On The Borel Inseparability of Game Tree Languages

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University of Warsaw

STACS 2009

Hierarchies

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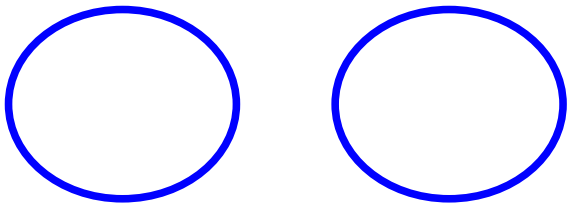

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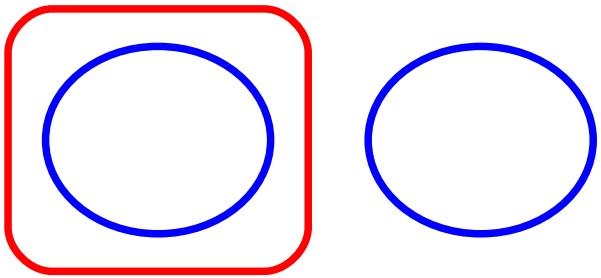

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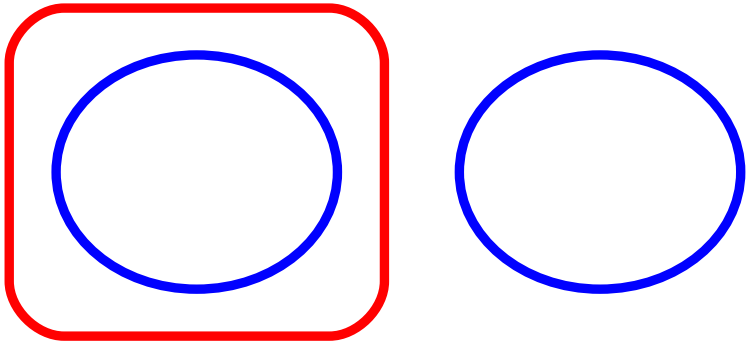

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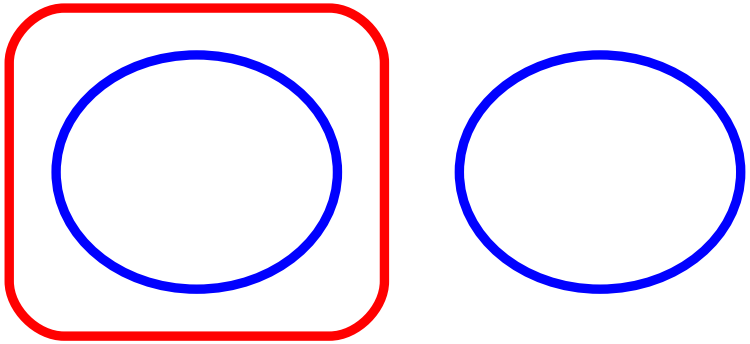

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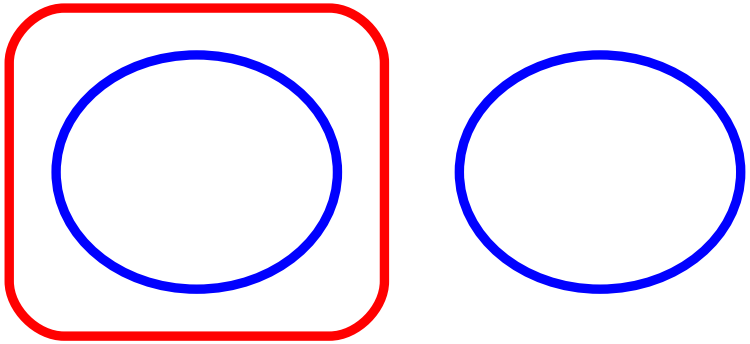

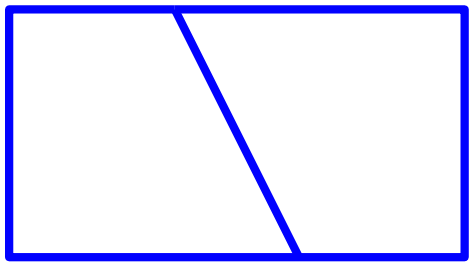

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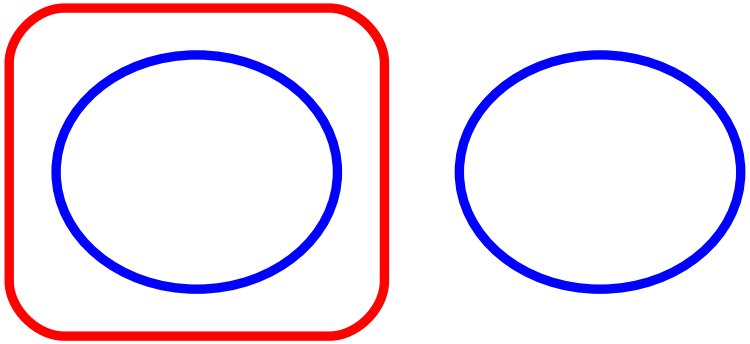

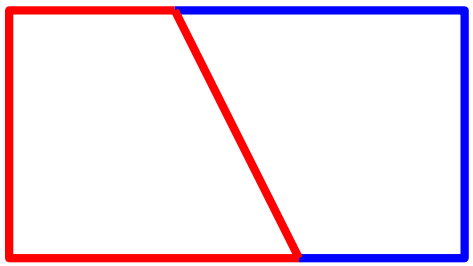

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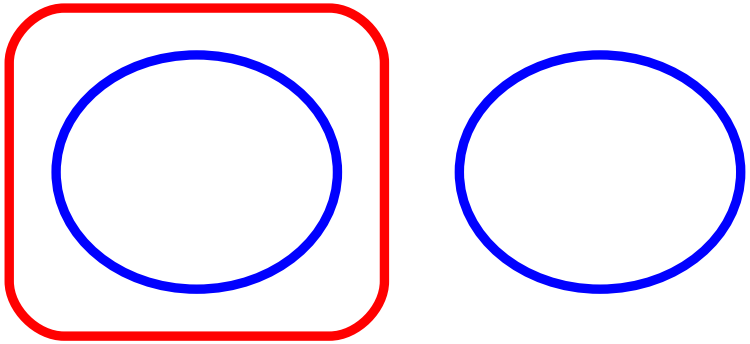

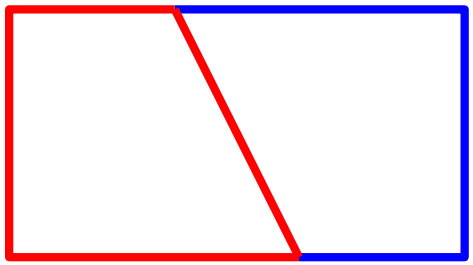

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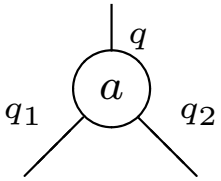

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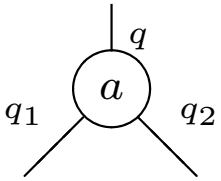

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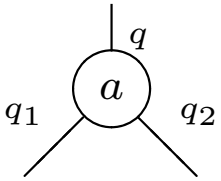

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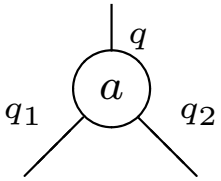

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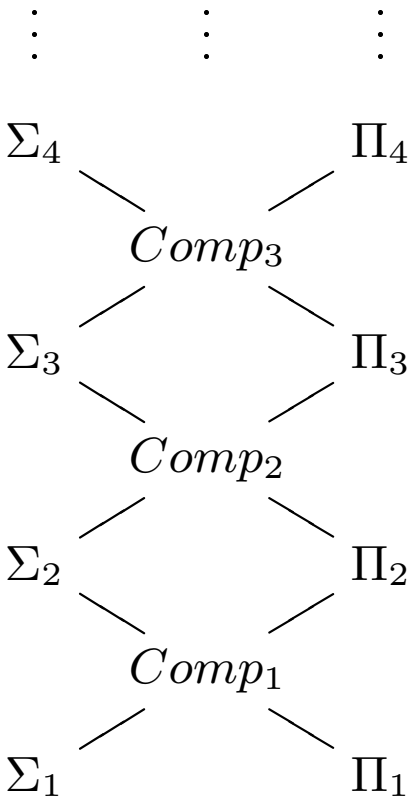
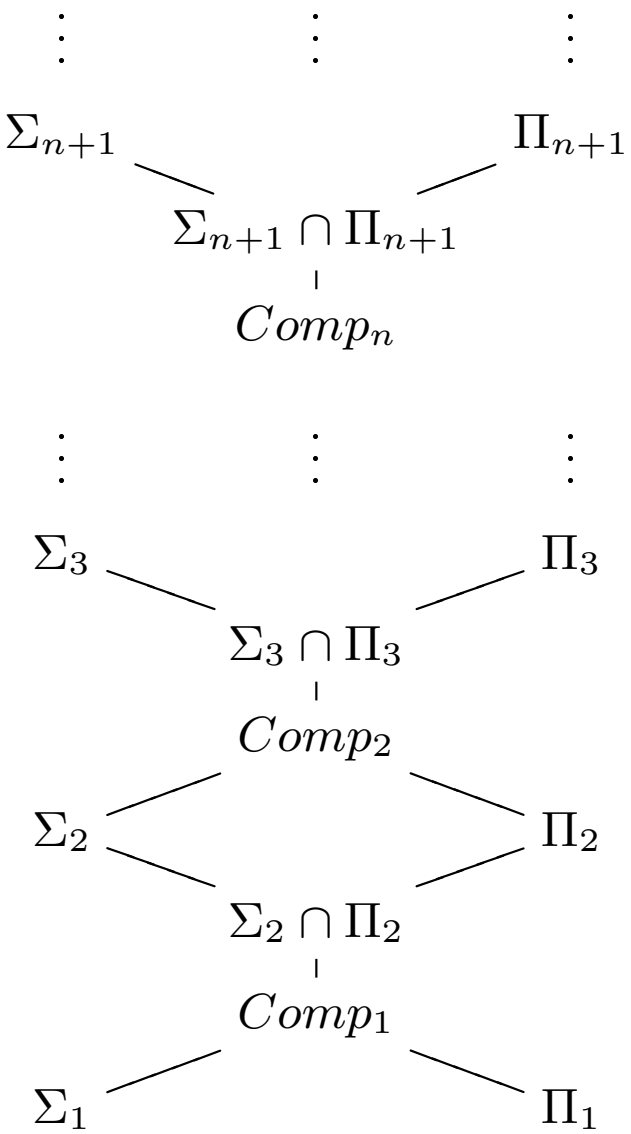


