 53, 59], of generalizing even the well-understood and fairly basic concept of tree width to digraphs have failed to produce digraph parameters that come even near the wide spectrum of algorithmic applications that tree width has found. This has even led to claims that this program cannot be successful and that such measures for digraphs cannot exist [31].

The main conceptual contribution of this paper is to finally give a positive example of a digraph parameter that satisfies the conditions of the program outlined above: we identify a very general type of digraph classes which have a similar set of algorithmic tools available as their undirected counterparts. We believe that these classes finally give a positive answer to the question whether interesting graph parameters can successfully be generalized to the directed setting. We support this claim by algorithmic applications described below.

The classes of digraphs we study are classes of *directed bounded expansion* and *nowhere crownful classes* of digraphs. These notions were defined in [41] where basic properties of these classes were developed. The first improvement of these initial results appeared in [40], where classes of digraphs of bounded expansion were studied and their relation to a certain form of generalized coloring numbers was established. These papers are the starting point for our investigation in this paper. Furthermore, we introduce a new type of digraph classes called *bounded crownless expansion* which have the broadest set of algorithmic tools among the three types of digraph parameters.

Nowhere crownful and directed bounded expansion classes are modeled after the concept of *bounded expansion* and *nowhere denseness* developed by Nešetřil and Ossona de Mendez [49, 51]. Bounded expansion and the related concept of nowhere denseness was introduced to capture structural sparseness of graphs. On undirected graphs, classes of bounded expansion are very general and contain, for instance, planar graphs or more generally classes with excluded (topological) minors. But the concept goes far beyond excluded minor classes.

Following [49, 51], many papers have shown that algorithmic results for many problems on classes of graphs excluding a fixed minor can be generalized to classes of bounded expansion [8, 11, 19, 20, 21, 22, 29, 33, 37, 39, 42, 47, 61]. These new algorithms not only work on much larger classes of graphs than those excluding a minor. Often they also become conceptually simpler as they do not rely on deep, but sometimes cumbersome to use, structure theorems for classes with excluded minors. Furthermore, Demaine et al. [17] analyzed a range of real-world networks and showed that many of them indeed fall within the framework of bounded expansion. This shows that the concept of bounded expansion captures many types of real world instances. An interesting property of classes of bounded expansion and classes which are nowhere dense is that they can equivalently be defined in many different and seemingly unrelated ways: by the density of *bounded depth minors*, by *low tree depth colourings* [49], by *generalized coloring numbers* [64], by wideness properties such as *uniformly quasi wideness* [50], by *sparse neighbourhood covers* [32, 33], and many more. Each of these different aspects of bounded expansion classes comes with its own set of algorithmic tools and many of the more advanced algorithmic results on bounded expansion classes mentioned above crucially rely on a combination of several of these techniques.

In this paper we study suitable generalizations of bounded expansion and nowhere dense classes of graphs to the directed setting. See Section Section 2 for details.

We show that classes of digraphs of directed bounded expansion, and especially classes of *bounded crownless expansion* that we introduce in this paper, have very similar characterizations as their undirected counterpart: they have *low directed tree depth colorings*, they have *bounded directed weak coloring numbers*, a concept that has been ground breaking in the undirected setting, they have *low neighborhood complexity* and *bounded VC dimension* and many more. As a consequence, we are able to show that most of the algorithmic tools that were developed for undirected bounded expansion have their directed counterpart. Thus, we obtain powerful algorithmic tools for directed bounded expansion that are similar to the tools available in the undirected setting.

Note that we cannot combine our tools as freely as in the undirected setting. For example, nowhere crownfulness does not imply that bounded depth minors are sparse, or directed bounded expansion does not imply directed uniform quasi-wideness. Hence it is only natural to combine the requirements of nowhere crownfulness with that of bounded expansion, as we do to obtain classes of bounded crownless expansion, to obtain classes which behave algorithmically as nice as their undirected counterparts.

To the best of our knowledge, this is the first time that the goal of generalizing one of the widely studied and very general type of classes of undirected graphs to the directed setting has been really successful and has produced a digraph concept with a similarly broad set of algorithmic tools as its undirected counterpart. We are therefore optimistic that classes of bounded directed expansion and classes of bounded crownfree expansion will find a wide range of applications. We support this believe by a range of algorithmic results we describe next.

On the more technical level, we demonstrate the power of the new concepts by showing that several common problems on digraphs, which do not admit efficient solutions in general, can be solved efficiently on classes of directed bounded expansion or classes of bounded crownless expansion.

We first consider the Directed Steiner Tree (Dst) problem, which is defined as follows. As input we are given a digraph G, a root $r \in V(G)$, a set $T \subseteq V(G) \setminus \{r\}$ of terminals and an integer k. The problem is to decide if there is a set $S \subseteq V(G) \setminus (\{r\} \cup T)$ such that in $G[\{r\} \cup S \cup T]$ there is a directed path from r to every terminal T. The Steiner Tree problem is one of the most intensively studied graph problems in computer science with many important applications. We refer to the textbook of Prömel and Steger [55] for more background. While the parameterized complexity of Steiner Tree parameterized by the number of terminals is well understood, not much is known about the parameterization by the number of non-terminals in the solution tree. It is known for this parameterization that both the directed and the undirected versions are W[2]-hard on general graphs [45], and even on graphs of degeneracy two [35]. On the positive side, it is proved in [35] that the problem is fixed-parameter tractable when parameterized by the number of non-terminals on graphs excluding a topological minor. This result is based on a preprocessing rule which allows to contract strongly connected subsets of terminal vertices to individual vertices. The authors furthermore show that if the subgraph induced by the terminals is required to be acyclic, then the problem becomes fixed-parameter tractable on graphs of bounded degeneracy. In this case, the strongly connected subsets of terminals have diameter 0. This suggests to consider the problem parameterized by the number k of non-terminals plus the maximal diameter s of a strongly connected component in the subgraph induced by the terminals. We prove that with respect to this parameter the problem becomes fixed-parameter tractable on every class of directed bounded expansion. As bounded expansion classes are much more general than graphs with excluded topological minors and are stable under bounded diameter contractions, we believe that this result may provide the "true" explanation for the earlier result of [35].

We then turn our attention to the Distance-r Dominating Set problem. Given a digraph G and an integer k, we are asked to decide whether there exists a set $D \subseteq V(G)$ such that every vertex $v \in V(G)$ is reachable by a directed path of length at most r from a vertex $d \in D$. The dominating set problem (and its variations) is one of the most important problems in algorithmic graph theory. It is NP-complete in general [36], and (under standard complexity theoretical assumptions) cannot be approximated better than up to a factor $\mathcal{O}(\log n)$ [56]. This situation is different on sparse graph classes, it admits a PTAS, e.g., on planar graphs [3], and a constant factor approximation on classes of undirected bounded expansion [20]. Most generally, it admits an $\mathcal{O}(\log k)$ approximation on graphs of bounded VC-dimension [9] (where k is the size of an optimal dominating set). We study the VC-dimension of set systems corresponding to r-neighborhoods in digraphs of bounded expansion and derive an $\mathcal{O}(k \log k)$-approximation algorithm for the Distance-r Red-Blue Dominating Set problem and an $\mathcal{O}(k^2 \log k)$-approximation algorithm for the Strongly Connected Distance-r Dominating Set problem on classes of directed bounded expansion. Our analysis is strongly based on a characterization of bounded expansion classes in terms of generalized coloring numbers which was provided in [40], and which enables us to capture local separation properties in classes of bounded expansion.

Finally, we study classes which have both bounded expansion and which are nowhere crownful, a property that we call *bounded crownless expansion*. We study the kernelization problem from the Distance-r Dominating Set problem. Recall that a kernelization algorithm is a polynomial-time preprocessing algorithm that transforms a given instance into an equivalent one whose size is bounded by a function of the parameter only, independently of the overall input size. We are mostly interested in kernelization algorithms whose output guarantees are polynomial in the parameter. The existence of a kernelization algorithm immediately implies that a problem is fixed-parameter tractable, and hence, as the dominating set problem is $\mathsf{W}[2]$-hard in general, there cannot exist a kernelization algorithm in general (under the standard assumption that $\mathsf{W}[2] \neq \mathsf{FPT}$). A key ingredient to kernelization results for dominating sets on undirected sparse graph classes [19] is a duality theorem proved by Dvořák [20], which states that on a graph G from a class of undirected bounded expansion the size of a minimum distance-r dominating set $\gamma_r(G)$ is only constantly larger than the size of a maximum packing of disjoint balls of radius r, $\alpha_r(G)$. We prove that no such duality theorem (with any functional dependence between $\gamma_r(G)$ and $\alpha_r(G)$) can hold in graphs of directed bounded expansion. However, if we additionally assume that the class is nowhere crownful, we can employ methods which were recently developed in stability theory [43] to derive a polynomial duality theorem between domination and packing number. We remark that the application of stability theory in classes of bounded crownless expansion is not straight forward. It is known that a class of (di)graphs which is closed under taking subgraphs is stable, if and only if, its underlying class of undirected graphs is nowhere dense [1]. However, classes of bounded crownless expansion do not necessarily have this property. We have to carefully establish a situation in which stability is applicable, which then allows us to derive the polynomial duality theorem. Our kernelization algorithm then follows the approach of [19].

2 Directed Minors and Directed Bounded Expansion

In this section we fix our notation. We refer to [4] for standard notation and background on digraph theory.

Let G be a digraph, let $v \in V(G)$ and let $r \geq 1$ be an integer. The *r-out-neighborhood* of v, denoted by $N^+_{G,r}(v)$, or just $N^+_r(v)$ if G is understood, is defined as the set of vertices u in G such that G contains a directed path of length at most r from v to u. We write $N^+(v)$ for $N^+_1(v) \setminus \{v\}$. The *r-in-neighborhood* $N^-_{G,r}(v)$ and $N^-(v)$ are defined analogously. The *out-degree* of a vertex $v \in V(G)$ is $d^+(v) := |N^+(v)|$, its *in-degree* is $d^-(v) := |N^-(v)|$ and its *degree* is $d(v) := |N^+(v)| + |N^-(v)|$. The *minimum out-degree* of G is defined as $\delta^+(G) := \min\{d^+(v) \ : \ v \in V(G)\}$, *minimum in-degree* and *minimum degree* are defined analogously. A set $U \subseteq V(G)$ is *r-scattered* if there is no $v \in V(G)$ and $u_1 \neq u_2 \in U$ with $u_1, u_2 \in N^+_r(v)$. If the arc relation of a digraph G is symmetric, i.e. if $(u,v) \in E(G)$ implies $(v,u) \in E(G)$, then we speak of an *undirected graph*. If G is a digraph, we write $\bar{G}$ for the *underlying undirected graph of* G, which has the same vertices as G and for each arc $(u,v) \in E(G)$ we have $(u,v) \in E(\bar{G})$ and $(v,u) \in E(\bar{G})$. Note that $|E(G)| \leq |E(\bar{G})| \leq 2\,|E(G)|$.

