On The Borel Inseparability of Game Tree Languages

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In contexts where we study hierarchies (Descriptive Set Theory, Automata Theory, Logic, Complexity Theory) we ask about:

\[
\begin{array}{c}
\Sigma_n \\
\uparrow \\
\Delta_n \\
\downarrow \\
\Pi_n \\
\uparrow \\
\Sigma_2 \\
\uparrow \\
\Delta_2 \\
\downarrow \\
\Sigma_1 \\
\uparrow \\
\Delta_1 \\
\end{array}
\]
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  e.g. **separation property**
Hierarchies

In contexts where we study hierarchies (Descriptive Set Theory, Automata Theory, Logic, Complexity Theory) we ask about:

- strictness of the hierarchy,
- structural properties of the hierarchy e.g. separation property
Separation vs. Simplification

Given two classes of subsets of some universe $\mathcal{U}$:
- $\mathcal{L}$ — ”large”
- $S$ — ”small”

we define two notions:

Definition (Separation)
Any two disjoint sets $L, M \in \mathcal{L}$ are separated by some set $K$ in $S$ (i.e., $L \subseteq K \subseteq \mathcal{U} - M$).

Definition (Simplification)
Whenever $L$ and its complement $\overline{L}$ are both in $L$, they are also in $S$.

Separation implies simplification, but in general not vice versa.

We consider only $S$ classes closed under complement.
Given two classes of subsets of some universe $\mathcal{U}$:
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- We consider only $\mathcal{S}$ classes closed under complement
Nondeterministic tree automaton (parity condition)

\[ A = (A, Q, q_I, \delta, rank) \]
- **Nondeterministic tree automaton** *(parity condition)*

\[ \mathcal{A} = (A, Q, q_1, \delta, \text{rank}) \]

- Transitions in \( \delta \) have the form:

![Diagram of a tree automaton node with transitions](image)
Nondeterministic tree automaton (parity condition)

\[ \mathcal{A} = (A, Q, q_I, \delta, rank) \]

- Transitions in \( \delta \) have the form:

\[ q \xrightarrow{a} q_1 \quad \text{and} \quad q \xrightarrow{a} q_2 \]

- \( rank : Q \rightarrow \mathbb{N} \) (priorities)
Nondeterministic tree automaton (parity condition)

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- **Acceptance:** on every path
  maximal priority occurring infinitely often is **even**
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**Index hierarchy** (Rabin-Mostowski)
Level \( n \) — automata use \( n \) priorities (alternations):
- \( \Sigma_n \) – greatest priority is odd,
- \( \Pi_n \) – greatest priority even,
- \( \text{Comp}_n \supseteq \Sigma_n \cup \Pi_n \) (compositional class)
ALTERNATING INDEX HIERARCHY

\[
\begin{array}{c}
\cdots \\
\Sigma_{n+1} \\
\Sigma_{n+1} \cap \Pi_{n+1} \\
\downarrow \\
Comp_n \\
\cdots \\
\Sigma_3 \\
\Sigma_3 \cap \Pi_3 \\
\downarrow \\
Comp_2 \\
\cdots \\
\Sigma_2 \\
\Sigma_2 \cap \Pi_2 \\
\downarrow \\
Comp_1 \\
\cdots \\
\Sigma_1 \\
\downarrow \\
\Pi_1
\end{array}
\]
**Index Hierarchy — Separation**

**Alternating Index Hierarchy**

\[
\vdots \\
\Sigma_{n+1} \quad \Pi_{n+1} \\
\Sigma_{n+1} \cap \Pi_{n+1} \\
\vdots \\
\Sigma_{3} \quad \Pi_{3} \\
\Sigma_{3} \cap \Pi_{3} \\
\vdots \\
\Sigma_{2} \quad \Pi_{2} \\
\Sigma_{2} \cap \Pi_{2} \\
[\text{Rabin'70}] \\
\vdots \\
\Sigma_{1} \quad \Pi_{1}
\]

Separation and even simplification fail \[\text{[S&A 2005]}\] 
Simplification holds \[\text{[Rabin'70]}\] 
Separation holds \[\text{[this paper]}\] 
Adjustment of Rabin’s proof of simplification
Index Hierarchy — Separation

Alternating Index Hierarchy

\[
\begin{align*}
\Sigma_{n+1} & \quad \Pi_{n+1} \\
\Sigma_{n+1} \cap \Pi_{n+1} & \quad ? = II \\
| & \\
Comp_n &
\end{align*}
\]

\[
\begin{align*}
\Sigma_3 & \quad \Pi_3 \\
\Sigma_3 \cap \Pi_3 & \quad ? = II \\
| & \\
Comp_2 &
\end{align*}
\]

\[
\begin{align*}
\Sigma_2 & \quad \Pi_2 \\
\Sigma_2 \cap \Pi_2 & \quad II \\
| & \\
Comp_1 &
\end{align*}
\]

\[
\begin{align*}
\Sigma_1 & \quad \Pi_1
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**Index Hierarchy — Separation**