- Index hierarchy (Rabin-Mostowski)
  Level n — automata use n priorities (alternations):
  - Σ_n – greatest priority is odd,
  - Π_n – greatest priority even,
  - $Comp_n \supseteq \Sigma_n \cup \Pi_n$ (compositional class)

Alternating index hierarchy



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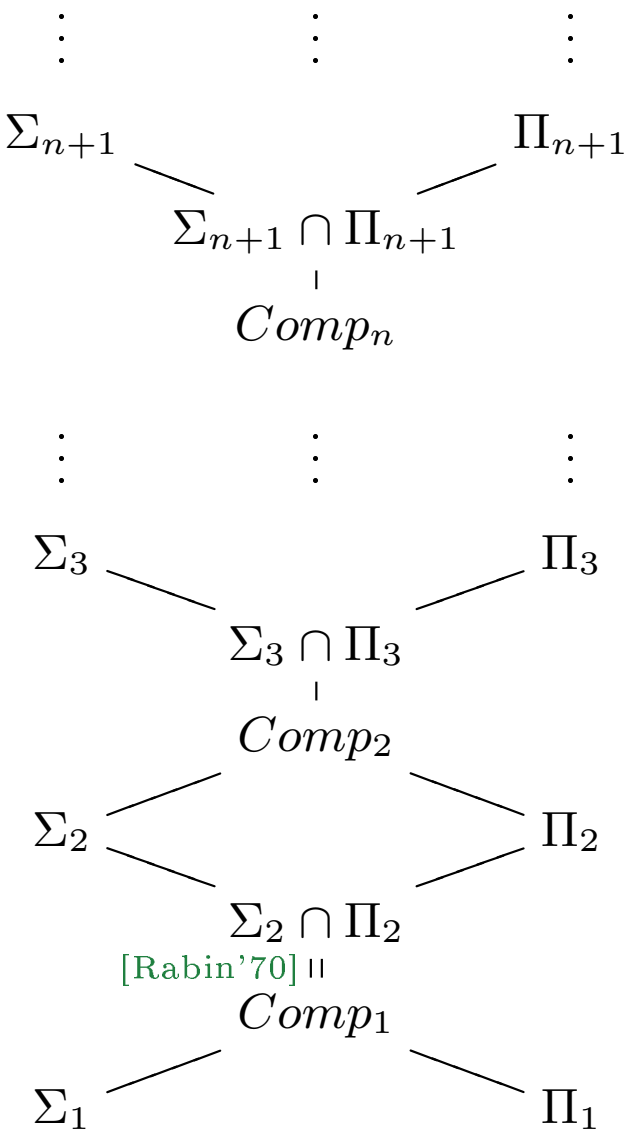

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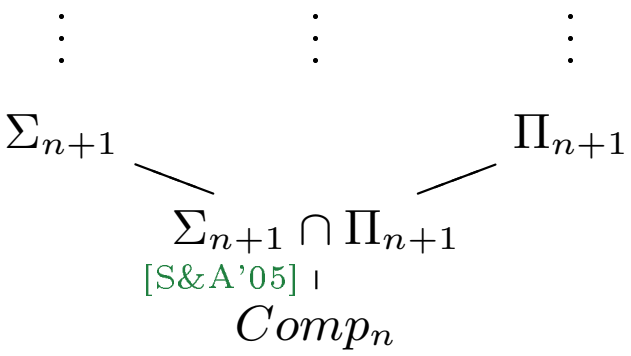

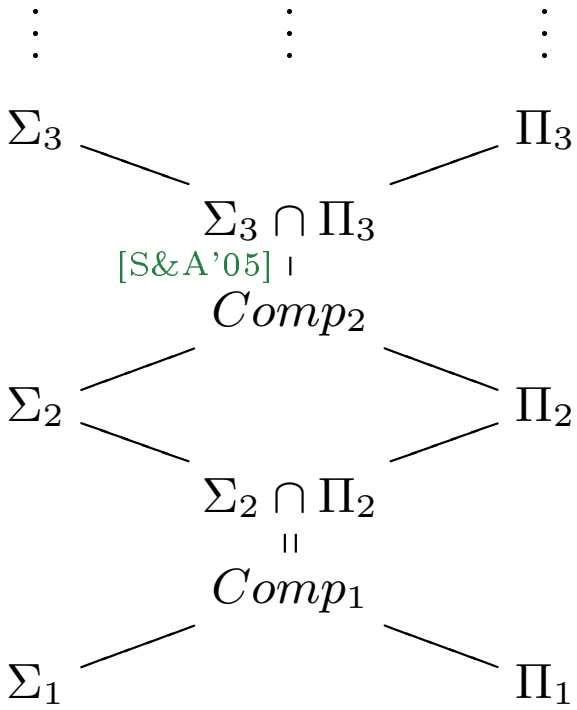

ALTERNATING INDEX HIERARCHY

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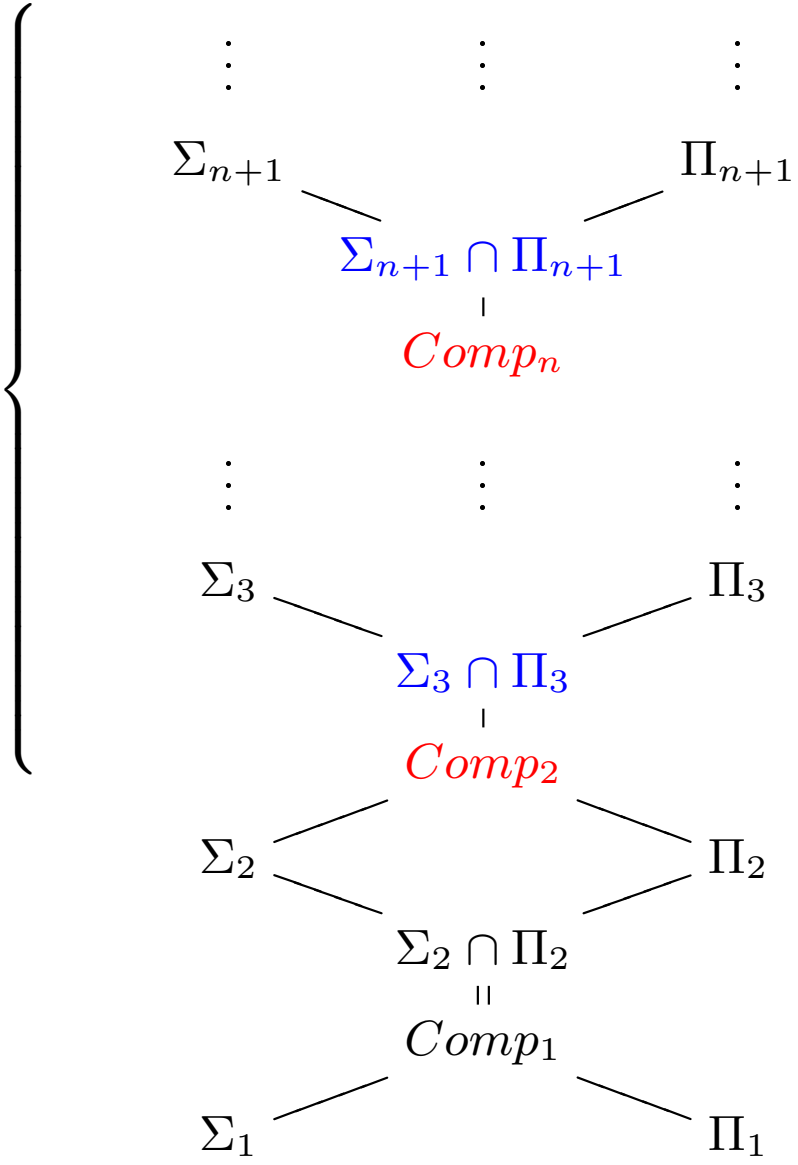
[S&A 2005]



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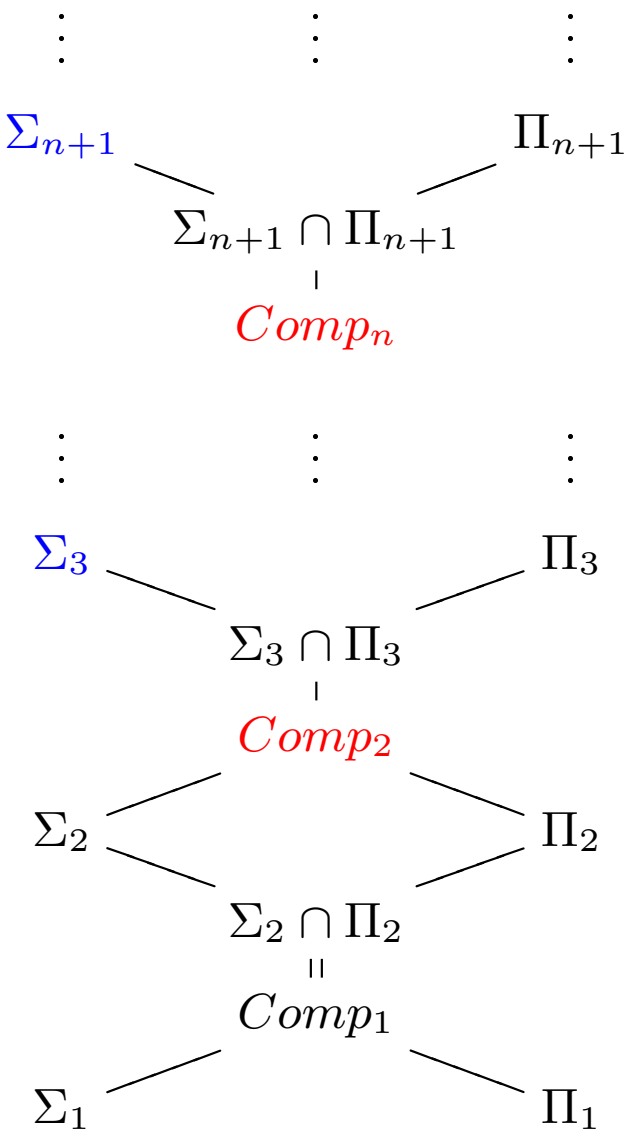