Directed minors. We are going to work with directed minors and directed topological minors. The following definition of directed minors is from [41]. A digraph H has a *directed model* in a digraph G if there is a function δ mapping vertices $v \in V(H)$ of H to sub-graphs $\delta(v) \subseteq G$ and arcs $e \in E(H)$ to arcs $\delta(e) \in E(G)$ such that (1) if $v \neq u$, then $\delta(v) \cap \delta(u) = \emptyset$; (2) if $e = (u,v)$ and $\delta(e) = (u',v')$ then $u' \in \delta(u)$ and $v' \in \delta(v)$. For $v \in V(H)$ let $\mathrm{in}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e=(u,v)\in E(H)} V(\delta(e))$ and $\mathrm{out}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e=(v,w)\in E(H)} V(\delta(e))$; (3) we require that for every $v \in V(H)$ (3.1) there is a directed path in $\delta(v)$ from every $u \in \mathrm{in}(\delta(v))$ to every $u' \in \mathrm{out}(\delta(v))$; (3.2) there is at least one source vertex $s_v \in \delta(v)$ that reaches (by a directed path) every element of $\mathrm{out}(\delta(v))$; (3.3) there is at least one sink vertex $t_v \in \delta(v)$ that can be reached (by a directed path) from every element of $\mathrm{in}(\delta(v))$. We write $H \preccurlyeq G$ if H has a directed model in G and call H a *directed minor* of G. We call the sets $\delta(v)$ for $v \in V(H)$ the *branch-sets* of the model.

For $r \geq 0$, a digraph H is a *depth-r minor* of a digraph G, denoted as $H \preccurlyeq_r G$, if there exists a directed model of H in G in which the length of all the paths in the branch-sets of the model are bounded by r. Note that every subgraph of G is a depth-0 minor of G.

Directed topological minors. A digraph H is a *topological minor* of a digraph G if there is an injective function δ mapping vertices $v \in V(H)$ to vertices of $V(G)$ and arcs $e \in E(H)$ to directed paths in G such that if $e = (u,v) \in E(H)$, then $\delta(e)$ is a path from $\delta(u)$ to $\delta(v)$ in G which is internally vertex disjoint from all vertices $\delta(w)$ (for $w \in V(H)$) and all paths $\delta(e')$ (for $e' \in E(H)$, $e' \neq e$). For $r \geq 0$, H is a *topological depth-r minor* of G, written $H \preccurlyeq_r^t G$, if it is a topological minor and all paths $\delta(e)$ have length at most $2r$.

Grads, bounded expansion and crowns. Let G be a digraph and let $r \geq 0$. The *greatest reduced average density of rank* r (short *grad*) of G is

$$\nabla_r(G) := \max\left\{ \frac{|E(H)|}{|V(H)|} \;:\; H \preccurlyeq_r G \right\}$$

and its *topological greatest average density of rank* r (short *top-grad*) is

$$\widetilde{\nabla}_r(G) := \max\left\{ \frac{|E(H)|}{|V(H)|} \;:\; H \preccurlyeq_r^t G \right\}.$$

Note that $\nabla_0(G)$ is also known as the *degeneracy* of G. As the following theorem shows, the densities of depth-r minors and depth-r topological minors are functionally related.

Theorem 2.1 ([40]) *Let $r, d \geq 1$ and let $p = 32 \cdot (4d)^{(r+1)^2}$. Let G be a digraph. If $\nabla_r(G) \geq p$, then $\widetilde{\nabla}_r(G) \geq d$.*

Definition 2.2 A class $\mathcal{C}$ of digraphs has *bounded expansion* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $r \geq 0$ we have $\nabla_r(G) \leq f(r)$ (or equivalently, $\widetilde{\nabla}_r(G) \leq f(r)$) for all $G \in \mathcal{C}$.

A *crown* of order q is a 1-subdivision of a clique of order q with all arcs oriented away from the subdivision vertices, that is, the digraph S_q with vertex set $\{v_1, \ldots, v_q\} \cup \{v_{ij} : 1 \leq i < j \leq q\}$ and arc set $\{(v_{ij}, v_i), (v_{ij}, v_j) : 1 \leq i < j \leq q\}$.

Definition 2.3 A class $\mathcal{C}$ of digraphs has *bounded crownless expansion* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $r \geq 0$ we have $\nabla_r(G) \leq f(r)$ and $S_{f(r)} \not\preccurlyeq_r G$ for all $G \in \mathcal{C}$.

3 Steiner trees

Definition 3.1 The Directed Steiner Tree (Dst) problem is defined as follows. The input is a tuple (G, r, T, k) where G is a digraph, $r \in V(G)$ is a vertex (a root), a set $T \subseteq V(G) \setminus \{r\}$ of terminals and k is an integer. The problem is to decide if there is a set $S \subseteq V(G) \setminus (\{r\} \cup T)$ of size at most k such that in $G[\{r\} \cup S \cup T]$ there is a directed path from r to every terminal T.

The Dst problem has been widely studied in the area of approximation algorithms as it generalizes several routing and domination problems. We are interested in the parameterized complexity of this problem. It follows from an algorithm by Nederlof [46] and Misra et al. [44], that the problem can be solved in time $2^{|T|} \cdot p(n)$, for some polynomial $p(n)$. In this paper, we are interested in the standard parameterization in parameterized complexity, where as parameter we take the solution size, i.e. we take the number k of non-terminals as parameter. This models the case where we need to pay for any node we add to the solution and we want to keep the bound k on these nodes as small as possible without any restriction on the number of terminals to connect.

In [35], Jones et al. show that Dst with this parameterization is fixed-parameter tractable on any class of digraphs such that the class of underlying undirected graphs excludes a fixed graph H as an undirected topological minor. In this section we show that this result can be extended to classes of bounded directed expansion, but with one restriction on the structure of the terminals.

Theorem 3.2 *Let $\mathcal{C}$ be a class of digraphs of bounded expansion. Dst is fixed parameter tractable on $\mathcal{C}$ parameterized by the number k of non-terminals in the solution plus the maximal diameter s of the strongly connected components in the subgraph induced by the terminals.*

Proof. Let $G' \in \mathcal{C}$ be a digraph, $r \in V(G')$ be the root node, $T' \subseteq V(G') \setminus \{r\}$ be the set of terminals and let $k \geq 0$ be an integer. Let s be the maximal diameter of a strongly connected component of $G'[T']$.

As a first reduction step, we contract every strongly connected component of $G'[T']$ into a single vertex. Let G be the resulting digraph and let T be the set of terminals which, for every strong component of $G'[T']$ contains the corresponding new vertex obtained by contraction. It is easily seen that any set $S \subseteq V(G)$ is a solution of (G, r, T, k) if, and only if, S is a solution of (G', r, T', k). Let $d := 2\nabla_0(G)$. Note that G is a depth-s minor of G', as every strong component that was contracted to obtain G has diameter at most s. It follows that $d \leq 2\nabla_s(G')$ is bounded by a constant only depending on s (and the expansion of $\mathcal{C}$).

Let $T_0 \subseteq T$ be the set of terminals that have in-degree 0 in $G[T]$. Since for every terminal $t \in T$, the graph $G[T]$ contains a path from some $t_0 \in T_0$ to t, we have for every set $S \subseteq V(G)$ the property that in $G[\{r\} \cup S \cup T]$ there is a directed path from r to every $t \in T$ if and only if there is a directed path from r to every $t_0 \in T_0$. We now prove the following claim (see [35, Lemma 2]).

Claim 1. There is an algorithm which, given a digraph G, $r \in V(G)$, $T \subseteq V(G) \setminus \{r\}$ and $T_0 \subseteq T$, computes a minimum size set $S \subseteq V(G)$ such that there is a path from r to every $t \in T_0$ in $G[\{r\} \cup T \cup S]$ in time $2^{|T_0|} \cdot p(n)$, for some fixed polynomial $p(n)$ where $n = |V(G)|$.

Proof. We first add an edge from every $t \in T_0$ to every node $u \in V(G) \setminus T$ which is reachable from t by a path whose internal vertices are all in T. Let G'' be the resulting graph. Then any set S is a solution for the problem from the claim if, and only if, it is a solution for Dst(G'', r, T_0, k). We can now call the algorithm of Misra et al. [44], to solve the instance Dst(G'', r, T_0, k) in time $2^{|T_0|} \cdot p(n)$, where $p(n)$ is a fixed polynomial. This immediately implies the claim. ⊣

We are now ready to present a recursive algorithm for deciding whether Dst(G, r, T, k) has a solution. In the recursive calls we are additionally given a partial solution $Y \subseteq V(G) \setminus (T \cup \{r\})$ as input and we need to decide if $\textsc{Dst}(G, r, T, k)$ has a solution extending Y. Note that a solution extending Y is simply a solution of $\textsc{Dst}(G, r, T \cup Y, k - |Y|)$.

Let G, r, T, k, Y be given as above. Clearly, if $|Y| > k$ we can reject immediately. Recall that $d = 2\nabla_0(G)$. Let $N \coloneqq V(G) \setminus (T \cup Y \cup \{r\})$ be the set of non-terminal vertices. Let $k_Y \coloneqq k - |Y|$ be the number of non-terminals we can still choose for our solution. Let $T_Y \coloneqq \{t \in T : t \in N^+(Y \cup \{r\})\}$ be the set of terminals dominated by Y or r. Let $\bar{T}_Y \coloneqq T_0 \setminus T_Y$.

Let $S_{>d} \coloneqq \{v \in N : |N^+(v) \cap \bar{T}_Y| > d\}$ be the set of non-terminals dominating more than d elements in $\bar{T}_Y$ and let $S_{\leq d} \coloneqq N \setminus S_{>d}$ be the non-terminals dominating at most d elements of $\bar{T}_Y$. Similarly, let $T_{>d} \coloneqq \{t \in \bar{T}_Y : t \in N^+(S_{>d})\}$ be the set of source terminals dominated by $S_{>d}$ and let $T_{\leq d} \coloneqq \{t \in \bar{T}_Y : t \notin N^+(Y \cup T_{>d})\}$ be the set of source terminals not dominated by either Y or $T_{>d}$.

Clearly, if $|T_{\leq d}| > d \cdot k_Y$, then we can reject the input, as all vertices in $T_{\leq d}$ can be dominated only by non-terminals from $S_{\leq d}$ and each $v \in S_{\leq d}$ can dominate only d elements of $T_{\leq d}$. Hence we can assume that $|T_{\leq d}| \leq d \cdot k_Y$. It follows that if $S_{>d}$ is empty and therefore $T_{>d}$ is empty, we can apply the algorithm in Claim 1 to decide whether $\textsc{Dst}(G, r, T \cup Y, k_Y) = \textsc{Dst}(G, r, T_{\leq d} \cup Y, k_Y)$ has a solution extending Y in time $2^{|T_{\leq d}|} \cdot p(n) \leq 2^{d \cdot k_Y} \cdot p(n)$.

Thus, we can assume that $|T_{\leq d}| \leq d \cdot k_Y$ and $S_{>d} \neq \emptyset$. Now choose among all vertices in $T_{>d}$ a vertex $v \in T_{>d}$ which minimizes $d_v \coloneqq |S_{>d} \cap N^-(v)|$. We claim that $d_v \leq d$. To see this, consider the subgraph H of G with vertex set $S_{>d} \cup T_{>d}$ and all edges of G from $S_{>d}$ to $T_{>d}$. As this is a subgraph of G it follows that $2\nabla_0(H) \leq 2\nabla_0(G) = d$. Hence, there must be a vertex in $T_{>d}$ of in-degree at most d and therefore, by the choice of v, we have $d_v \leq d$.

