Description

Table 1 presents state of the art of six different hierarchies considered separately for weak languages, deterministic languages and regular languages of infinite trees. A row for ω-words is given for the reference.

In each cell of the table we give information about the shape of the hierarchy. We are in particular interested in the strictness of the hierarchy. If the hierarchy occurs not to be strict in particular setting, we want to know which levels are inhabited by the languages from given class. The second information included in the table is decidability of the hierarchy, i.e. given an automaton of respective type (weak, deterministic, non-deterministic, depending on the row), can it be computed on which level of respective hierarchy (depending on the column) it lies.

It is well known that the expressive powers of alternating and non-deterministic tree automata are the same ([MSS86]). It is although possible that a language is recognized by an alternating automaton of lower index than the index of any non-deterministic automaton recognizing it (we will later see that it actually is the case). This is why it is reasonable to distinguish between those two hierarchies.

For each deterministic language we can, of course, ask about its position in deterministic hierarchy. But it is also legitimate to ask of its non-deterministic (or alternating) index. The answer to this question gives us an understanding of how hard the deterministic languages are for non-deterministic automata. Another words, what is the power of non-determinism in the context of parity tree automata. As we will see in the sequel, the answer to this question is surprising.

Discussion

First we note that very little is known about hierarchies for regular (non-deterministic) languages. We know that alternating and non-deterministic hierarchies are strict; we know where in the Borel hierarchy most complex regular languages are; but nothing is known about decidability of any of the hierarchies.

The situation is a bit more optimistic for weak languages. Hierarchies are shorter here, and a bit more recognized. But still no algorithms are known.

Thanks to the recent contribution of Niwiński, Walukiewicz and Murlak, we know almost everything about all the hierarchies for deterministic languages. An interesting phenomenon here is, so called, *gap property*. A deterministic language is on at most third level of the Borel hierarchy or is coanalytic complete, hence non-Borel. The same pattern occurs in the case of weak hierarchy of deterministic languages.

Now, we will concentrate on the already mentioned interrelations between the hierarchies.

First, we look at the characteristics of the Borel hierarchy for weak languages and deterministic languages, to see that there are many weak languages that are not deterministic, and vice versa. The class of deterministic languages includes some non-Borel sets, that cannot be weak; whereas there are weak languages $\mathbf{\Sigma}^0_3$-hard in the Borel hierarchy, that are not deterministic, thanks to the *gap property*.

Another interesting result states that there exist deterministic languages on arbitrary high levels of the non-deterministic hierarchy. This is the mentioned surprising result concerning the power of non-determinism.

There is another example of non-trivial relation between hierarchies in the context of deterministic languages. Sets of $\mathbf{\Delta}^0_3$ topological complexity can be found arbitrarily high in the deterministic index hierarchy; on the other hand, there exists a coanalytic-complete set recognized by deterministic automaton of index $(0,1)$ (*finitely many 1's on each branch*).

However, there is also an example of exact correspondence between hierarchies, in the same context (of deterministic languages). The hierarchies are: Borel hierarchy and weak index hierarchy. A deterministic language is recognized by a weak automaton of index $(0,n)$ if and only if it is of Borel class $\mathbf{\Pi}^0_n$; is recognized by a weak automaton of index $(1,n)$ if and only if it is of $\mathbf{\Sigma}^0_n$ class; and is non-Borel if and only if it is not weak ([Mu05]).

In [Mu08, section 2.10], Filip Murlak states the conjecture that similar correspondence exists for weak languages.

[1] There probably are better lower bounds for the height, but nothing is published yet.

Table 1: Hierarchies for the languages of infinite trees

	Borel Hierarchy + (Projective H.)	**Wadge Hierarchy**	**Deterministic Hierarchy**	**Weak Hierarchy**	**Nondeterministic Hierarchy**	**Alternating Hierarchy**
Weak Languages	$<\mathbf{\Delta}^0_\omega$, strict up to this level ([Sk93]) **Decidability:** $\boxed{\textbf{OPEN}}$	at least ε_0 $(=Fix(x\mapsto\omega^x))$ ([DM07]) **Decidability:** $\boxed{\textbf{OPEN}}$	Not each weak language is deterministic, e.g. any weak $\mathbf{\Sigma}^0_3$- complete language (such exist, by [Sk93]) is not det.	strict ([Mo91] using weakMSO [T82]; another proof [NW03]) **Decidability:** $\boxed{\textbf{OPEN}}$	$\leq(0,1)\cap(1,2)$ (weak aut. recognize exactly those lang. ([R70, MSS86]))	$\Rightarrow$ this hierarchy also collapses
Deterministic Languages	*Gap property:* $\mathbf{\Pi}^1_1$- compl. or $\leq\mathbf{\Pi}^0_3$ (strict up to the level $\mathbf{\Pi}^0_3$) [NW03] **Decidability:** Yes ([Mu05, NW03])	height $\omega^{\omega\cdot 3}+3$, embedding into the general hier. is known [Mu06] **Decidability:** Yes (*canonical automata*, [Mu06])	strict (arbitrary high one can find $\mathbf{\Delta}^0_3$ sets, [Ni86]) **Decidability:** Yes ((ι,κ)- *flower*, [NW98])	*Gap property:* $\leq(0,2)$ or non-weak (*split*); up to the level $(0,3)$ strict (corresponds to the Borel hier.) [NW03] **Decidability:** Yes (*flowers*, *weak fl.*, [Mu08, NW03]), earlier: weak- det. hier. [Mu05]	unbounded (arb. high one can find det. lang.) **Decidability:** Yes ([NW05], earlier for the level $(1,2)$ [U00])	$\leq(0,1)$ (folklore, proof e.g. in [Mu08, Prop.1])
Regular Languages	$\leq\mathbf{\Delta}^1_2$ ([R69, Compl. Lemma]) **Decidability:** $\boxed{\textbf{OPEN}}$	known sequence of lang. constituting hier. of length ω ([AN06]); but length of whole hier. $\geq\varepsilon_0$ (contains all weak lang., [DM07])[1] **Decidability:** $\boxed{\textbf{OPEN}}$	Not every reg. lang. is det.; e.g. language ,,exists branch labeled only with a" (index $(0,0)$)	Not every reg. lang. is weak; e.g. any non-Borel lang. (see above)	strict ([Ni86]) **Decidability:** $\boxed{\textbf{OPEN}}$	strict (*game languages* $W_{(\iota,\kappa)}$, [Br99, Ar99]) — also of index (ι,κ) in non-det. hier.!; constitute strict Wadge'a hier. **Decidability:** $\boxed{\textbf{OPEN}}$
Languages of Infinite words (regular)	$<\mathbf{\Delta}^0_3$, even $Boole(\mathbf{\Sigma}^0_2)$; strict up to this level ([L69]) **Decidability:** Yes (automata model assigned to each level, [L69])	length ω^ω (Wagner hierarchy, [W79]), embedding into general hierarchy: $\omega^k n_k+\ldots+\omega n_1+n_0 \rightsquigarrow \omega_1^k n_k+\ldots+\omega_1 n_1+n_0$ **Decidability:** Yes ([W79])	strict ([W77]) **Decidability:** Yes ($\in P$, [NW98])		$\leq(1,2)$ (each ω-regular language can be recognized by non-det. Büchi aut.; proof e.g. [T97])	

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