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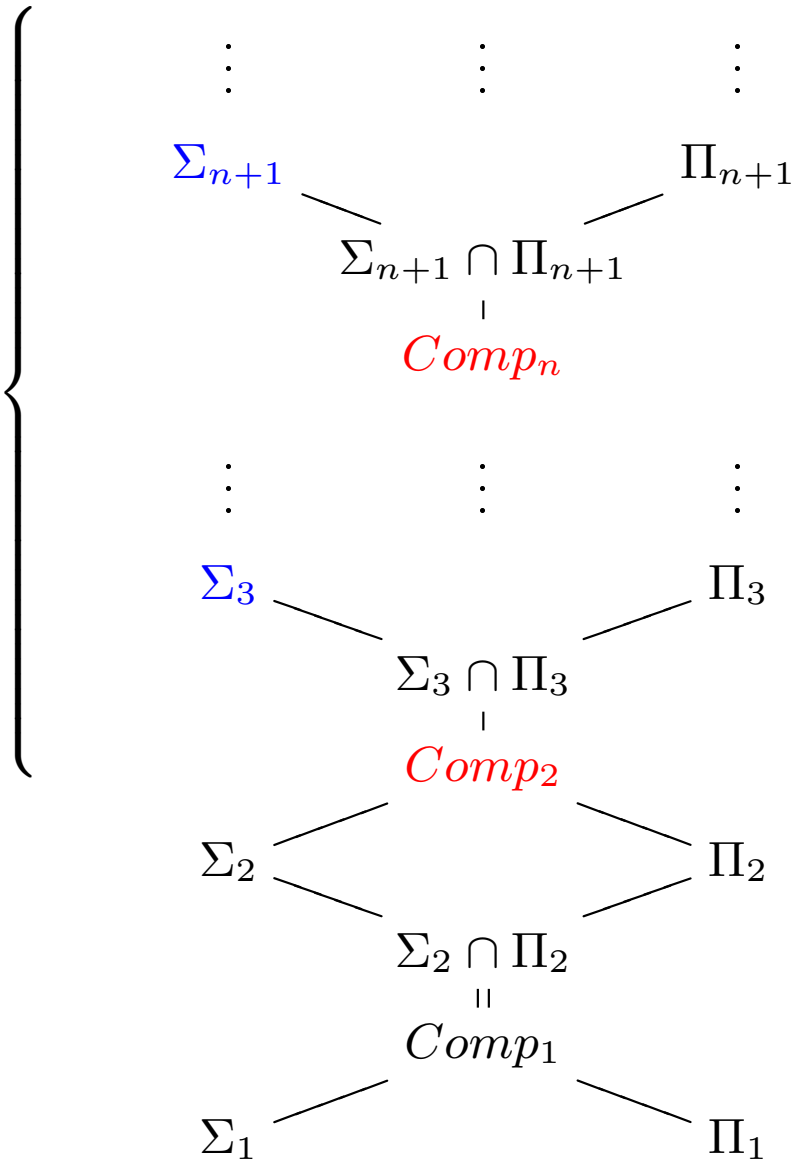

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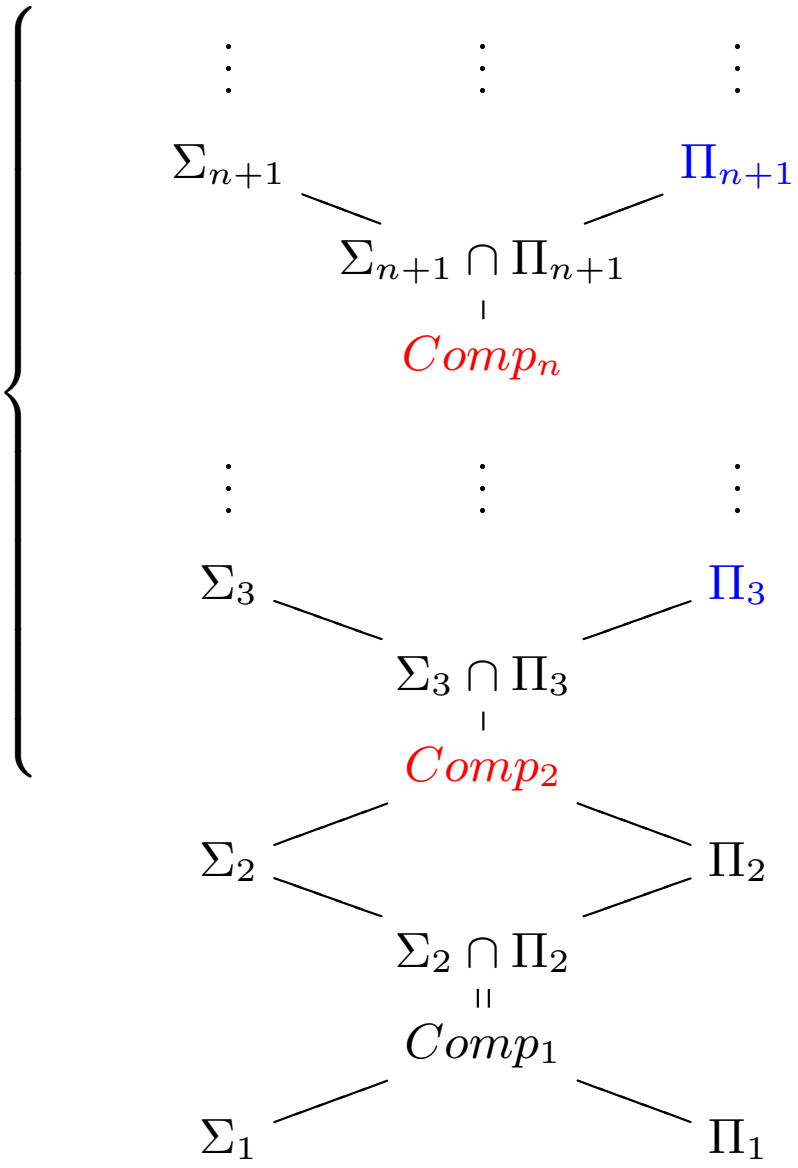

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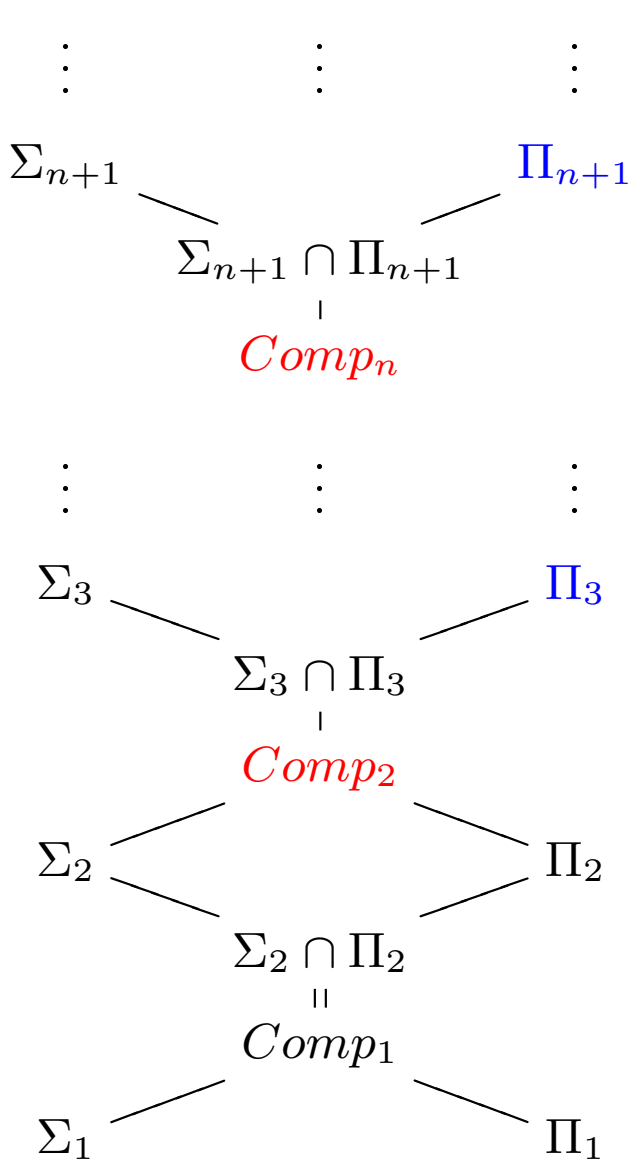