Clearly, any solution to $\textsc{Dst}(G, r, T, k)$ extending Y must contain a vertex dominating v. We can explore all possibilities by branching into $d + 1$ recursive calls: for each $s \in (S_{>d} \cap N^-(v))$ we call $\textsc{Dst}(G, r, T \cup Y \cup \{s\}, k_Y - 1)$ recursively. If one of these calls is successful, then we return the solution. Otherwise we know that there is no solution in which v is dominated by a high degree vertex and hence, $\textsc{Dst}(G, r, T \cup Y, k_Y)$ has a solution if, and only if, $\textsc{Dst}(G - (S_{>d} \cap N^-(v)), r, T \cup Y, k_Y)$ has a solution. Hence, the last branch is to recursively call $\textsc{Dst}(G - (S_{>d} \cap N^-(v)), r, T \cup Y, k_Y)$. If this returns a solution extending Y, we are done and return this solution, otherwise we can reject the input. Note that in this recursive instance, v is no longer a vertex in $T_{>d}$ as it is not dominated anymore by any non-terminal which dominates at least d nodes in $\bar{T}_Y$.

We show next that the algorithm terminates sufficiently fast on all inputs. In every recursive call, either the set Y increases and $|T_{\leq d}|$ does not change or Y is not changed but $|T_{\leq d}|$ increases by one, as the vertex v in the last branch is now added to $T_{\leq d}$ in the recursive call. For every node x in the recursion tree we can therefore define the complexity of x as $|Y| + |T_{\leq d}|$, where Y and $T_{\leq d}$ are the corresponding sets of the $\textsc{Dst}$-instance solved at this node. Hence, every recursive call increases the complexity.

As the algorithm terminates as soon as $|T_{\leq d}| > d \cdot (k - |Y|)$ or $|Y| > k$, this means that every branch of the recursion tree has length at most $k + d \cdot k = k \cdot (d + 1)$. As at every node we do at most $d + 1$ recursive calls, this means that the entire search tree has at most $(d + 1)^{k \cdot (d+1)} = 2^{k \cdot (d+1) \cdot \log(d+1)}$ nodes. Clearly, the computation at every node can be done in polynomial time. Hence, the entire algorithm runs in time $\mathcal{O}(2^{k \cdot (d+1) \cdot \log(d+1)}) \cdot p(n)$ for some fixed polynomial $p(n)$. This completes the proof of the theorem. □

The proof of the theorem has the following immediate consequences.

Corollary 3.3 *Let $\mathcal{C}$ be a class of digraphs closed under taking directed minors for which $\nabla_0(G) \leq c$ for a constant c for all $G \in \mathcal{C}$. Then $\textsc{Dst}(G, r, T, k)$ can be solved for all $G \in \mathcal{C}$, $r \in V(G)$, $T \subseteq V(G) \setminus \{r\}$ and k in time $2^{\mathcal{O}(k)} \cdot p(n)$, for some fixed polynomial $p(n)$.*

Note that this strictly generalizes classes of undirected graphs excluding a fixed minor.

Another consequence of this is the following result, which immediately follows from the well-known observation in parameterized complexity (see e.g. [35, Lemma 7]), that for all functions $g(n) = o(\log n)$ there is a function $f(k)$ such that $f(k) \leq 2^{g(n)\cdot k}$, for all k and all n.

Corollary 3.4 *Let $\mathcal{C}$ be a class of digraphs such that $\nabla_{|G|}(G) \cdot \log \nabla_{|G|}(G) \leq o(\log n)$ for all $G \in \mathcal{C}$. Then* Dst *is fixed-parameter tractable on $\mathcal{C}$ with parameter k.*

Finally, the result also implies an fpt factor-2-approximation algorithm for the Strongly Connected Steiner Subgraph problem, Scss, on classes of bounded directed expansion. In the Scss we are given a digraph G, a number k, and a set T of terminals and we are asked to compute a set S of at most k non-terminals such that $G[T \cup S]$ is strongly connected.

Theorem 3.5 *Let $\mathcal{C}$ be a class of digraphs of bounded expansion. There is an fpt factor-2-approximation algorithm for* Scss *on $\mathcal{C}$ parameterized by the number k of non-terminals in the solution plus the maximal diameter s of a strongly connected component in the subgraph of G induced by the terminal nodes.*

Proof. Note first that if H is obtained from a digraph G by reversing the orientation of all edges of G, then for all r, $\nabla_r(H) = \nabla_r(G)$. Now, given a digraph G, a number k and a set T of terminals, we can fix a terminal $t \in T$ and solve $P_1 := \text{Dst}(G, t, T \setminus \{t\}, k)$ and $P_2 := \text{Dst}(H, t, T \setminus \{t\}, k)$, where H is obtained from G by reversing all edges. We then take the union of the two solutions S_1 and S_2 for P_1 and P_2. Clearly, if $\text{Scss}(G, k, T)$ has a solution S of size k then S is also a solution for the two subproblems. Hence, $|P_1|, |P_2| \leq k$ and therefore $|S_1 \cup S_2| \leq 2k$ as claimed. □

We close the section by showing that for bounded expansion classes, the parameterization $k + s$ in Theorem 3.2 cannot be replaced by taking only k as parameter. This follows immediately from a result of [35] where it is shown that Set Cover can be reduced to Dst on 2-degenerate graphs. It is straight forward to modify this example so that the resulting class of graphs has bounded directed expansion.

Theorem 3.6 *The* Dst*-problem restricted to classes of digraphs of bounded expansion parameterized by the solution size k is $W[2]$-hard.*

4 VC-dimension and domination

We come to another algorithmic application on graphs of directed bounded expansion, namely, the approximation of the Distance-r Dominating Set problem. We study the VC-dimension of set systems corresponding to r-neighborhoods in digraphs of bounded expansion and derive an $\mathcal{O}(k \log k)$-approximation algorithm for the Distance-r Red-Blue Dominating Set problem and an $\mathcal{O}(k^2 \log k)$-approximation algorithm for the Strongly Connected Distance-r Dominating Set problem on classes of directed bounded expansion.

4.1 VC-dimension and neighborhood complexity

Let $\mathcal{F} \subseteq 2^A$ be a family of subsets of a set A. For a set $X \subseteq A$, we denote $X \cap \mathcal{F} = \{X \cap F : F \in \mathcal{F}\}$. The set X is *shattered by* $\mathcal{F}$ if $X \cap \mathcal{F} = 2^X$. The *Vapnik-Chervonenkis dimension*, short *VC-dimension*, of $\mathcal{F}$ is the maximum size of a set X that is shattered by $\mathcal{F}$.

Note that if $\mathcal{F}$ has VC-dimension d, then also $B \cap \mathcal{F}$ for every subset $B \subseteq A$ of the ground set has VC-dimension at most d. The following theorem was first proved by Vapnik and Chervonenkis [63], and rediscovered by Sauer [60] and Shelah [62]. It is often called the Sauer-Shelah lemma in the literature.

Theorem 4.1 *If $|A| \leq n$ and $\mathcal{F} \subseteq 2^A$ has VC-dimension d, then $|\mathcal{F}| \leq \sum_{i=0}^{d} \binom{n}{i} \in \mathcal{O}(n^d)$.*

Definition 4.2 In the Distance-r Red-Blue Dominating Set problem, we are given a digraph G and two sets $R, B \subseteq V(G)$ and an integer k, and asked whether there exists a subset $D \subseteq B$ of at most k *blue* vertices such that each *red* vertex from R is at distance at most r to a vertex in D. We allow that R and B intersect.

The study of the distance-r dominating set problem in context of bounded VC-dimension motivates the following definition. Let G be a digraph and $r \geq 1$. The *distance-r VC-dimension* of G is the VC-dimension of the set family $\{N_r^-(v) \colon v \in V(G)\}$ over the set $V(G)$.

According to Theorem 4.1, the distance-r VC-dimension of a graph is bounded if the *distance-r neighborhood complexity* of its sets is polynomially bounded. Let G be a digraph, let $X \subseteq V(G)$ and let $r \geq 1$. The *distance-r neighborhood complexity of X in G*, denoted $\nu^-(G)$, is defined by

$$\nu^-(G, X) \coloneqq \left|\{N_r^-(v) \cap X \ : \ v \in V(G)\}\right|.$$

Analogously, one can define the *distance-r out-neighborhood complexity* when using $N_r^+(v)$ and the *distance-r mixed neighborhood complexity* when using $(N_r^+(v) \cup N_r^-(v))$ in the above definition and our proofs can be analogously carried out for these measures.

It was proved in [57] that a class $\mathcal{C}$ of undirected graphs has bounded expansion, if and only if, for every $r \geq 1$ there is a constant c_r such that for all $G \in \mathcal{C}$ and all $X \subseteq V(G)$ we have $\nu(G, X) \leq c_r \cdot |X|$. The analogous statement for classes of directed graphs does not hold, not even for $r = 1$, as pointed out in [40]. However, we prove that the distance-r neighborhood complexity of a digraph can be bounded in terms of its weak r-coloring numbers.

The weak r-coloring numbers for undirected graphs were introduced by Kierstead and Yang [38] and they play a key role in the algorithmic theory of graphs of undirected bounded expansion, ever since Zhu [64] proved that these classes can be characterized by the weak coloring numbers.

Let G be a digraph. By $\Pi(G)$ we denote the set of all linear orders of $V(G)$. For $r \geq 0$, we say that u is *weakly r-reachable* from v with respect to an order $L \in \Pi(G)$ if there is a path P of length at most r, connecting u and v, *in either direction*, such that u is minimum among the vertices of P with respect to L. By $\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]$ we denote the set of vertices that are weakly r-reachable from v with respect to L. We define the *weak r-coloring number* $\mathrm{wcol}_r^{\rightleftarrows}(G)$ of G as

$$\mathrm{wcol}_r^{\rightleftarrows}(G) \coloneqq \min_{L \in \Pi(G)} \max_{v \in V(G)} \left|\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]\right|.$$

Note that $\mathrm{wcol}_r^{\rightleftarrows}(G)$ is a *monotone parameter*, in the sense that if $H \subseteq G$, then $\mathrm{wcol}_r^{\rightleftarrows}(H) \leq \mathrm{wcol}_r^{\rightleftarrows}(G)$.

Theorem 4.3 ([40]) *A class $\mathcal{C}$ of digraphs has bounded expansion if, and only if, there is $f \colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq f(r)$ for all $G \in \mathcal{C}$ and all $r \geq 1$.*

The next lemma shows that the weak r-coloring numbers are very useful to describe local separation properties in graphs of bounded expansion.