**Alternating Index Hierarchy**

\[
\begin{array}{c}
\vdots \\
\Sigma_{n+1} \quad \Pi_{n+1} \\
\Sigma_{n+1} \cap \Pi_{n+1} \\
\text{[S&A'05]} \\
Comp_n \\
\vdots \\
\Sigma_3 \quad \Pi_3 \\
\Sigma_3 \cap \Pi_3 \\
\text{[S&A'05]} \\
Comp_2 \\
\Sigma_2 \quad \Pi_2 \\
\Sigma_2 \cap \Pi_2 \\
Comp_1 \\
\Sigma_1 \quad \Pi_1
\end{array}
\]
Alternating Index Hierarchy

\[ \Sigma_n \cap \Pi_n \] is closed under \( \neg \)

\( \downarrow \) separation

and even simplification fail

[S&A 2005]
NONDETERMINISTIC INDEX HIERARCHY

\[
\begin{array}{ccc}
\Sigma_{n+1} & \cap & \Pi_{n+1} \\
\Sigma_{n+1} \cap \Pi_{n+1} & \downarrow & \text{Comp}_n \\
\Sigma_3 & \cap & \Pi_3 \\
\Sigma_3 \cap \Pi_3 & \downarrow & \text{Comp}_2 \\
\Sigma_2 & \cap & \Pi_2 \\
\Sigma_2 \cap \Pi_2 & \downarrow & \text{Comp}_1 \\
\Sigma_1 & \cap & \Pi_1 \\
\end{array}
\]
**Index Hierarchy — Separation**

**Non-deterministic Index Hierarchy**

\[
\begin{align*}
\Sigma_{n+1} & \quad \Pi_{n+1} \\
\Sigma_{n+1} \cap \Pi_{n+1} & \\
\Sigma_3 & \quad \Pi_3 \\
\Sigma_3 \cap \Pi_3 & \\
\Sigma_2 & \quad \Pi_2 \\
\Sigma_2 \cap \Pi_2 & \\
\Sigma_1 & \quad \Pi_1
\end{align*}
\]

Separation and even simplification fail [S&A 2005]
Index Hierarchy — Separation

NonDeterministic Index Hierarchy

Separation and even simplification fail [S&A 2005]
COMMENTARY ON NONDETERMINISTIC INDEX HIERARCHY

separation and even simplification fail

[S&A 2005]

 separation (and simplification) hold

[S&A 2005]
Index Hierarchy — Separation

NonDeterministic Index Hierarchy

\[
\begin{align*}
\Sigma_{n+1} \searrow & \quad \Pi_{n+1} \\
\Sigma_{n+1} \searrow \cap & \quad \Pi_{n+1} \\
\Sigma_3 \searrow & \quad \Pi_3 \\
\Sigma_3 \searrow \cap & \quad \Pi_3 \\
\Sigma_2 \searrow & \quad \Pi_2 \\
\Sigma_2 \searrow \cap & \quad \Pi_2 \\
\Sigma_1 \searrow & \quad \Pi_1
\end{align*}
\]

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**Index Hierarchy — Separation**

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\[ \begin{array}{c}
\vdots \\
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\vdots \\
\Sigma_3 \quad \Pi_3 \\
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\vdots \\
\Sigma_2 \quad \Pi_2 \\
\Sigma_2 \cap \Pi_2 \\
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- Separation (and simplification) hold [S&A 2005]
- Separation holds [Rabin’70]

Adjustment of Rabin’s proof of simplification
INDEX HIERARCHY — SEPARATION

NONDETERMINISTIC INDEX HIERARCHY

\[
\begin{align*}
\vdots & \quad \vdots & \quad \vdots \\
\Sigma_{n+1} & \quad \Pi_{n+1} \\
\Sigma_{n+1} \cap \Pi_{n+1} & \quad \text{Comp}_n \\
\vdots & \quad \vdots & \quad \vdots \\
\Sigma_3 & \quad \Pi_3 \\
\Sigma_3 \cap \Pi_3 & \quad \text{Comp}_2 \\
\vdots & \quad \vdots & \quad \vdots \\
\Sigma_2 & \quad \Pi_2 \\
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\ldots & \quad \ldots & \quad \ldots \\
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\Sigma_{n+1} \cap \Pi_{n+1} & \quad Comp_n \\
\ldots & \quad \ldots & \quad \ldots \\
\Sigma_3 & \quad \Pi_3 \\
\Sigma_3 \cap \Pi_3 & \quad Comp_2 \\
\ldots & \quad \ldots & \quad \ldots \\
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Inseparable Pair