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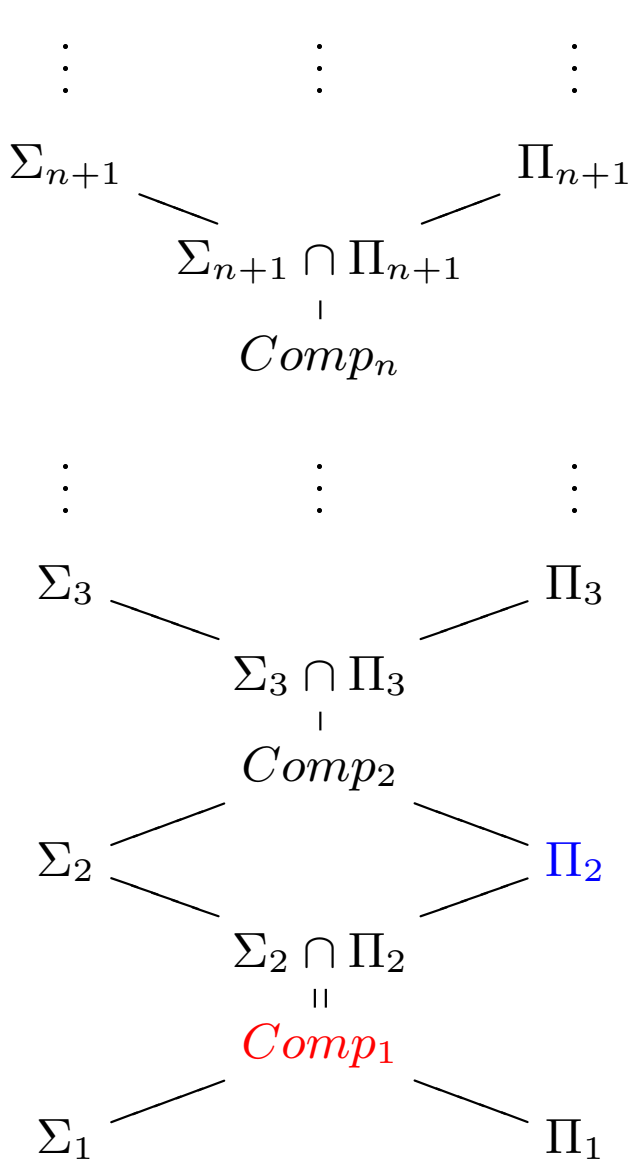

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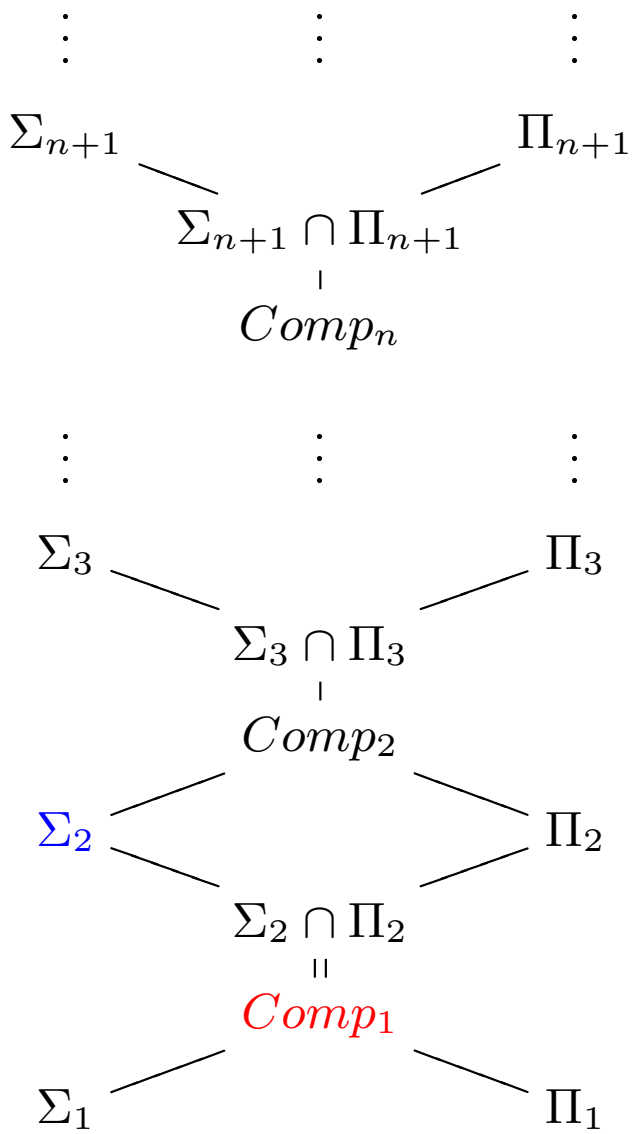

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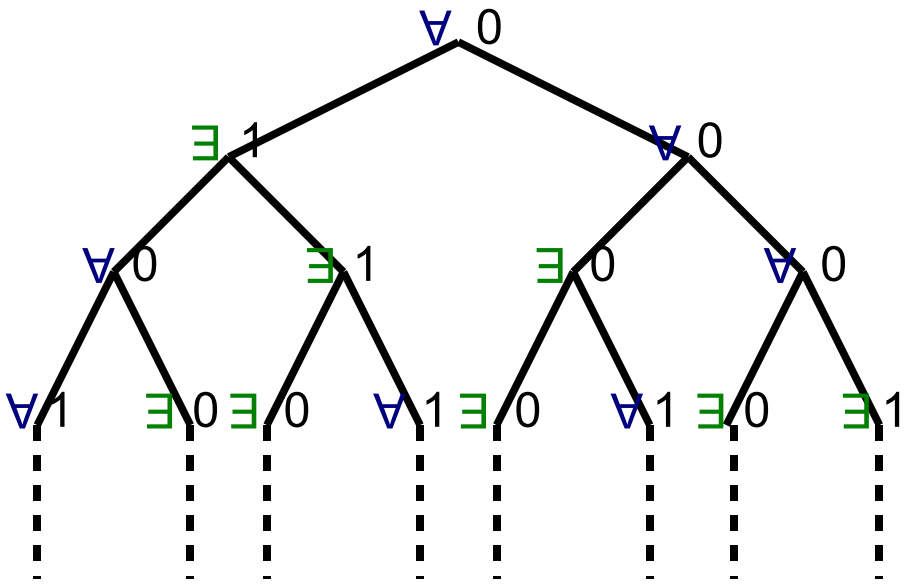

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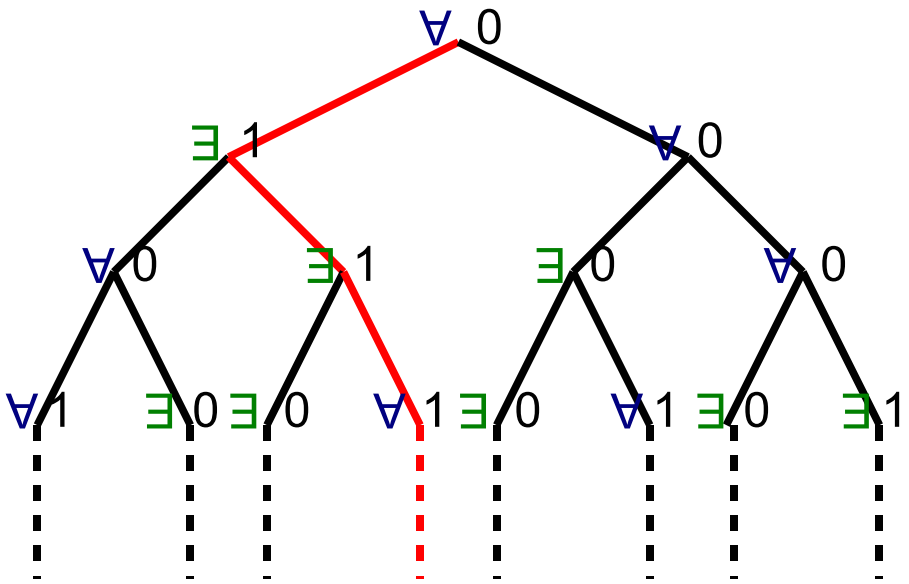

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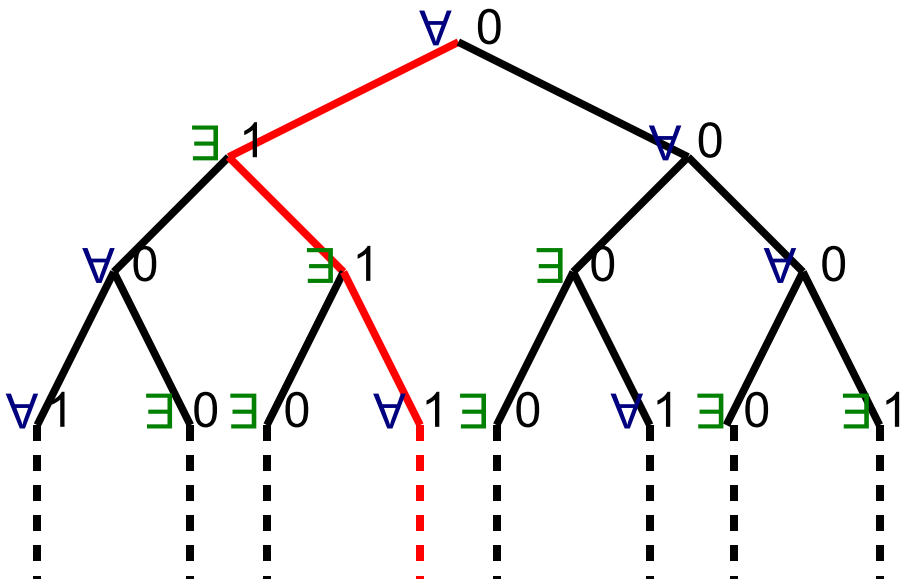