Lemma 4.4 *Let G be a digraph and let $r \geq 1$. Let P be a path of length at most r with endpoints u and v in either direction. Let L be an order of $V(G)$. Then $\mathrm{WReach}_r^{\rightleftarrows}[G, L, u] \cap \mathrm{WReach}_r^{\rightleftarrows}[G, L, v]$ contains a vertex of P.*

Proof. Let z be the minimal vertex of P with respect to L. Then $z \in \mathrm{WReach}_r^{\rightleftarrows}[G, L, u]$ and $z \in \mathrm{WReach}_r^{\rightleftarrows}[G, L, v]$. □

Using Lemma 4.4 we can well control the interaction of distance-r neighborhoods with a set X. Let G be a digraph and let L be a linear order on $V(G)$ and let $r \geq 1$. Let $A \subseteq V(G)$ be enumerated as $a_1, \ldots, a_{|A|}$, consistently with the order. For $v \in V(G)$ let $D_r^-(v, A)$ denote the *distance-r vector* of v and A, that is, the vector $(d_1, \ldots, d_{|A|})$, where $d_i = \text{dist}(a_i, v)$ if $0 \leq \text{dist}(a_i, v) \leq r$, and ∞ otherwise. Here $\text{dist}(a_i, v)$ is the length of a shortest path from a_i to v.

Lemma 4.5 *Let G be a digraph, let $X \subseteq V(G)$ and let $r \geq 1$. Let $c := \text{wcol}_r^{\rightleftarrows}(G)$. Then the number of distinct distance-r vectors $D_r^-(v, X)$ is bounded by $((r+2) \cdot c \cdot |X|)^c$, and in particular,*

$$\nu_r^-(G, X) \leq ((r+2) \cdot c \cdot |X|)^c.$$

Proof. Let $W := \text{WReach}_r^{\rightleftarrows}[G, L, X] = \bigcup_{x \in X} \text{WReach}_r^{\rightleftarrows}[G, L, x]$. We claim that if

$$D_r^-(u, W \cap \text{WReach}_r^{\rightleftarrows}[G, L, u]) = D_r^-(v, W \cap \text{WReach}_r^{\rightleftarrows}[G, L, v]),$$

then

$$N_r^-(u) \cap X = N_r^-(v) \cap X.$$

To see this, fix u and v with $D_r^-(u, W \cap \text{WReach}_r^{\rightleftarrows}[G, L, u]) = D_r^-(v, W \cap \text{WReach}_r^{\rightleftarrows}[G, L, v])$ and some $x \in X$. Assume that $x \in N_r^-(u) \cap X$. We prove that $x \in N_r^-(v) \cap X$. Let P be a shortest path from x to u. By Lemma 4.4, P contains a vertex z of $W \cap \text{WReach}_r^{\rightleftarrows}[G, L, u]$ and because P is a shortest path, the subpath of P between z and u is of minimal distance, say distance $r' \leq r$. Because $D_r^-(u, W \cap \text{WReach}_r^{\rightleftarrows}[G, L, u]) = D_r^-(v, W \cap \text{WReach}_r^{\rightleftarrows}[G, L, v])$, also v is at distance at most r' to z. Then the initial part of P from x to z together with the path from z to v witnesses that $x \in N_r^-(v) \cap X$. The case that $x \in N_r^-(v) \cap X$ is symmetric.

Now, since $|W| \leq c \cdot |X|$ and we have $|W \cap \text{WReach}_r^{\rightleftarrows}[G, L, v]| \leq c$ for all $v \in V(G)$, we have $|\{D_r^-(v, W \cap \text{WReach}_r^{\rightleftarrows}[G, L, v]) : v \in V(G)\}| \leq (c \cdot |X|)^c \cdot (r+2)^c$. As argued above, this number of distinct distance profiles bounds the number of neighborhoods in $\nu_r^-(G, X)$. ☐

Corollary 4.6 *Let G be a digraph and $r \geq 1$. Then the distance-r VC-dimension of G is bounded by $(r+2) \cdot (2\text{wcol}_r^{\rightleftarrows}(G))^2$.*

Proof. Let $c := \text{wcol}_r^{\rightleftarrows}(G)$ and let $X \subseteq V(G)$ be the largest set which is shattered by $\{N_r^-(v) : v \in V(G)\}$. Let $\mathcal{F} = \{N_r^-(v) \cap X : v \in V(G)\}$. As X is shattered, $\mathcal{F}$ has VC-dimension $|X|$ and contains $2^{|X|}$ elements. On the other hand, according to Lemma 4.5 we have $\nu_r^-(G, X) \leq ((r+2) \cdot c \cdot |X|)^c$. Hence $\mathcal{F}$ has at most $((r+2) \cdot c \cdot |X|)^c$ elements, which implies that $2^{|X|} \leq ((r+2) \cdot c \cdot |X|)^c$. Assuming $|X| \geq ((r+2) \cdot c)$, we get

$$2^{|X|} \leq |X|^{2c} \Leftrightarrow |X|/\log|X| \leq 2c$$

which is violated for $|X| \geq (2c)^2$, as $c \geq 2$ (unless G is an edgeless graph in which case the claim trivially holds). ☐

4.2 Approximation of distance-r red-blue dominating sets

For our approximation algorithm we will make use of the following algorithm of Brönnimann and Goodrich.

Theorem 4.7 (Brönnimann and Goodrich [9]) *For every fixed dimension $d \geq 1$, there is a polynomial-time algorithm for finding a hitting set in a set system $\mathcal{F}$ of VC-dimension d of size $\mathcal{O}(k \cdot \log k)$, where k is the size of a minimum hitting set for $\mathcal{F}$.*

Theorem 4.8 *Let $\mathcal{C}$ be a class of bounded expansion and let $r \geq 1$. There is a polynomial time algorithm which on input $G \in \mathcal{C}$, $R, B \subseteq V(G)$ computes a distance-r red-blue dominating set of G of size $\mathcal{O}(k \cdot \log k)$, where k is the size of a minimum distance-r red-blue dominating set in G.*

Proof. Let $B \subseteq V(G)$ and $R \subseteq V(G)$. Let $\mathcal{F} = \{N_r^-(v) \cap B : v \in R\}$. Then a hitting set of $\mathcal{F}$ is a blue distance-r dominating set of R. According to Corollary 4.6, $\mathcal{F}$ has bounded VC-dimension on any bounded expansion class. Now conclude with Theorem 4.7. ☐

With slightly more effort we can approximate the connected distance-r dominating set problem.

Theorem 4.9 *Let $\mathcal{C}$ be a class of bounded expansion and let $r \geq 1$. There is a polynomial time algorithm which on input $G \in \mathcal{C}$ computes a strongly connected distance-r dominating set of G of size $\mathcal{O}(k^2 \cdot \log k)$, where k is the size of a minimum strongly connected distance-r dominating set of G.*

Proof. Assume we know the size k of an optimal strongly connected distance-r dominating set (we will incrementally test all values $1, \ldots, k$ until we find a valid solution). We guess one vertex $v \in V(G)$ which is a central vertex of an optimal strongly connected distance-r dominating set D. Observe that D has radius at most k, and hence, we can restrict our search for an approximate solution to the strongly connected k-neighborhood of v, which we color blue. We color the rest of the graph red and now search for an approximate distance-r red-blue dominating set using Theorem 4.8. We find a solution D' of size $\mathcal{O}(k \cdot \log k)$, which may not be connected. Because we restricted the blue vertices to the strong k-neighborhood of v, for each $w \in D'$ there is a closed walk $W_{v,w}$ of length at most k which contains both v and w. Now taking the union of all vertices of the walks $W_{v,w}$ for $w \in D'$ gives us a dominating set of size at most k times larger than D'. Hence, we compute an $\mathcal{O}(k^2 \cdot \log k)$ approximation to a strongly connected distance-r dominating set. ☐

5 Kernelization on classes of bounded crownless expansion

A powerful method in parameterized complexity is *kernelization.* A kernelization algorithm is a polynomial-time preprocessing algorithm that transforms a given instance into an equivalent one whose size is bounded by a function of the parameter only, independently of the overall input size. We are mostly interested in kernelization algorithms whose output guarantees are polynomial in the parameter. As the dominating set problem is W[2]-hard in general, we cannot expect a kernelization algorithm in general. Again, the situation is quite different in sparse graphs. The Dominating Set problem admits linear kernels on planar graphs [2], bounded genus graphs [7], apex-minor free graphs [26], H-minor free graphs [27] and H-topological minor free graphs [28]. It admits polynomial kernels on graphs of bounded degeneracy [54]. The Distance-r Dominating Set problem admits a linear kernel on classes of bounded expansion [19], and almost linear kernel on nowhere dense classes of graphs [39]. We are not aware of any kernelization results on directed graphs, though, it is easy to observe that the result of [54] also holds on digraphs of bounded degeneracy.

We prove that for every fixed value of $r \geq 1$, the distance-r dominating set problem admits a polynomial kernel on every class of bounded crownless expansion. For this, we first prove in Section 5.1 a polynomial duality theorem between the size of a largest r-scattered set and a smallest distance-r dominating set in these classes. We show that this duality theorem does not hold in classes of directed bounded expansion without the additional assumption on crown-freeness. In Section 5.2 we then adapt a method developed in [19] for kernelization on classes of undirected bounded expansion to the directed case.

5.1 A polynomial duality theorem

Denote by $\gamma_r(G)$ the size of a smallest set X such that $N_r^+(X) = V(G)$. Denote by $\alpha_r(G)$ the size of a largest set Y such that for all $x, y \in Y$ there is no $u \in V(G)$ with $x, y \in N_r^+(u)$, that is, the largest set which is r-scattered. Clearly, $\gamma_r(G) \geq \alpha_r(G)$, because no vertex in G can r-dominate more than one vertex of Y. As shown by Dvořák [20], in a class $\mathcal{C}$ of undirected bounded expansion, it holds that $\gamma_r \leq c \cdot \alpha_r(G)$ for some constant c depending only on $\mathcal{C}$. This is not true for directed graphs.

Theorem 5.1 *There is a class of directed bounded expansion such that for every constant c we have $\gamma_1(G) \geq c$ for infinitely many $G \in \mathcal{C}$ and $\alpha_1(G) = 2$ for all $G \in \mathcal{C}$.*

Proof. Let $n \in \mathbb{N}$. Let G_n be the digraph with vertex set $\{v_1, \dots, v_n\} \cup \{w_{ij} : 1 \leq i < j \leq n\} \cup \{a\}$ and arc set $\{(w_{ij}, v_i), (w_{ij}, v_j) : 1 \leq i < j \leq n\} \cup \{(a, w_{ij}) : 1 \leq i < j \leq n\}$, that is, G_n is obtained from a 1-subdivision of a clique of size n with all subdivision arcs pointing away from the subdivision vertex, together with an apex vertex adjacent to all subdivision vertices. Then $\gamma_1(G_n) = \lceil n/2 \rceil + 1$. Every subdivision vertex can dominate 2 principle vertices. The apex vertex dominates all subdivision vertices, and this is best possible. We have $\alpha_1(G) = 2$, as for all $x, y \in V(G_n) \setminus \{a\}$ we either have $(x, y) \in E(G_n)$, or there is $u \in V(G_n)$ with $x, y \in N(u)$. ☐

If we however have a class of graphs which has bounded crownless expansion, then γ_r and α_r are polynomially related. We will apply the algorithm of Dvořák [20] to the digraph G and prove that it finds both a dominating set and a polynomially smaller independent set. Our proof is inspired by a recent result of Malliaris and Shelah on stability theory [43], which allows us to apply a Ramsey type argument, but with polynomial instead of exponential bounds.