- $\Sigma = \{\exists, \forall\} \times \{0, 1\}$
- $T_\Sigma$ — class of $(0, 1)$-game trees
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\[ T_\Sigma \] — class of \((0, 1)\)-game trees
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  - $W_{0,1}$ — Set of trees where $\exists$ has a strategy to force only 0’s from some moment on
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\[ \Sigma = \{\exists, \forall\} \times \{0, 1\} \]

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- The inseparable pair:
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  - \( W'_{0,1} \) — obtained from \( W_{0,1} \) by interchanging \( \forall \leftrightarrow \exists \) and \( 0 \leftrightarrow 1 \)
**Inseparable Pair**

- $\Sigma = \{\exists, \forall\} \times \{0, 1\}$
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- The inseparable pair:
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- We consider standard topology on $T_\Sigma$ (first difference metric)
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- We consider standard topology on $T_\Sigma$ (first difference metric)
- We use these to prove something stronger than needed:
**Main Result**

**Definition**

The Borel sets constitute the least family containing *open sets* and closed under *complement* and *countable union*.
**Main Result**

**Definition**

The Borel sets constitute the least family containing open sets and closed under complement and countable union.

**Theorem**

There is no Borel set separating $W_{0,1}$ and $W_{0,1}'$. 

$W_{0,1}$ and $W_{0,1}'$ are recognized by nondeterministic automata with co-Büchi condition.
**Main Result**

**Definition**

The Borel sets constitute the least family containing open sets and closed under complement and countable union.

**Theorem**

There is no Borel set separating $W_{0,1}$ and $W_{0,1}'$.

- $W_{0,1}$ and $W_{0,1}'$ are recognized by nondeterministic automata with co-Büchi condition.
- $\text{Comp}_1 \subseteq \text{Borel}$

**Corollary**

There exists a pair of disjoint sets recognized by nondeterministic $\Sigma_2$ automata, that is not separated by any $\text{Comp}_1$-recognized set.
We show that our pair has a capacity to describe every Borel set.
Core of the Proof

- We show that our pair has a capacity to describe every Borel set

Lemma

Let $B \subseteq T_\Sigma$ be an arbitrary Borel set. There exists a continuous function $F_B : T_\Sigma \rightarrow T_\Sigma$ such that:

\[
\begin{align*}
t \in B & \implies F_B(t) \in W_{0,1} \\
t \notin B & \implies F_B(t) \in W'_{0,1}
\end{align*}
\]
Core of the Proof

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**Lemma**

Let $B \subseteq T_{\Sigma}$ be an arbitrary Borel set. There exists a continuous function $F_B : T_{\Sigma} \to T_{\Sigma}$ such that:

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Proof of the Lemma

Consider class $C$ of sets $B$ for which there is such function $F_B$
Proof of the Lemma

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It suffices to show that $C$

- includes all clopen sets
- is closed under complementation
- is closed under countable unions
Proof of the Lemma

- Consider class $C$ of sets $B$ for which there is such function $F_B$
- It suffices to show that $C$
  - includes all clopen sets — characteristic function
  - is closed under complementation
  - is closed under countable unions

![Diagram showing the meta-game construction](image-url)
Consider class $C$ of sets $B$ for which there is such function $F_B$

It suffices to show that $C$

- includes all clopen sets — characteristic function
- is closed under complementation — by symmetry of $W_{0,1}$ and $W'_{0,1}$
- is closed under countable unions
Proof of the Lemma

- Consider class $C$ of sets $B$ for which there is such function $F_B$
- It suffices to show that $C$
  - includes all clopen sets — characteristic function
  - is closed under complementation — by symmetry of $W_{0,1}$ and $W'_{0,1}$
  - is closed under countable unions — meta-game construction

$$B = \bigcup_{i \in \mathbb{N}} B_i$$
**Definition**

Class $\mathcal{L}$ has **First Separation Property** if separation property holds for $\mathcal{L}$ and $S = \{X : X, \overline{X} \in \mathcal{L}\}$.

- For such $\mathcal{L}$ and $S$ simplification holds trivially.

**Diagram**

```
  Alternating hierarchy
    : : : :
  \Sigma_{n+1} \quad \Pi_{n+1}
    \Sigma_{n+1} \cap \Pi_{n+1}
    Comp_n
    : : : :
  \Sigma_3 \quad \Pi_3
    \Sigma_3 \cap \Pi_3
    Comp_2
    : : : :
  \Sigma_2 \quad \Pi_2
    \Sigma_2 \cap \Pi_2
    Comp_1
    : : : :
  \Sigma_1 \quad \Pi_1
```

**Theorem**

First Separation Property holds Büchi class.