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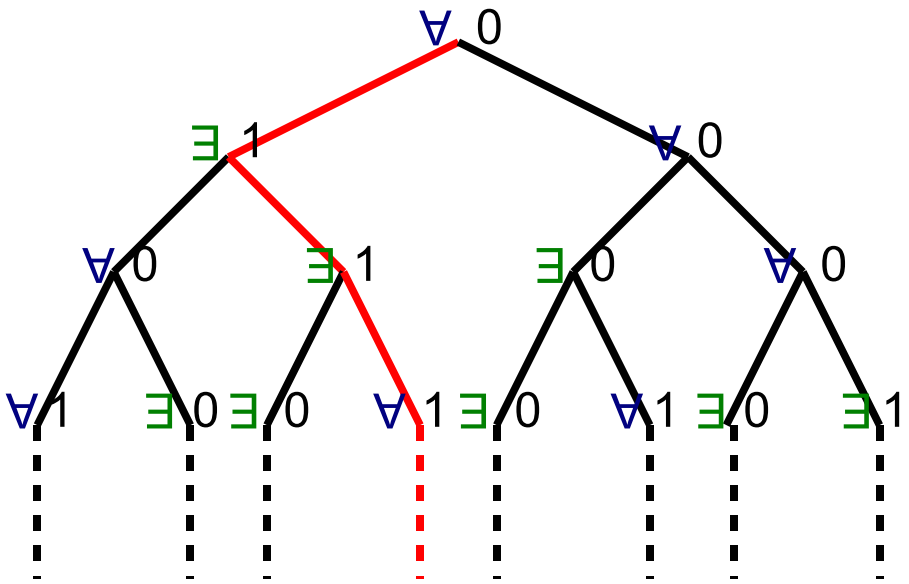

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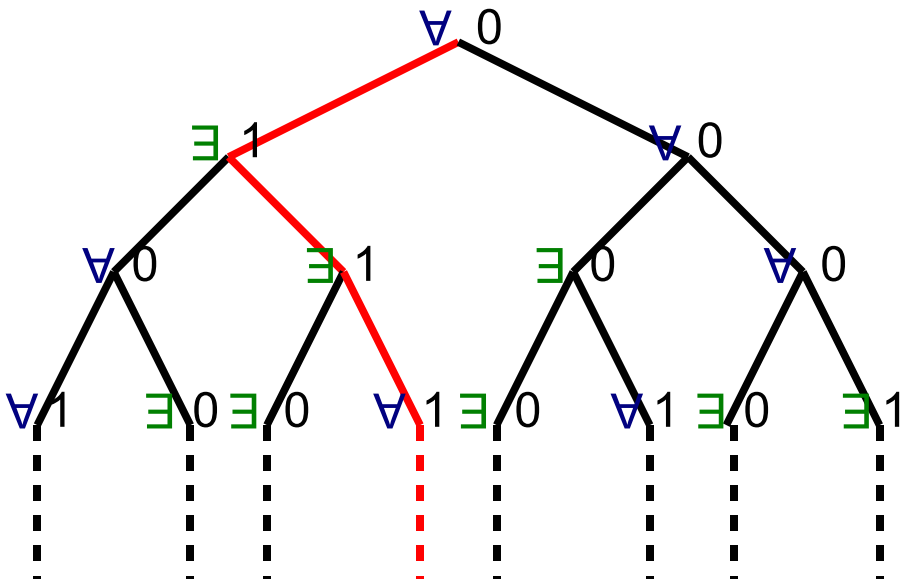

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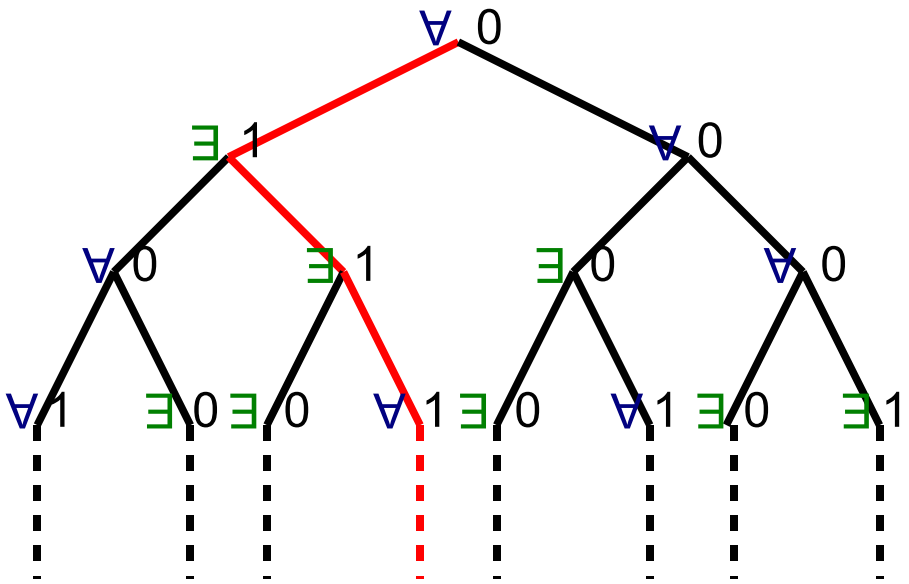

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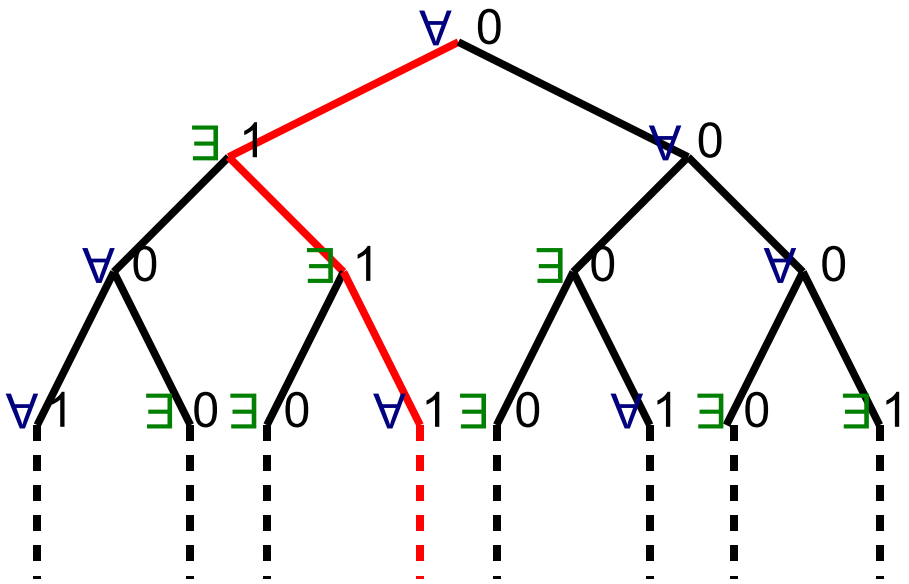

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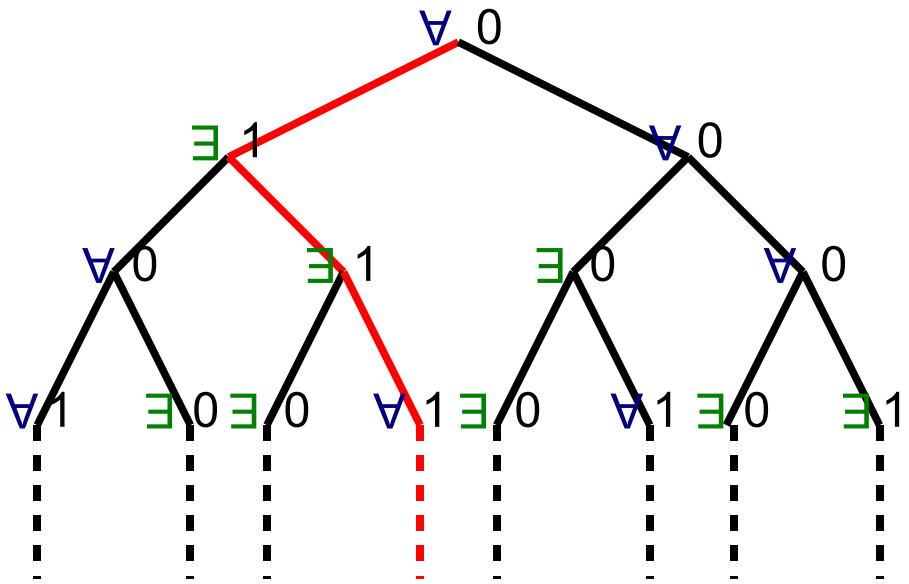

Definition

The Borel sets constitute the least family containing *open sets* and closed under *complement* and *countable union*.

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There is no Borel set separating $W_{0,1}$ and $W'_{0,1}$.

- $W_{0,1}$ and $W'_{0,1}$ are recognized by nondeterministic automata with co-Büchi condition.
- $Comp_1 \subseteq Borel$

Corollary

There exists a pair of disjoint sets recognized by nondeterministic Σ_2 automata, that is not separated by any $Comp_1$-recognized set.

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Let $B \subseteq T_\Sigma$ be an arbitrary Borel set. There exists a continuous function $F_B : T_\Sigma \to T_\Sigma$ such, that:

$$t \in B \implies F_B(t) \in W_{0,1}$$
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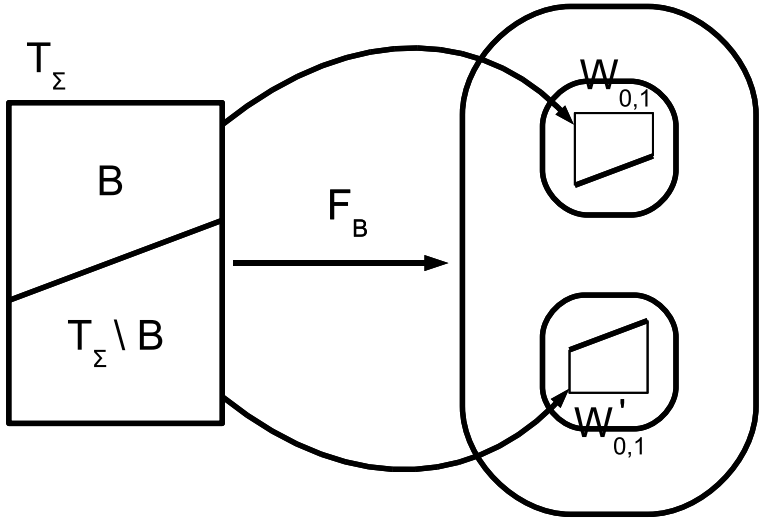