Let T be a (rooted) binary tree, where each vertex (except the root) is marked as a left or right successor of its predecessor. We call w a *left (right) descendant* of v if the first successor on the unique v-w path in T is a left (right) successor.

Fix an enumeration $a_1, \dots, a_\ell$ of a set $A \subseteq V(G)$. The *r-independence tree* of $(a_1, \dots, a_\ell)$ is a binary tree which is constructed recursively as follows. We make a_1 the root of the tree. Assume that $a_1, \dots, a_i$ have already been inserted into the tree. In order to insert the next element a_{i+1}, we follow a root-leaf path to find a position for it. Starting from the root a_1, at each point we are at some node a_j and we have to decide whether we continue along the left or to the right branch at a_j. If there is an element u such that $a_j, a_{i+1} \in N_r^+(u)$, we continue along the right branch at a_j, otherwise we follow the left branch. If there is no right successor (or left successor, respectively), we insert a_{i+1} as a right (or left child, respectively) of a_j.

Lemma 5.2 *Let T be a rooted binary tree and let $t \geq 1$ be an integer. Assume that no root-leaf path in T contains a sub-sequence $a_1, \dots, a_t$ (of pairwise distinct elements) such that a_j is a right descendant of a_i for all $1 \leq i < j \leq t$. If T has height at most h, then T has at most h^{t+1} vertices.*

Proof. We can describe the position of each node u of T by a set $P \subseteq \{0, \dots, h\}$ such that $h(u) \in P$, where $h(u)$ denotes the height of u in T, and $h(w) \in P$ for all w such that u is a right descendant of w in T. The position of u in T is then found by following a path from the root which turns right at the smallest $|P| - 1$ levels which are contained in P and which stops at the largest level in P (which is the number $h(u)$). It hence suffices to count the number of possible such sets P. Since by assumption, no path in T contains a sub-sequence $a_1, \dots, a_t$ such that a_j is a right descendant of i for all $1 \leq i < j \leq t$, every set P describing a position in T has at least 1 and at most $t + 1$ entries. We conclude that there are at most

$$\sum_{i=1}^{t+1} \binom{h}{i} \leq h^{t+1}$$

elements in T. ☐

Unfortunately, we cannot completely avoid the usage of Ramsey arguments, however, the numbers will be fixed constants depending only on the radius r, the density c at depth r and the order of the crown that we assume is excluded at depth r.

Lemma 5.3 (Finite Canonical Ramsey Theorem [23]) *For every integer k there exists an integer n with the following property: Given any $f : [n] \times [n] \to \mathbb{N}$, there exists an subset $C \subseteq [n]$ of size k such that either of the following holds.*

1. *For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2)$.*
2. *For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2) \Leftrightarrow a_1 = a_2$.*
3. *For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2) \Leftrightarrow b_1 = b_2$.*
4. *For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2) \Leftrightarrow (a_1 = a_2$ and $b_1 = b_2)$.*

Lemma 5.4 *For all integers r, c, K there exists an integer N such that the following property holds. Let G be a digraph with maximum out-degree at most c and let S, T be subsets of vertices of G, such that $|T| \geq N$ and for each $t, t' \in T$ there exist a vertex $s = s(t, t') \in S$, a directed path $P_{s,t}$ of length at most r from s to t and a directed path $P_{s,t'}$ of length at most r from s to t'. Then G contains a crown of order K as a depth-r minor.*

Proof. First note that we can assume that the paths from $s(t, t')$ to t and t' are non-intersecting shortest paths, that $s(t, t')$ are chosen in such a way that the sum of the length of the paths to t and t' is minimum, and that if the path from $s(t, t')$ to t intersects the path from $s(t, t'')$ to t, then they share the same subpath after they first meet.

We order T as $t_1, \ldots, t_n$. For every $1 \leq i < j \leq n$ we denote by $\lambda(i, j)$ the vector formed by the internal vertices of the path from $s(t_i, t_j)$ to t_i in reverse order, followed by $s(t_i, t_j)$, then by the internal vertices of the path from $s(t_i, t_j)$ to t_j. By a standard Ramsey argument, if T is sufficiently large we can extract a large subset $T_1 \subseteq T$ such that for all $t_i, t_j \in T_1$ with $i < j$ the path from $s(t_i, t_j)$ to t_i and the path from $s(t_i, t_j)$ to t_j have the same lengths ℓ_1, and ℓ_2, respectively, independently of the choice of $i < j$. Denote by $\lambda_k(i, j)$, $1 \leq k \leq \ell_1 + \ell_2$, the kth component of the vector $\lambda(i, j)$.

By applying iteratively for $k = 1, 2, \ldots$, Lemma 5.3 there is a large subset $T_2 \subseteq T_1$ such that for every k, on T_2 either all vertices $\lambda_k(i, j)$ are the same or all distinct for all pairs (i, j), or $\lambda_k(i, j)$ is an injective function of i or an injective function of j. First note that no function $\lambda_k(i, j)$ of (i, j) can be constant if $|T_2| > c^r$ as every vertex can reach at most c^r vertices in T_2 in at most r steps. Similarly, for $k \leq \ell_1$, the function $\lambda_k(i, j)$ cannot be an injective function of j and, for $k \geq \ell_1$, the function $\lambda_k(i, j)$ cannot be an injective function of i. In particular, $(i, j) \neq (i', j')$ implies $s(t_i, t_j) \neq s(t_{i'}, t_{j'})$.

Moreover, as we assume that if the path from $s(t, t')$ to t intersects the path from $s(t, t'')$ to t, then they share the same subpath after they first meet, the general situation will be as follows: from the first coordinate to some coordinate $a < \ell_1$ the vertices are given by an injective function of i, then until some coordinate $b > \ell_1$ the vertices are different on all $\lambda_k(i, j)$, then after b the vertices are given by injective functions of j. Let $t_i^- := \lambda_a(i, j)$ (resp. $t_i^+ := \lambda_b(i, j)$) be the element at coordinate a (resp. b) of the vectors λ corresponding to t_i.

Now observe that the assumption that paths are shortest paths imply that the paths from $s(t_i, t_j)$ to t_i^- (resp. from $s(t_i, t_j)$ to t_j^+) are vertex disjoint. However these two families may intersect, but each path of a family intersects at most r paths from the other. By a standard Ramsey argument we can assume (again by considering smaller T_3) that they do not intersect. Also for $i \neq j$, the path from t_i^- to t_i cannot intersect the path from t_j^+ to t_j as their intersection would contradict the minimality assumption on $s(t_i, t_j)$. Furthermore, if the path from t_i^- to t_i intersects the path from t_i^+ to t_i, then they share their subpath after they first meet.

Contracting the paths from t_i^- to t_i and the paths from t_i^+ to t_i, as well as all remaining paths from $s(t_i, t_j)$ to t_i^- and t_j^+ (excluding these vertices) we get the required crown shallow minor. □

Theorem 5.5 *Let G be a digraph with $\mathrm{wcol}_r^{\rightleftarrows}(G) \le c$ and $S_q \not\preccurlyeq_r G$. Then there exists $N = N(c, q, r)$ such that $\gamma_r(G) \in \mathcal{O}(\alpha_r(G)^N)$.*

Proof. Fix an order L witnessing that $\mathrm{wcol}_r^{\rightleftarrows}(G) \le c$. We compute a distance-$r$ dominating set D as follows. Initialize $D := \emptyset$, $A := \emptyset$ and $N := V(G)$. While there is a vertex $v \in N$, the set of non-dominated vertices, pick the smallest such vertex v with respect to L. Add v to A and $\mathrm{WReach}_{2r}^{\rightleftarrows}[G, L, v]$ to D. Mark all newly dominated vertices, that is, remove $N_r^+[\mathrm{WReach}_{2r}^{\rightleftarrows}[G, L, v]]$ from N. If $N = \emptyset$, return D. Clearly, D is a distance-r dominating set of G. We prove that we find a large r-independent subset of A.

Construct the undirected graph H with vertex set A such that two vertices $a, b \in A$ are connected in H if there is $u \in V(G)$ such that $a, b \in N_r^+(u)$. An independent set in H corresponds to an r-scattered subset of A in G.

We claim that every vertex $u \in V(G)$ satisfies $|N_r^+(u) \cap A| \le c$. Fix $u \in V(G)$. Assume towards a contradiction that $|N_r^+(u) \cap A| > c$. For each $a \in N_r^+(u) \cap A$ fix a path P_{ua} of length at most r from u to a. For each path P_{ua}, denote by m_{ua} its minimal element with respect to L. Since $\mathrm{wcol}_r^{\rightleftarrows}(G) \le c$, we have $|\{m_{ua} : N_r^+(u) \cap A\}| \le c$. Since we have more than c paths P_{ua}, there must be two paths P_{ua_1}, P_{ua_2}, $a_1 \neq a_2$, which have the same element m as their minimal element. Without loss of generality assume that $a_1 < a_2$. Since m is the smallest vertex on the path P_{ua}, the subpath of P_{ua_1} between m and a_1 certifies that m is weakly r-reachable from a_1. Hence, when a_1 was added to A, the element m was added to the set D. Now, the subpath of P_{ua_2} between m and a_2 shows that a_2 is at distance at most r from m, and hence a_2 is marked as dominated at this point. This again proves $a_2 \notin A$, a contradiction.

We now build the r-independence tree T of $a_1, \ldots, a_\ell$ (the enumeration of A with respect to L). Using Lemma 5.4, we conclude that there is $N' = (c, r, q)$ such that T does not contain a path with $s = N'$ right descendants. Let $N = N' + 1$.

Hence, by Lemma 5.2, if we have $|A| > (m + N)^N$, then we find a sequence of length m with all left descendants. This set is an r-scattered set, which proves the theorem. □

Clearly, the r-independence tree of a sequence of vertices can be computed in polynomial time, which gives us the following corollary.

Corollary 5.6 *Let $\mathcal{C}$ be a class of digraphs which has bounded crownless expansion. Then for every $r \in \mathbb{N}$, there is a polynomial time algorithm which computes a distance-r dominating set D with $|D| \le p(\gamma_r(G))$ for some polynomial p.*

Furthermore, we will need the duality theorem for subsets of vertices. The following theorem is proved exactly as Theorem 5.5, starting the algorithm with $N = X$.

Theorem 5.7 *Let G be a digraph with $\mathrm{wcol}_r^{\rightleftarrows}(G) \le c$ and $S_q \not\preccurlyeq_r G$. Let $X \subseteq V(G)$. Denote by $\gamma_r(G, X)$ the size of a smallest set $D \subseteq V(G)$ with $X \subseteq N_r^+(D)$ and by $\alpha_r(G, X)$ the size of a largest set $Y \subseteq X$ such that for all $y \neq y' \in Y$ there is no $u \in V(G)$ with $y, y' \in N_r^+(u)$. Then there is $N = N(c, r, q)$ such that $\gamma_r(G, X) \in \mathcal{O}(\alpha_r(G, X)^N)$. Furthermore, there is a polynomial time algorithm which on input G, $X \subseteq V(G)$ and $k \ge 1$ either computes a distance-r dominating set D of X with $|D| \le p(k)$ for some polynomial p (of degree N), or outputs that no such set of size k exists, together with an r-scattered set of size $k + 1$.*

In Theorem 5.7, we need to be able to compute a good weak r-coloring order L in order to find the described distance-r dominating set. We prove that this is possible in the next section.