**Corollary**

First Separation Property fails for co-Büchi class.

Weaker version of our theorem (our languages even nondet. co-Büchi).
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First Separation Property **fails for co-Büchi class**.

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**First Separation Property** holds Büchi class.

- alternating Büchi class = nondeterministic Büchi class.
**Definition**

Class $\mathcal{L}$ has **First Separation Property** if separation property holds for $\mathcal{L}$ and $S = \{X : X, \overline{X} \in \mathcal{L}\}$.

- For such $\mathcal{L}$ and $S$ simplification holds trivially.

**Corollary**

First Separation Property *fails* for co-Büchi class.

- Weaker version of our theorem (our languages even nondet. co-Büchi).

**Theorem**

First Separation Property *holds* Büchi class.

- alternating Büchi class = nondeterministic Büchi class.
- First Separation Property for higher levels — open
- Borel sets constitute a hierarchy

\[
\begin{array}{c}
\Sigma_0^0 \quad \Sigma_1^0 \quad \Delta_0^0 \\
\Sigma_2^0 \quad \Delta_2^0 \quad \Pi_2^0 \\
\Sigma_1^1 \quad \Delta_1^0 \\
\Sigma_1^2 \quad \Pi_1^0 \\
\vdots \\
\omega_1
\end{array}
\]
Borel sets constitute a hierarchy

Above this hierarchy:

- Analytic sets ($\Sigma^1_1$) — projections of Borel sets
- Coanalytic sets ($\Pi^1_1$) — complements of analytic sets

\[
\begin{align*}
\Sigma^1_1 & \to \Pi^1_1 \\
\Sigma^0_n & \to \Pi^0_n \\
\Delta^0_n \\
\Sigma^0_2 & \to \Pi^0_2 \\
\Delta^0_2 \\
\Sigma^0_1 & \to \Pi^0_1 \\
\Delta^0_1 \\
\omega_1 &
\end{align*}
\]
- Borel sets constitute a hierarchy
- Above this hierarchy:
  - **Analytic** sets ($\Sigma_1^1$) — projections of Borel sets
  - **Coanalytic** sets ($\Pi_1^1$) — complements of analytic sets
- $\Sigma_1^1 \cap \Pi_1^1 = \text{Borel}$
Borel sets constitute a hierarchy

Above this hierarchy:
- Analytic sets ($\Sigma^1_1$) — projections of Borel sets
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$\Sigma^1_1 \cap \Pi^1_1 = \text{Borel}$

For $\Sigma^1_1$ First Separation Property holds.
Borel sets constitute a hierarchy

Above this hierarchy:
- **Analytic sets** ($\Sigma^1_1$) — projections of Borel sets
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- For $\Sigma^1_1$ First Separation Property **holds**.
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Above this hierarchy:
- **Analytic sets** ($\Sigma^1_1$) — projections of Borel sets
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$\Sigma^1_1 \cap \Pi^1_1 = \text{Borel}$

- For $\Sigma^1_1$ First Separation Property **holds**.
- For $\Pi^1_1$ First Separation Property **fails**.

It gives some analogy between
- $\Sigma^1_1$ and Büchi class
- $\Pi^1_1$ and co-Büchi class
Broken Analogy

Other potential candidate pair:
Other potential candidate pair:

- $W_{0,1}$ and $W'_{0,1}$ are $\Pi^1_1$-complete (coanalytic complete) sets
- Classical Borel inseparable $\Pi^1_1$ pair (translated to our context):

  \[WF = \{ t \in T_{\{0,1\}} : \text{every path has only finitely many 1’s} \}\]
  \[UB = \{ t \in T_{\{0,1\}} : \text{exactly one path has infinite number of 1’s} \}\]

  But $UB$ is not even recognized by alternating co-Büchi automaton
Broken Analogy

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\end{align*}$$

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\]

But $UB$ is not even recognized by alternating co-Büchi automaton.

Broken analogy:

- Büchi sets are all in $\Sigma^1_1$.
- But not every regular $\Sigma^1_1$ set is in Büchi class (consider $UB$).
Other potential candidate pair:

- $W_{0,1}$ and $W'_{0,1}$ are $\Pi_1^1$-complete (coanalytic complete) sets
- Classical Borel inseparable $\Pi_1^1$ pair (translated to our context):

  $$WF = \{ t \in T_{\{0,1\}} : \text{every path has only finitely many 1's} \}$$
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Broken analogy:

- Büchi sets are all in $\Sigma_1^1$
- But not every regular $\Sigma_1^1$ set is in Büchi class (consider $UB$)
- Above breaks the analogy between analytic and Büchi classes that one could deduce from previous slide