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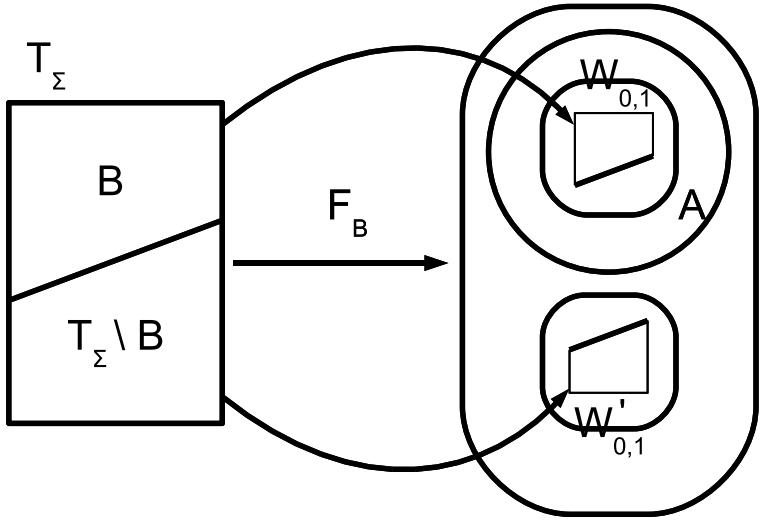

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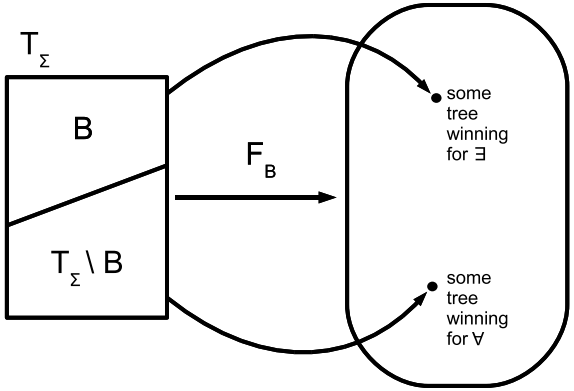

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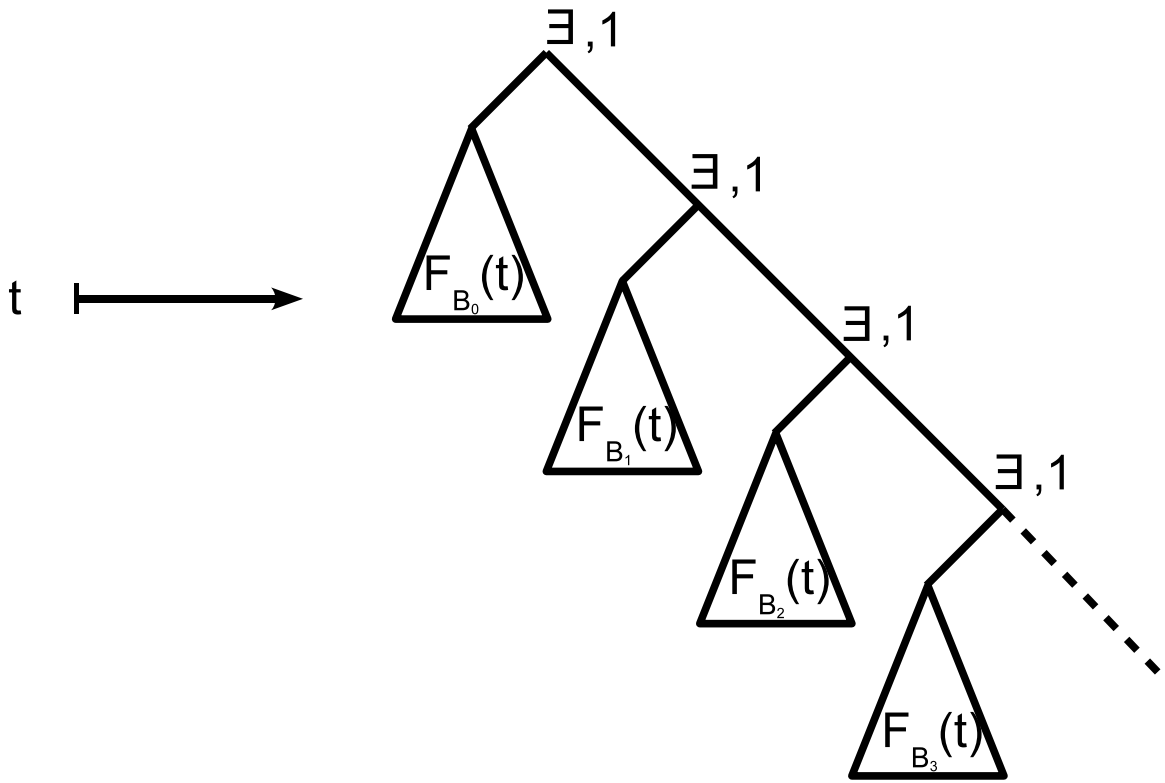
$$B = \bigcup_{i \in \mathbb{N}} B_i$$



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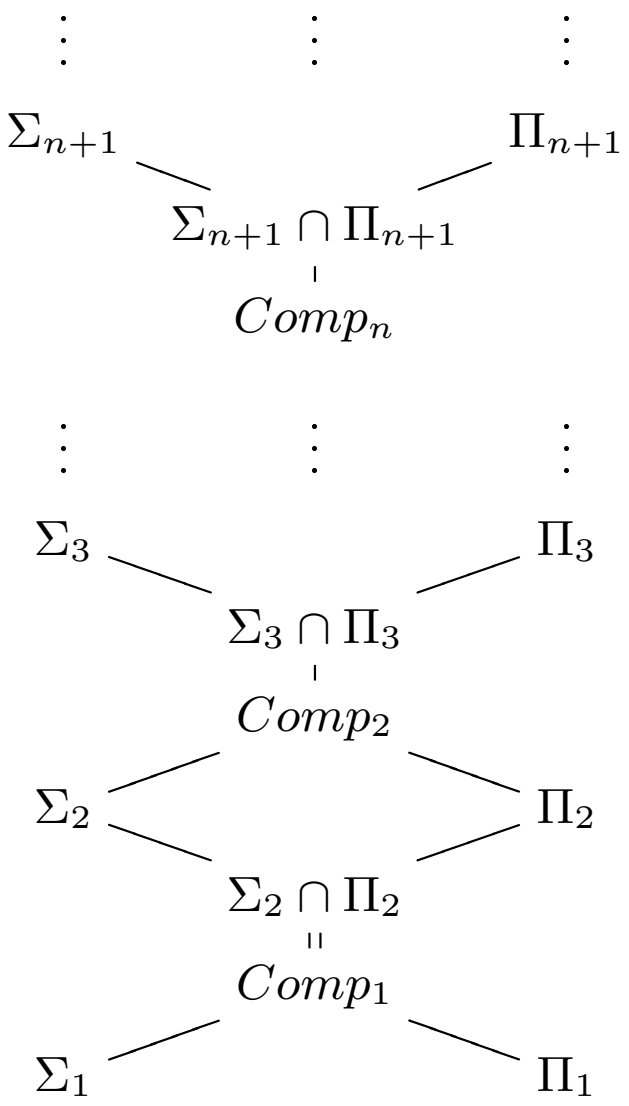
Class $\mathcal{L}$ has First Separation Property if separation property holds for $\mathcal{L}$ and $\mathcal{S} = \{X : X, \overline{X} \in \mathcal{L}\}$.

- For such $\mathcal{L}$ and $\mathcal{S}$ simplification holds trivially.

Alternating hierarchy



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First Separation Property *fails for co-Büchi class.*

- Weaker version of our theorem (our languages even nondet. co-Büchi).

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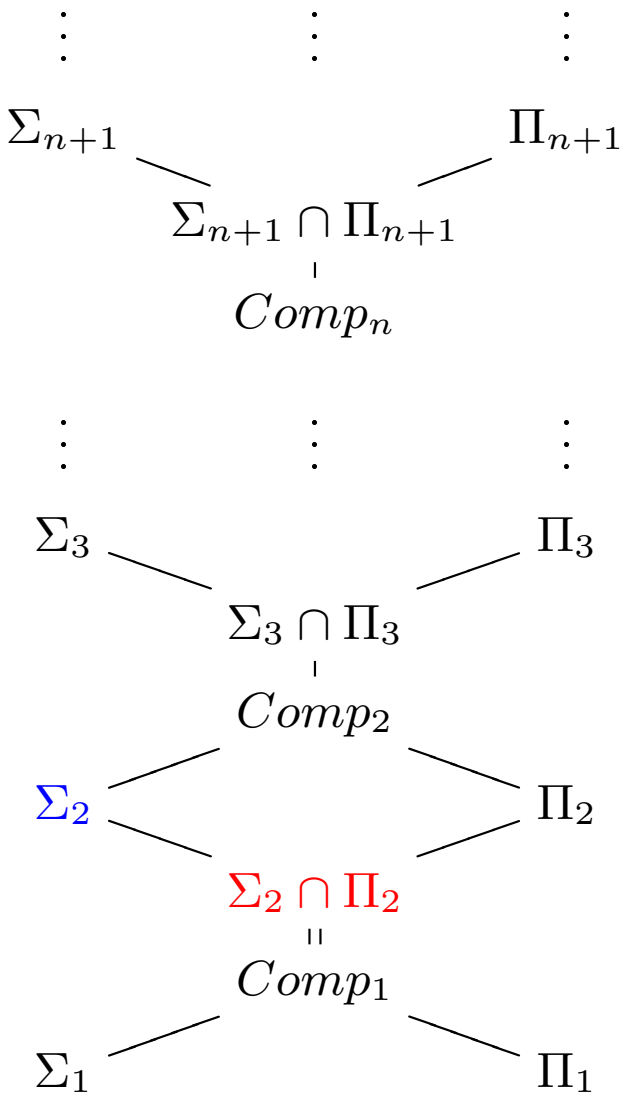