5.2 Building the kernel

We prove that on classes of bounded crownless expansion we can for every fixed value of r find a polynomial kernel for the directed distance-r dominating set problem. In the following, fix $\mathcal{C}$ and r.

Definition 5.8 (r-domination core) Let G be a digraph. A set $Z \subseteq V(G)$ is an r-*domination core* in G if every minimum-size set which r-dominates Z also r-dominates G.

Clearly, the set $V(G)$ is an r-domination core. We will show how to iteratively remove vertices from this trivial core, to arrive at smaller and smaller domination cores, until finally, we arrive at a core of polynomial size in k. Observe that we do not require that every r-dominating set for Z is also an r-dominating set for G; there can exist dominating sets for Z which are not of minimum size and which do not dominate the whole graph.

Lemma 5.9 *There exists a polynomial p and a polynomial-time algorithm that, given an r-domination core $Z \subseteq V(G)$ with $|Z| > p(k)$, either correctly decides that G cannot be dominated by k vertices, or finds a vertex $z \in Z$ such that $Z \setminus \{z\}$ is still an r-domination core.*

Hence, by gradually reducing $|Z|$, we arrive at the following theorem.

Theorem 5.10 *There exists a polynomial p and a polynomial-time algorithm that, given an instance (G, k) where $G \in \mathcal{C}$, either correctly decides that G cannot be dominated by k vertices, or finds an r-domination core $Z \subseteq V(G)$ with $|Z| \leq p(k)$.*

Proof. (of Lemma 5.9) Let p_1 be the polynomial of the algorithm of Theorem 5.7. Given Z, we first apply the algorithm of that theorem to G, Z, and the parameters r and k. Thus, we either find a distance-r dominating set Y_1 of Z such that $|Y_1| \leq p_1(k)$, or we find a subset $W \subseteq Z$ of size at least $k+1$ that is r-scattered in G. In the latter case, since W is an obstruction to an r-dominating set of size at most k, we may terminate the algorithm and provide a negative answer. Hence, assume that Y_1 has been successfully constructed.

In the first phase, we inductively construct sets $X_1, Y_2, X_2, Y_3, X_3, \ldots$ with $Y_1 \subseteq X_1 \subseteq Y_2 \subseteq X_2 \subseteq \ldots$ as follows:

- If Y_i is already defined, then set $X_i = \mathrm{WReach}_r^{\rightleftarrows}[G, L, Y_i]$.
- If X_i is already defined, then apply the algorithm of Theorem 5.7 to $G - X_i$, $Z \setminus X_i$, and the parameters r and $q(|X_i|)$, where $q(x) = (k+1) \cdot ((r+2) \cdot c \cdot x)^c$.
  1. Suppose the algorithm finds a set $W \subseteq Z \setminus X_i$ that is r-scattered in $G - X_i$ and has cardinality greater than $q(|X_i|)$. Then we let $X = X_i$, terminate the procedure and proceed to the second phase with the pair (X, W).
  2. Otherwise, the algorithm has found an r-dominating set D_{i+1} in the graph $G - X_i$ of size at most $p_1(q(|X_i|))$. Then define $Y_{i+1} \coloneqq X_i \cup D_{i+1}$ and proceed.

Set $q'(x) = c \cdot (q(x) + p_1(x))$. As $\mathrm{wcol}_r^{\rightleftarrows}(G)$ is bounded by c, an induction shows that $|X_i| \leq (q'(k))^i$. Hence, the algorithm consecutively finds r-dominating sets $D_2, D_3, D_4, \ldots$ and constructs sets $X_2, X_3, X_4, \ldots$ up to the point when case (1) is encountered. Then the construction is terminated. We claim that case (1) is always encountered after a constant number of iterations, more precisely, after at most c many iterations.

Towards proving this, assume that a vertex z lies in $Z \setminus X_i$ for some $i \geq 1$. For each D_j, which is an r-dominating set of $Z \setminus X_{j-1}$ in $G - X_{j-1}$, fix a shortest path P_j in $G - X_{j-1}$ from some $d_j \in D_j$ to z. The smallest vertex x_j on that path is weakly r-reachable from d_j, as well as from z, and hence is added to the set X_j (see Lemma 4.4). Hence, after i iterations, we have constructed a set $\{x_1, \ldots, x_i\}$ of vertices weakly r-reachable from z, which proves that the procedure must stop after at most $\mathrm{wcol}_r^{\rightleftarrows}(G) = c$ many steps. Therefore, the construction terminates within c iterations with a pair (X, W) with the following properties:

- $|X| \leq (q'(k))^c$ and $|W| > q(|X|)$;
- X r-dominates Z (because $Y_1 \subseteq X$);
- $W \subseteq Z \setminus X$ and W is r-scattered in $G - X$.

We now define an equivalence relation $\simeq$ on W (recall that $D_r^-(u, X)$ denotes the distance vector of u towards X): for $u, v \in W$, let

$$u \simeq v \quad \Leftrightarrow \quad D_r^-(u, X) = D_r^-(v, X).$$

According to Lemma 4.5, $\simeq$ has at most $q(x) = (k+1) \cdot ((r+2) \cdot c \cdot |X|)^c$ equivalence classes. Since we have that $|W| > q(|X|)$, we infer that there is a class κ of relation $\simeq$ with $|\kappa| > k+1$. Note that we can find such a class κ in polynomial time, by computing the classes of $\simeq$ directly from the definition and examining their sizes. Let z be an arbitrary vertex of κ. We claim that $Z \setminus \{z\}$ is an r-domination core.

To see this, let $Z' = Z \setminus \{z\}$. Take any minimum-size set D which r-dominates Z' in G. If D also dominates z, then D is a minimum-size set which r dominates Z, hence, as Z is an r-domination core, also D is an r-dominating set in G, and the claim follows. Hence, towards a contradiction, assume that z is not r-dominated by D.

Every vertex $s \in \kappa \setminus \{z\}$ is r-dominated by D. For each such s, let $v(s)$ be a vertex of D that r-dominates s, and let $P(s)$ be a path of length at most r that connects $v(s)$ with s. We claim that for each $s \in \kappa \setminus \{z\}$, the path $P(s)$ does not pass through any vertex of X (in particular $v(s) \notin X$). Also, vertices $v(s)$ for $s \in \kappa \setminus \{z\}$ are pairwise distinct. Suppose otherwise and let w be the vertex of $V(P(s)) \cap X$ that is closest to s on $P(s)$. Then, as $D_r^-(s, X) = D_r^-(z, X)$, also z is r-dominated by w, contradicting our assumption that z is not r-dominated by D.

For the second part of the claim, suppose $v(s) = v(s')$ for some distinct $s, s' \in \kappa \setminus \{z\}$. Then $v(s)$ together with the paths $P(s)$ and $P(s')$ would contradict with the fact that W is r-scattered in $G - X$.

This however is not possible, as κ has more than k elements. Hence D' is a Z-dominator, which gives us the desired contradiction. □

Now that it remains to dominate a subset Z, we may keep one representative from each equivalence class in the equivalence relation: $u \cong_{Z,r} v \Leftrightarrow N_r^+(u) \cap Z = N_r^+(v) \cap Z$. As before, there are only polynomially many equivalence classes, hence from a polynomial domination core we can construct a polynomial kernel.

Theorem 5.11 *Let $\mathcal{C}$ be a class of bounded expansion. There is a polynomial time algorithm which on input G, k and r computes a subgraph $G' \subseteq G$ and a set $Z \subseteq V(G')$ such that G can be r-dominated by k vertices if, and only if, Z can be r-dominated by k vertices in G' and $|Z| \leq p(k)$.*

Formally, in Theorem 5.11 we are not computing a kernel for distance-r dominating set, as we do not compute an instance of the distance-r dominating set problem on G' but rather a red-blue instance. As observed by Drange et al. [19], such an annotated instance can be translated back to the standard problem in the following way: add two fresh vertices w, w', add a directed path of length r from w to w', and add a directed path of length r from w to each vertex of $V(G') \setminus Z$. Then the obtained graph G'' has a distance-r dominating set of size at most $k+1$ if, and only if, G' admits a set of at most k vertices that r-dominates Z.

6 Generalized coloring numbers

This section contains more results about the generalized coloring numbers, which, except for Theorem 6.12, are not necessary for the results presented in the main body of the paper. The results are mainly concerned with structural properties of classes of bounded expansion and the weak coloring numbers.

For many algorithmic applications it is useful to compute an order of the vertices as a preprocessing step. For example, the coloring number $\mathrm{col}(G)$ of a digraph G is the minimum integer k such that there exists a linear ordering L of $V(G)$ such that each vertex v has at most k smaller neighbors. It is easily seen that the coloring number of G is equal to its degeneracy. Recall that a graph G is k-degenerate if every subgraph of G has a vertex of degree at most k. Hence the coloring number is a structural measure that measures the edge density of subgraphs of G. The coloring number gets its name from the fact that it bounds the chromatic number – we can simply color the vertices in the order L such that every uncolored vertex gets a color not used by its at most $\mathrm{col}(G)$ smaller neighbors. This bound is very useful, as computing the chromatic number of G is NP-complete, whereas the coloring number can be computed by a greedy algorithm in linear time.

This section is structured as follows. We already defined the weak coloring numbers $\mathrm{wcol}_r^{\rightleftarrows}(G)$ in Section 4, we will define the related measure $\mathrm{adm}_r^{\rightleftarrows}(G)$ in Section 6.1. In Section 6.2, we study the limit parameter $\mathrm{wcol}_\infty^{\rightleftarrows}(G)$. In undirected graphs, this parameter is equal to the well known structural measure tree-depth, which motivates us to call this new measure *directed tree-depth*. In Section 6.3, we introduce the concept of low directed tree-depth colorings, generalizing the very successful concepts of low tree-depth colorings for undirected graphs [48] to directed graphs. This concept allows us to decompose a more complex graph into a few parts whose structure is simpler and whose interaction is highly regular. We prove that classes of directed bounded expansion are exactly those classes which admit low directed tree-depth colorings. Finally, in Section 6.4, we introduce the concept of directed transitive fraternal augmentations, which we use to compute generalized coloring orders in linear time.

6.1 Generalized coloring numbers

For a digraph G and a natural number r, the *r-admissibility* $\mathrm{adm}_r^{\rightleftarrows}[G, L, v]$ of v with respect to L is the maximum size k of a family $\{P_1, \ldots, P_k\}$ of paths of length at most r with one end v, and the other end at a vertex w with $w \le_L v$, and which satisfy $V(P_i) \cap V(P_j) = \{v\}$ for all $1 \le i < j \le k$. As for $r > 0$ we can always let the paths end in the first vertex smaller than v, we can assume that the internal vertices of the paths are larger than v. Note that $\mathrm{adm}_r^{\rightleftarrows}[G, L, v]$ is an integer, whereas $\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]$ is a vertex set. The *r-admissibility* $\mathrm{adm}_r^{\rightleftarrows}(G)$ of G is

$$\mathrm{adm}_r^{\rightleftarrows}(G) \coloneqq \min_{L \in \Pi(G)} \max_{v \in V(G)} \mathrm{adm}_r^{\rightleftarrows}[G, L, v].$$

Note that $\mathrm{adm}_r^{\rightleftarrows}(G)$ and $\mathrm{wcol}_r^{\rightleftarrows}(G)$ are *monotone parameters*, in the sense that if H is a sub-digraph of G, then $\mathrm{adm}_r^{\rightleftarrows}(H) \le \mathrm{adm}_r^{\rightleftarrows}(G)$ and $\mathrm{wcol}_r^{\rightleftarrows}(H) \le \mathrm{wcol}_r^{\rightleftarrows}(G)$. As proved in [40], for all $r \ge 1$ it holds that $\mathrm{wcol}_r^{\rightleftarrows}(G) \le 2 \cdot \mathrm{adm}_r^{\rightleftarrows}(G)^r$.