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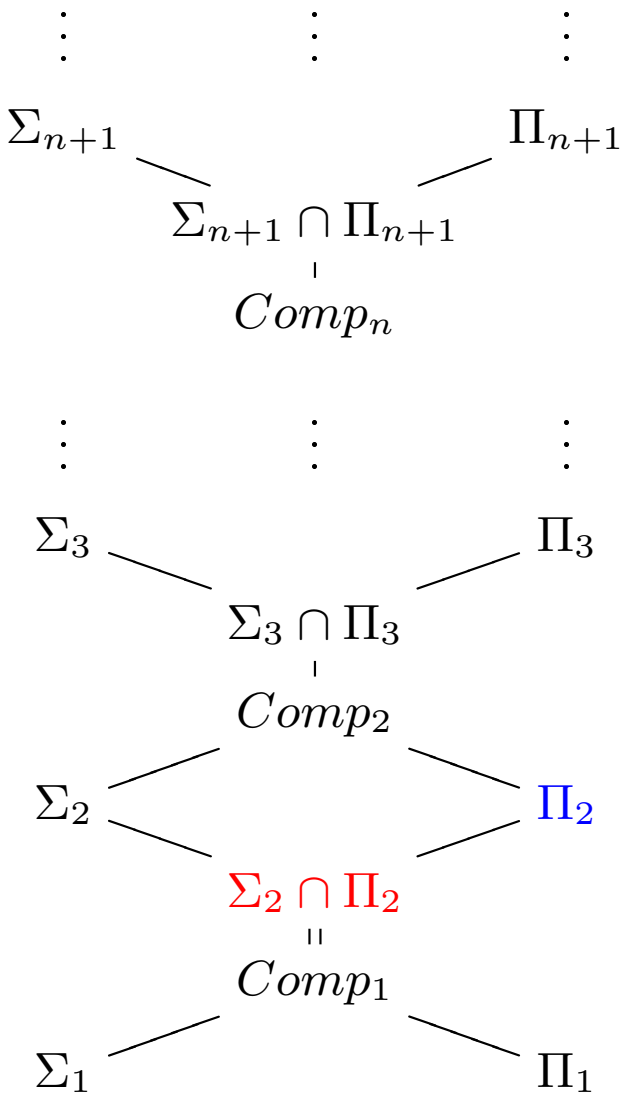

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- First Separation Property for higher levels — open

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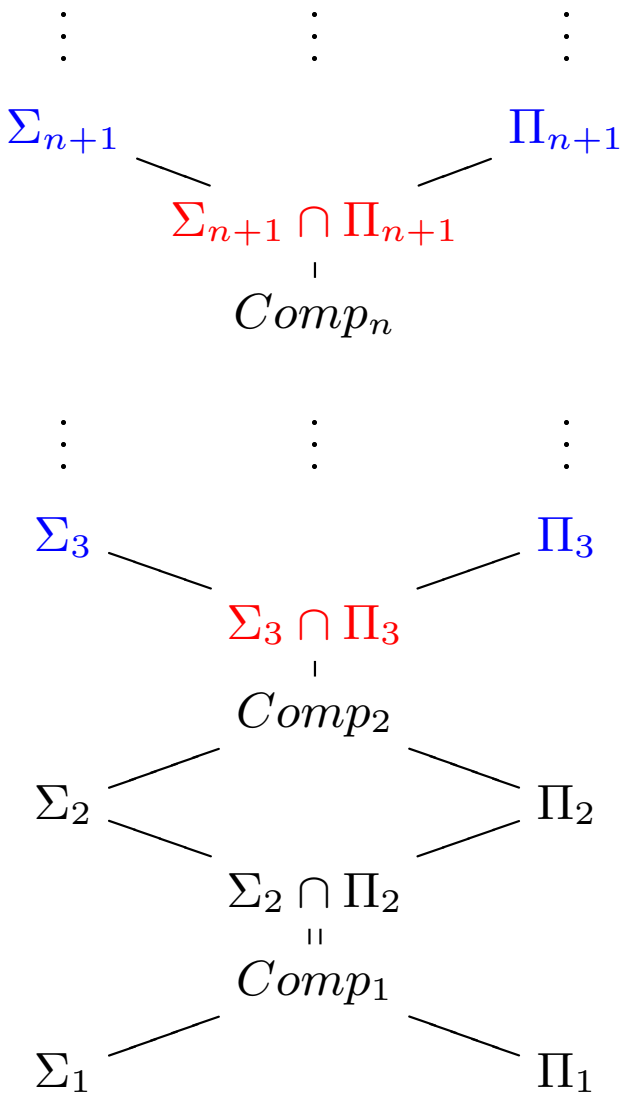

Borel Hierarchy

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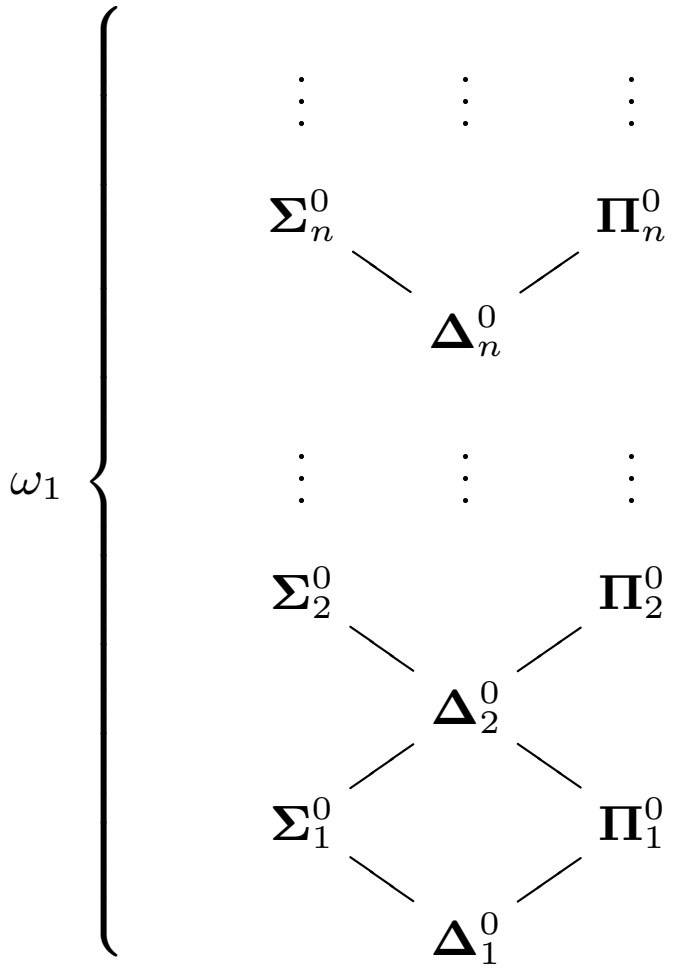

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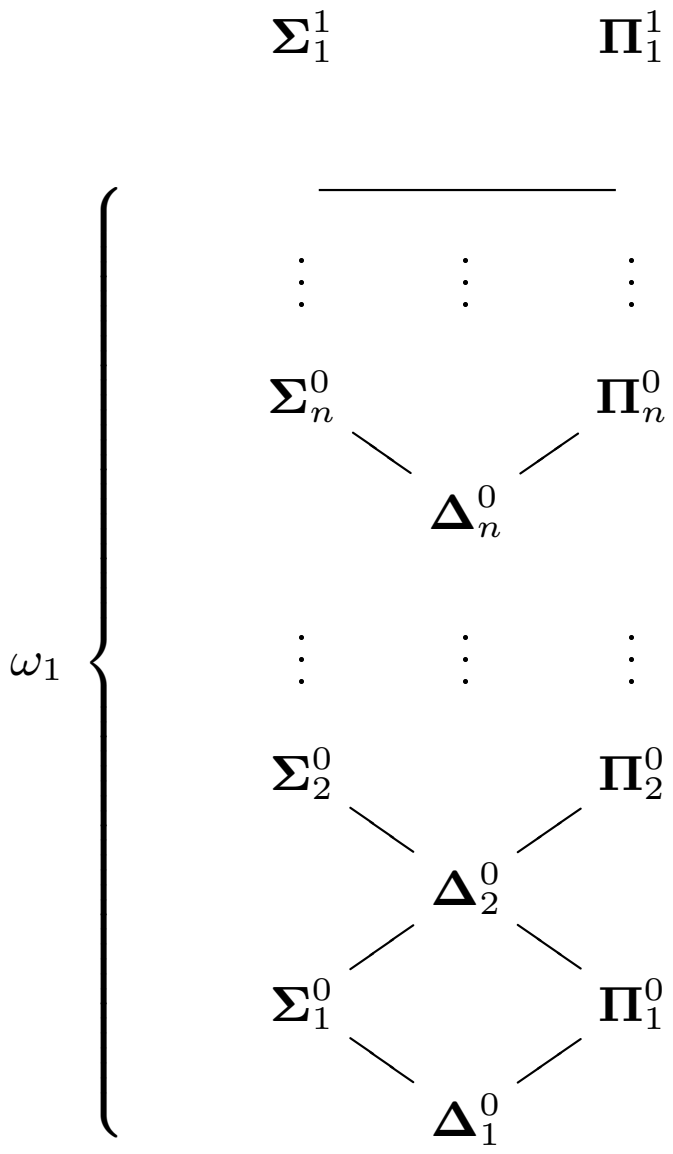