It will be very useful to work with the following characterization of classes of bounded expansion.

Theorem 6.1 ([40]) *A class $\mathcal{C}$ of digraphs has bounded expansion if, and only if, there is $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{wcol}_r^{\rightleftarrows}(G) \le f(r)$ for all $G \in \mathcal{C}$ and all $r \ge 1$.*

6.2 The limit parameter – directed tree-depth

If G is an n-vertex graph, we denote by $\mathrm{wcol}^{\rightleftarrows}_{\infty}(G)$ the number $\mathrm{wcol}^{\rightleftarrows}_{n}(G)$. If G is undirected, then $\mathrm{wcol}^{\rightleftarrows}_{\infty}(G) = \mathrm{td}(G)$, where $\mathrm{td}(G)$ denotes the tree-depth of G (see Lemma 6.5 in [52]). This motivates our study of the limit parameter in directed graphs, which we call the *directed tree-depth* of a digraph G. We start we an easy example.

Example 6.2 Let P_n be a directed path of order n (hence length $n-2$). Then

$$\mathrm{wcol}^{\rightleftarrows}_{\infty}(P_n) = \lceil \log_2(n+1) \rceil.$$

It follows from an easy induction that $\mathrm{wcol}^{\rightleftarrows}_{\infty}(P_n) \geq \lceil \log_2(n+1) \rceil$: Let k be minimal such that $n \leq 2^k - 1$ and let r be the minimum vertex of P_n (for a linear order witnessing $\mathrm{wcol}^{\rightleftarrows}_{\infty}(P_n)$). Note that r is weakly reachable from every vertex in P_n, and that the deletion of r breaks P_n into two directed path P_a and P_b, one of them (say P_a) having order at least $2^{k-1}-1$. Using the restriction of the linear order of P_n on this path we get

$$\mathrm{wcol}^{\rightleftarrows}_{\infty}(P_n) \geq \mathrm{wcol}^{\rightleftarrows}_{\infty}(P_{2^{k-1}-1}) + 1.$$

Conversely, ordering the vertex set of P^{2^n-1} with its mid vertex r as its minimum, then the mid vertices of the sub-directed paths obtained by deleting r, etc. we get an ordering of P_n witnessing $\mathrm{wcol}^{\rightleftarrows}_{\infty}(P_n) \leq n$.

Lemma 6.3 *Let G be a directed graph such that $\mathrm{wcol}^{\rightleftarrows}_{\infty}(G) \leq c$ for some constant c. Then G does not contain a directed path with length greater than $2^c - 2$ and all directed topological minors of G have arc density at most $4c$.*

Proof. If G contains a directed path P_{2^c} of length $2^c - 1$, then $\mathrm{wcol}^{\rightleftarrows}_{\infty}(G) \geq \mathrm{wcol}^{\rightleftarrows}_{\infty}(P_{2^c}) = c+1$, contradicting the assumption.

Now, let $r := 2^c$. Then every directed topological minor of G is a topological depth-r minor of G. Let H be the densest topological depth-r minor of G and assume towards a contradiction that $|E(H)|/|V(H)| > 4c$. Let $\bar{H}$ be the underlying undirected graph of H. We apply Proposition 1.2.2 of [18] to $\bar{H}$, that is, we iteratively remove small degree vertices of $\bar{H}$ to obtain a graph $\bar{H}'$ with minimum degree $\delta(\bar{H}') \geq |E(\bar{H})|/|V(\bar{H})| \geq 2c$. Let H' be a directed subgraph of H induced by $V(\bar{H}')$, where we remove for each pair of bi-directed arcs $(u,v),(v,u)$ one of the two arcs (but keep the other). Then H' has minimum degree (in-degree plus out-degree) at least c.

Let L be an order of $V(G)$ witnessing that $\mathrm{wcol}_{\infty}(G) \leq c$. This order also induces an order on $V(H')$. Let v be the largest vertex of G that corresponds to a vertex of H'. Then v weakly reaches more than c vertices in G; it weakly reaches at least one vertex on each path connecting v with the vertices of G corresponding to its neighbors in H. Together with the vertex v itself we get $\mathrm{wcol}_{\infty}(G) \geq \mathrm{wcol}_{\infty}(H') > c$, contradicting our assumption. ☐

6.3 Sparse directed tree-depth colourings

In this section, we use the nice properties of the generalized coloring numbers to decompose a more complex graph into a few parts whose structure is simpler and whose interaction is highly regular. More precisely, we prove that classes of directed bounded expansion are exactly those classes which admit low directed tree-depth colorings.

Theorem 6.4 *A class $\mathcal{C}$ of directed graphs has directed bounded expansion if, and only if, for every integer p there are integers $N(p)$, $\ell(p)$ and $d(p)$ such that every graph $G \in \mathcal{C}$ can be colored with $N(p)$ many colors such that the combination of any $i \leq p$ color classes induces a subgraph H which excludes a directed path of length $\ell(p)$ and all directed topological minors of H have density at most $d(p)$.*

The first direction of the theorem follows from the next lemma and Lemma 6.3.

Lemma 6.5 *Let G be a digraph, let p be an integer and assume that $\mathrm{wcol}^{\rightleftarrows}_{2^p}(G) \leq c$ for some constant c. Then G can be colored with c colors such that the combination of any $i \leq p$ color classes induces a subgraph H with $\mathrm{wcol}^{\rightleftarrows}_{\infty}(H) \leq i$.*

Proof. Let L be an order of $V(G)$ such that $|\mathrm{WReach}^{\rightleftarrows}_{2^p}[G, L, v]| \leq c$ for all $v \in V(G)$. Color the vertices of G greedily along the order L starting from the least element such that the color assigned to a vertex v is distinct from the colors assigned to $\mathrm{WReach}^{\rightleftarrows}_{2^p}[G, L, v] \setminus \{v\}$. As $|\mathrm{WReach}^{\rightleftarrows}_{2^p}[G, L, v]| \leq c$, c colors suffice for this.

Let H be a sub-digraph induced by $i \leq p$ colors. According to the coloring rule above it follows that $\mathrm{wcol}^{\rightleftarrows}_{2^p}(H) \leq i$.

If H contains a directed path of length 2^p then

$$\mathrm{wcol}^{\rightleftarrows}_{2^p}(H) \geq \mathrm{wcol}^{\rightleftarrows}_{2^p}(P_{2^p}) = \mathrm{wcol}^{\rightleftarrows}_{\infty}(P_{2^p}) > p,$$

contradicting $\mathrm{wcol}^{\rightleftarrows}_{2^p}(H) \leq i$. It follows that $\mathrm{wcol}^{\rightleftarrows}_{\infty}(H) = \mathrm{wcol}^{\rightleftarrows}_{2^p}(H) \leq i$. □

For the other direction of Theorem 6.4 we use the characterization of bounded expansion classes by bounded r-admissibility. For this, we need two more lemmas. The first lemma describes an obstruction for small r-admissibility in directed graphs.

Lemma 6.6 (see Lemma 4.7 of [40]) *Let G be a directed graph. If $\mathrm{adm}^{\rightleftarrows}_r(G) \geq c + 1$ for some constant c, then there exists a set $S \subseteq V(G)$ such that every $v \in S$ is connected to at least c other vertices of S via directed paths from v of length at most r intersecting only in v whose internal vertices belong to $V(G) \setminus S$.*

In the above lemma, when we say that a vertex v is connected to a vertex w by a directed path, we mean that v and w are the end-vertices of a directed path, the path may go in either direction.

The next lemma describes the interaction of high degree vertices with sets in bounded expansion classes. A variant of the lemma was originally developed for undirected graphs in [19] and proved in [40] for directed graphs.

Let G be a digraph, $X \subseteq V(G)$, $u \in V(G) \setminus X$ and $r \in \mathbb{N}$. The r*-projection* of u onto X is the set $M^G_r(u, X)$ of all vertices $v \in X$ such that there is a directed path between u and v in G (in either direction) of length at most r with all internal vertices in $V(G) \setminus X$.

Lemma 6.7 ([40]) *Let G be a digraph, $r \geq 0$ and $X \subseteq V(G)$. There exists a set $\mathrm{cl}^G_r(X) \subseteq V(G)$, called an* r-closure of X in G *with the following properties. Let $\xi := \lceil 2\nabla_{r-1}(G)\rceil$.*

1. *$X \cap \mathrm{cl}^G_r(X) = \emptyset$;*
2. *$|\mathrm{cl}^G_r(X)| \leq (r-1)\xi \cdot |X|$; and*
3. *$|M^{G-\mathrm{cl}^G_r(X)}_r(u, X)| \leq \xi$ for all $u \in V(G) \setminus (X \cup \mathrm{cl}^G_r(X))$.*

We can now prove the reverse direction of Theorem 6.4.

Lemma 6.8 *Let $\mathcal{C}$ be a class of directed graphs such that for every integer r there are integers $N(r), \ell(r)$ and $d(r)$ such that every graph $G \in \mathcal{C}$ can be colored with $N(r)$ many colors such that the combination of any $i \leq r$ color classes induces a subgraph H which excludes a directed path of length $\ell(r)$ and all directed topological minors of H have density at most $d(r)$. Then $\mathcal{C}$ has bounded expansion.*

Proof. Let $G \in \mathcal{C}$, and let $r \geq 0$. We color G with $N(r+1)$ many colors such that the combination of any $i \leq r+1$ color classes induces a subgraph H which excludes a directed path of length $\ell(r+1)$ and all directed topological minors of H have density at most $d(r+1)$. According to Theorem 2.1, all directed minors of H have density at most $q \coloneqq 32 \cdot (4d(r+1))^{(\ell(r+1)+1)^2}$. We want to prove that $\mathrm{adm}_r^{\rightleftarrows}(G) \leq c$ for a constant depending only on q and r to be determined in the course of the proof.

Assume towards a contradiction that $\mathrm{adm}_r^{\rightleftarrows}(G) \geq c+1$. According to Lemma 6.6, there exists a set $S \subseteq V(G)$ such that every $v \in S$ is connected to at least c other vertices of S via directed paths of length at most r intersecting only in v and whose internal vertices belong to $V(G) \setminus S$. Denote the set of all of these paths by $\mathcal{P}$. Since there are at most $x = \binom{N(r+1)}{r+1}$ possible ways to color a path of length at most r, we find a set $\mathcal{P}' \subseteq \mathcal{P}$ of paths all of which have the same set of colors of size $i \leq r+1$ and $|\mathcal{P}'| \geq (|S| \cdot c)/x$. Let H be the subgraph of G induced by the vertices of $\mathcal{P}'$. Let $S' \coloneqq S \cap V(H)$. By definition, H is colored with at most $r+1$ colors, and hence by assumption its directed minors have density at most q. Let $\xi \coloneqq \lceil 2q \rceil$.