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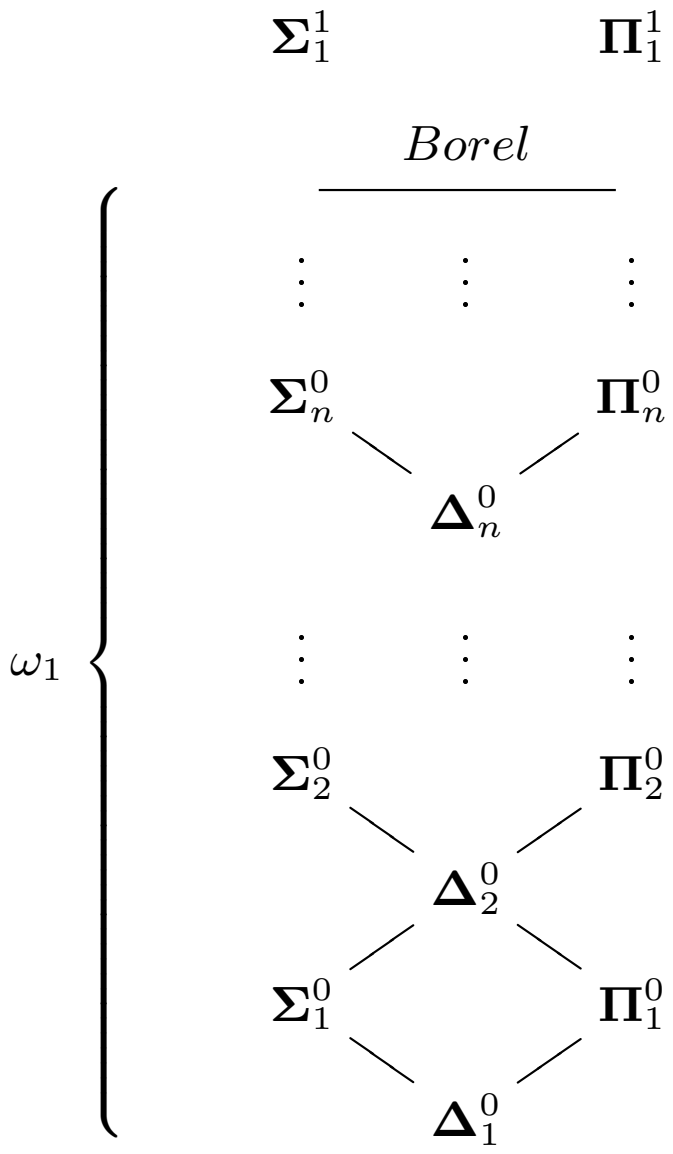

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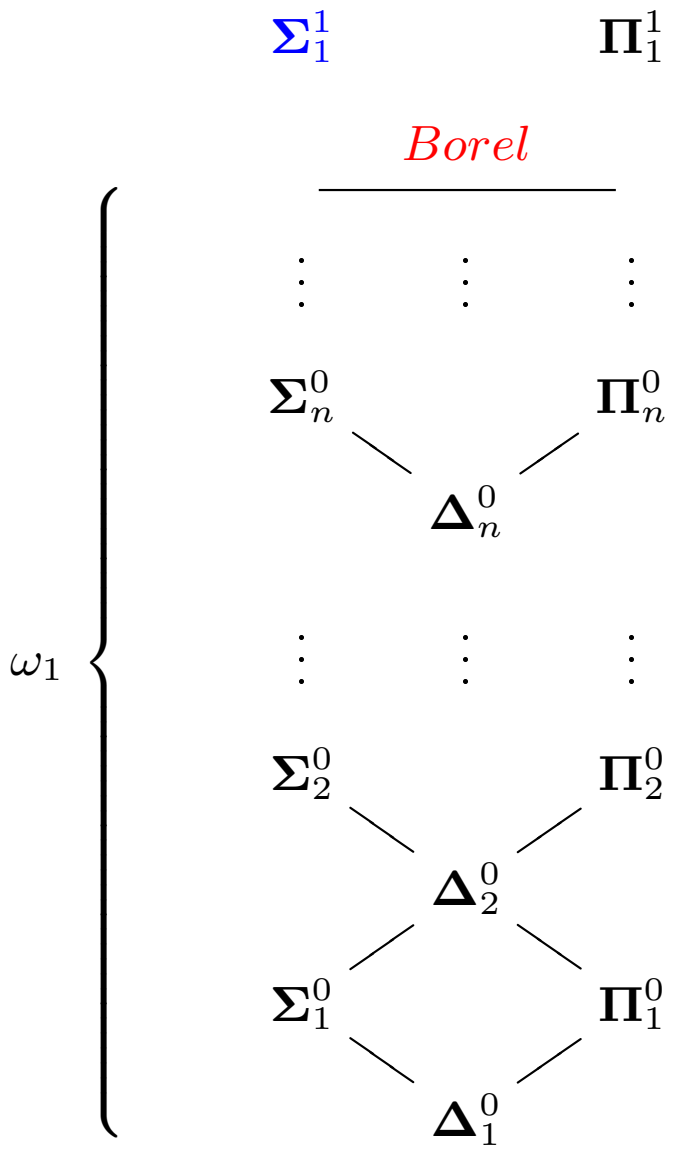

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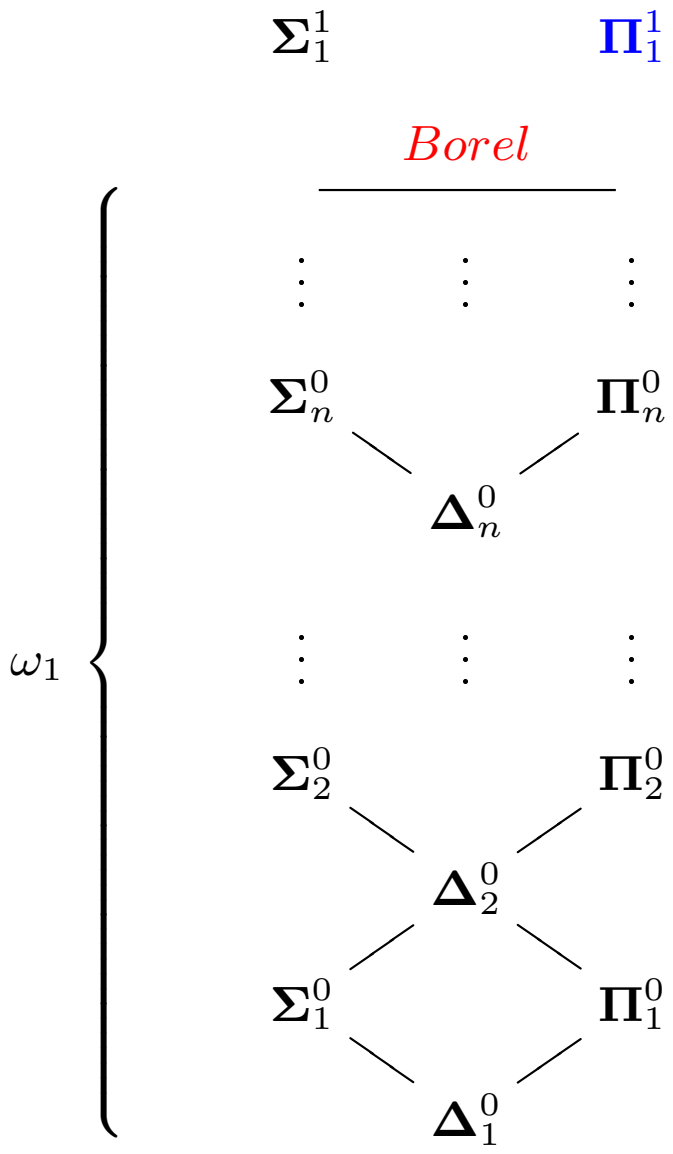

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- It gives some analogy between
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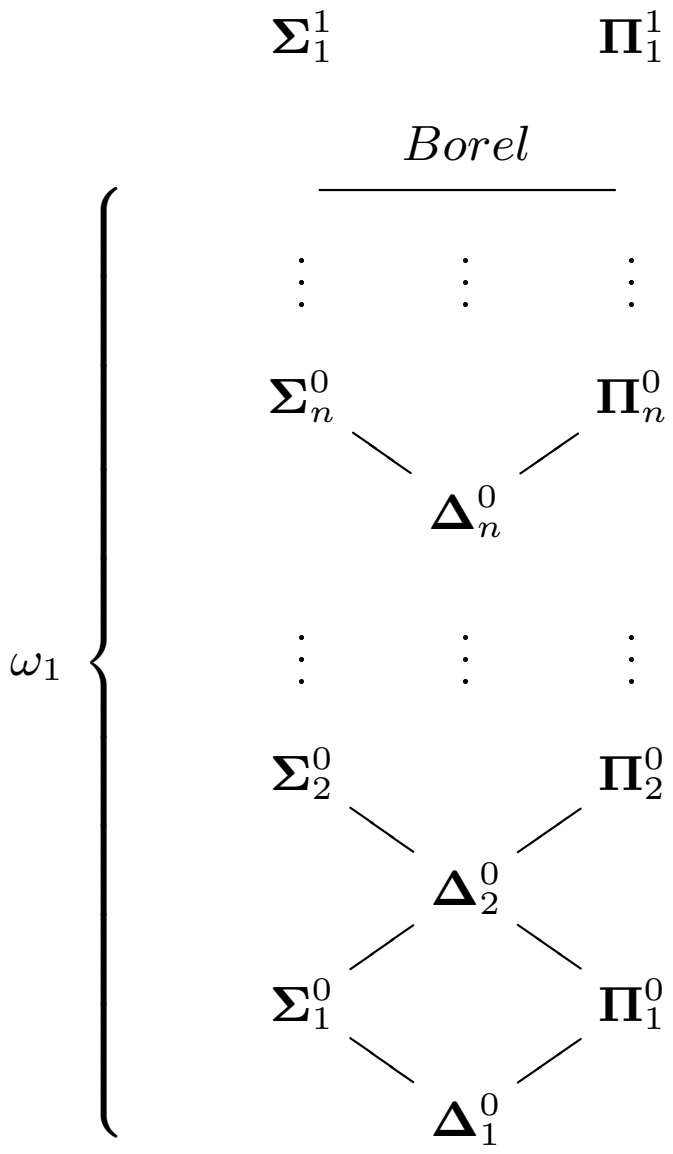

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- But not every regular Σ^1_1 set is in Büchi class (consider $\overline{UB}$)
- Above breaks the analogy between analytic and Büchi classes that one could deduce from previous slide