We construct $\mathrm{cl}_r^H(S')$ according to Lemma 6.7, which has size at most $(r-1)\xi \cdot |S'|$. We now iteratively contract short paths between S' and $\mathrm{cl}_r^H(S')$. For each path P in $\mathcal{P}'$ with an end-vertex $v \in S'$, let P_0 be the restriction of P between v and the vertex $w \in \mathrm{cl}_r^H(S') \cup S'$ which is the nearest to v, but not v itself. Let $\mathcal{P}_0 \coloneqq \{P_0 : P \in \mathcal{P}'\}$. If two paths of $\mathcal{P}_0$ have the same initial and terminal vertex (but are oriented in different directions), we remove one of them from $\mathcal{P}_0$. We hence have $|\mathcal{P}_0| \geq |S| \cdot c/(2x) \geq |S'| \cdot c/(2x)$.

Now, for $i = 0, 1, \ldots$, as long as there exists $P \in \mathcal{P}_i$, contract P to an arc and remove from $\mathcal{P}_i$ all paths which intersect P to obtain $\mathcal{P}_{i+1}$. We claim that every internal vertex u of P can intersect with at most ξ many other paths $P' \in \mathcal{P}_i$. This is because every path $P \in \mathcal{P}'$ which uses vertex u must have both their end-vertices in $M_r^{G-\mathrm{cl}_r^H(S')}(u, S')$ (by definition of $\mathcal{P}_0$ all paths are cut when hitting $\mathrm{cl}_r^H(S') \cup S'$). Hence, as every internal vertex u of P satisfies $|M_r^{G-\mathrm{cl}_r^H(S')}(u, S')| \leq \xi$ by assumption, P can intersect with at most $r\xi$ many other paths $P' \in \mathcal{P}_i$.

Hence, hence after $i+1$ contractions, we have $|\mathcal{P}_{i+1}| \geq \frac{c}{2}|S'| - (i+1)r\xi$. Note that we are constructing a graph $H^* \preccurlyeq_{r-1}^d H$ with vertex set $S' \cup \mathrm{cl}_r^H(S')$, that is, with at most $((r-1)\xi+1) \cdot |S'|$ vertices, which by assumption on q can have at most $\xi/2 \cdot ((r-1)\xi+1) \cdot |S'|$ many arcs. This gives a contradiction for $c > r\xi^2((r-1)\xi+1)\binom{N(r+1)}{r+1}$, e.g. for $c = r^2\xi^3\binom{N(r+1)}{r+1}$. ☐

6.4 Transitive fraternal augmentations

To approximate the weak coloring numbers, we use the transitive fraternal augmentation method which was employed also in the undirected setting (see [49] and Section 7.4 of the textbook [52]).

Let G be a digraph. An *re-orientation of* G is a digraph H such that for each arc $(u,v) \in E(G)$ exactly one of (u,v) or (v,u) is an arc of H. We also say that an arc set F is a re-orientation of an arc set E, if for each arc $(u,v) \in E$ exactly one of (u,v) or (v,u) is in F.

For $r \in \mathbb{N}$ and a digraph G, a *depth-r transitive fraternal augmentation of* G is a directed graph G_r with arc set $E(G_r)$ partitioned as $E_1 \cup \ldots \cup E_r$, such that

- the set E_1 is a re-orientation of $E(G)$;
- for every every arc $(u,v) \in E_i$, $1 \leq i \leq r$, there exists a directed path of length at most i with endpoints u, v (in either direction) in G;
- $(u,v) \in E(G_r)$ implies $(v,u) \notin E(G_r)$ for all $u, v \in V(G_r)$;
- for all $1 \leq i \leq j \leq r$ with $i+j \leq r$, and for all $u, v, w \in V(G)$, if $(w,u) \in E_i$ and $(w,v) \in E_j$ and there exists a directed path of length at most $i+j$ between u and v in G, then (u,v) or (v,u) belongs to $\bigcup_{k=1}^{i+j} E_k$.

- for all $1 \leq i \leq j \leq r$ with $i+j \leq r$, and for all $u, v, w \in V(G)$, if $(u,v) \in E_i$ and $(v,w) \in E_j$ there exists a directed path of length at most $i+j$ between u and w in G, then (u,w) or (w,u) belongs to $\bigcup_{k=1}^{i+j} E_k$.

Lemma 6.9 *Let G be an n-vertex digraph with $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq c$. Then we can compute a depth-r transitive fraternal augmentation H with $\Delta^+(H) \leq 4^{r-1}(2c)^{2^{r-1}}$ in time $\mathcal{O}(r \cdot 4^{r-1}(2c)^{2^{r-1}} \cdot n)$.*

Proof. We compute the sets $E_1, \ldots, E_r$ as follows. We re-orient $E(G)$ such that the out-degree of E_1 is at most c. This is possible, as $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq c$ in particular implies that G is c-degenerate. Hence E_1 satisfies the above conditions. Now, assume that $E_1, \ldots, E_i$ have been constructed and satisfy the above conditions. For all $1 \leq j_1 \leq j_2 \leq i+1$ with $j_1 + j_2 = i+1$, and for all $u, v, w \in V(G)$, if $(w,u) \in E_{j_1}$ and $(w,v) \in E_{j_2}$ and there exists a path of length at most $i+1$ between u and v in G, we introduce an undirected edge $\{u,v\}$ to be oriented appropriately to E_{i+1}. Also transitive arcs are introduced as undirected edges accordingly. We then orient the resulting edge set greedily to obtain E_{i+1}. Clearly, E_{i+1} again satisfies the above conditions, and hence, after r steps we have computed a depth-r transitive fraternal augmentation of G. It remains to prove the claimed degree bounds.

We define the following arc sets $F_1, \ldots, F_r$. Fix an order L witnessing that $\mathrm{wcol}_r^{\rightleftarrows}(G) \leq c$. The set F_i contains all arcs (u,v) such that there is a path of length at most i in G between u and v such that v is the smallest vertex of the path with respect L, that is, the sets F_i represent the weak r-reachability relation of L. For each vertex $v \in V(G)$ we will determine a bound f_i on the number of arcs which are oriented differently in the arc sets E_i and F_i, more precisely, f_i will be a bound on the number of arcs with one end v and which are arcs of $\left(\bigcup_{1 \leq j \leq i} E_j \setminus \bigcup_{1 \leq j \leq i} F_j\right) \cup \left(\bigcup_{1 \leq j \leq i} F_j \setminus \bigcup_{1 \leq j \leq i} E_j\right)$.

First, let $i = 1$. Since v has at most c smaller neighbors in the first greedy orientation, at most c arcs of E_1 may be directed away from v, which are all directed towards v in F_1. On the other hand, there may be c arcs in F_1 being directed towards v which are directed away from v in E_1. In total, we have at most $2c$ wrongly directed arcs and we define $f_1 = 2c$.

Now assume that we have defined the number f_i for some fixed $i \geq 1$. Consider $v \in V(G)$ and see how many undirected edges including v are introduced to E_{i+1}, which are not also edges of F_{i+1}. First observe that for each triple u, v, w, if we have only one wrongly directed arc, then we do not introduce an edge which is not also present in F_{i+1}. Consider e.g. the case that there is an arc $(w,v) \in F_{j_1} \cap E_{j_1}$ and an arc $(u,w) \in E_{j_2}$ with $(w,u) \in F_{j_2}$, $j_1 + j_2 = i+1$. Then we have $(v,u) \in F_{i+1}$ as a transitive arc, while we have $\{u,v\}$ as a fraternal edge to be directed in E_{i+1}. The other cases are similar.

Hence a wrongly oriented edge $\{u,v\}$ can only be introduced if there is a vertex $w \in V(G)$ such that both (v,w) and (u,w) are wrongly oriented. However, there are at most f_i such choices for w and each such w also has at most so many bad choices. Hence, every vertex has at most f_i^2 wrongly oriented edges in E_{i+1} which are not edges of F_{i+1}. Hence, as F_{i+1} is c-degenerate, E_{i+1} is $c + f_i^2$-degenerate and the greedy orientation procedure will produce an orientation which for every vertex coincides on all but $2(c + f_i^2) \leq 2(2f_i^2) = 4f_i^2$ edges, as in the case $i = 1$. We can hence define $f_i \coloneqq 4^{i-1}(2c)^{2^{i-1}}$ and conclude.

For the running time, observe that a greedy orientation of a graph with m edges can be computed in time $\mathcal{O}(m)$. As we have to compute r orientations on graphs with at most $4^{r-1}(2c)^{2^{r-1}} \cdot n$ edges, the claim follows. □

We now show that transitive fraternal augmentations can be used to compute good linear orders for the weak coloring numbers. We need one more lemma.

Lemma 6.10 *Let P be a directed path of length at most r in a digraph G with end vertices u, v. Then in every depth-r transitive fraternal augmentation H of G, either (u,v) or (v,u) are arcs of H, or there is $w \in V(G)$ such that (u,w) and (v,w) are arcs of H.*

Proof. Since the appropriate sub-paths of P witness the existence of paths of length i in the i-th augmentation step, we can argue as in the undirected case, compare to Lemma 7.9 of [52]. □

Lemma 6.11 *Let L be a greedy orientation of a depth-r transitive fraternal augmentation H of G with $\Delta^+(H) \leq d$, such that every vertex has at most c smaller neighbors with respect to L. Then*

$$|\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]| \leq (d+1)c + 1$$

for all $v \in V(G)$.

Proof. For each vertex $v \in V(G)$ we count the number of end-vertices of paths of length at most r from v such that the end-vertex is the smallest vertex of the path. This number is exactly $|\mathrm{WReach}_r^{\rightleftarrows}[G, L, v)]|$.

By Lemma 6.10, for each such path with end-vertex $w \neq v$, we either have an arc (v, w) or an arc (w, v) in H or there is u on the path and we have arcs (v, u), (w, u) in H. By assumption on L there are at most c arcs (v, w) or (w, v) such that $w <_L v$. Furthermore, we have at most d arcs (v, u), as v has out-degree at most d and for each such u there are at most c arcs (w, u) such that $w <_L u$ by assumption on L. These are exactly the pairs of arcs we have to consider, as no vertex on the path from v to w may be smaller than w. Hence in total we have $|\mathrm{WReach}_r^{\rightleftarrows}[G, L, v]| \leq c + d \cdot c + 1 = (d+1)c + 1$. □

Algorithmically, to obtain a good order from Lemma 6.11, we have to compute one final greedy orientation step. We hence obtain the following theorem.

Theorem 6.12 *If $\mathcal{C}$ is a class of digraphs of bounded expansion, then there is a function $f : \mathbb{N} \to \mathbb{N}$ and an algorithm which on input $G \in \mathcal{C}$ and $r \in \mathbb{N}$ computes an order L with $|\mathrm{WReach}_r^{\rightleftarrows}[G, L.v]| \leq f(r)$ for all $v \in V(G)$ in time $\mathcal{O}(f(r) \cdot n)$.*

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