SYMPLECTIC CONFIGURATIONS

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Abstract. We define a class of symplectic fibrations called symplectic configurations. They are natural generalization of Hamiltonian fibrations. Their geometric and topological properties are investigated. We are mainly concentrated on integral symplectic manifolds.

We construct the classifyng space $\mathbb{B}$ of symplectic integral configurations. The properties of the classifying map $\mathbb{B} \to \mathrm{BSymp}(M,\omega)$ are examined. The universal symplectic bundle over $\mathbb{B}$ has a natural connection whose holonomy group is the enlarged Hamiltonian group recently defined by McDuff.

The space $\mathbb{B}$ is identified with the classifying space of an extension of certain subgroup of the symplectomorphism group.

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1. Introduction

1.1. **Symplectic configurations.** The subject of this paper is certain class of symplectic fibrations called *symplectic configurations*. They are defined as follows.

Definition 1.1. Let (M,ω) and (W,ω_W) be symplectic manifolds. We say that a symplectic fiber bundle (fiber bundle with a structure group $\mathrm{Symp}(M,\omega)$) $M \to E \to B$ is a *W-symplectic configuration* if there exists a map $E \to W$ such that when restricted to any fiber of E is a symplectic embedding. If W is integral we say that E is an *integral configuration*. The fibration E is called an (integral) symplectic configuration if it is a W-configuration for some (integral) (W,ω_W).

Here is an alternative approach. Consider the space $\mathrm{Symp}(M,W)$ of symplectic embeddings $f\colon (M,\omega) \to (W,\omega_W)$. The group of all symplectomorphisms of the source acts freely on that space and the quotient is denoted by $\mathbb{B}_W$. We call it the *space of symplectic configurations of* (M,ω) *in* (W,ω_W). We get a principal fibration

$$\mathrm{Symp}(M,\omega) \to \mathrm{Symp}(M,W) \to \mathbb{B}_W$$

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and the associated symplectic one

$$(M,\omega) \to M_W \to \mathrm{B}_W.$$

Proposition 1.2. *Symplectic fibration E is a W-symplectic configuration if and only if it is a pull-back of the bundle M_W.*

Proof. Let $E \to W$ be a map such that it is a symplectic embedding on every fiber. Clearly, it defines a map $B \to \mathrm{B}_W$ such that E is a pull-back of M_W.

To prove the converse choose a point $p \in M$ and consider the following composition of two maps:

$$\mathrm{Symp}(M,W) \xrightarrow{\mathcal{P}} M_W \xrightarrow{ev} W,$$

where

$$\mathcal{P}(f) := [f,p] \quad \text{and} \quad ev[f,q] = f(q).$$

The map ev is well defined. Indeed M_W is the quotient of $\mathrm{Symp}(M,W) \times M$ by the relation $[f \circ \psi, q] \sim [f, \psi(q)]$ and $ev([f \circ \psi, q]) = f(\psi(q)) = ev([f, \psi(q)])$. The evaluation map is the required map for the fibration M_W. Since E is a pull-back of M_W then the proof is finished. □

1.2. **Closed connection 2-form.** Without loss of generality we will assume that the base of a fibration is a manifold thus we may speak of the differential forms on E.

Recall that if $(M,\omega) \to E \to B$ is a symplectic fibration then a closed two form $\Omega \in \Omega^2(E)$ is called a *closed symplectic connection 2-form* if it restricts to the symplectic form on any fiber (see Lemma 6.18 in [12] for the general definition). According to Thurston (Theorem 6.3 in [12]), the existence of a closed extension is purely cohomological condition, i.e. the form Ω exists if and only if the class $[\omega]$ belongs to the image of the map $H^2(E;\mathbb{R}) \to H^2(M;\mathbb{R})$.

The existence of a closed connection 2-form is important. It was first examined by Guillemin, Lerman and Sternberg [5] where they introduced the notion of a coupling form. It is a certain closed connection 2-form satisfying some uniqueness condition. Further, the fiber integrals of powers of the cohomology class of the coupling form gives a sequence of symplectic characteristic classes [7, 8].

Lemma 1.3. *A symplectic bundle $(M,\omega) \to E \to B$ is a symplectic configuration if and only if it admits a closed connection form.*

Proof. If $f\colon E \to W$ defines a configuration, then $f^*(\omega_W)$ is a closed connection 2-form on E.

To proof of the converse construct a map $f : (E,\Omega) \to (W,\omega_W) := \bigtimes_i (\mathbb{CP}^{n_i}, a_i\omega_{n_i})$ such that $f^*[\omega'] = [\Omega]$ for appropiate $a_i \in \mathbb{R}$, where ω_n is the standard symplectic form on $\mathbb{CP}^n$. To see that the map f exists express $[\Omega]$ as a linear combination $\sum_i a_i\eta_i$ of integral classes. The classifying map for a tuple $(\eta_1,\dots,\eta_m)$ has $\bigtimes_i(\mathbb{CP}^\infty, a_i\omega_\infty)$ as a target and factrors through a product of finite dimensional projective spaces as required. For sufficiently lagre $\sum n_i$ we may perturb f into an embedding preserving the 2-form, according to the h-principle (see Section 3.4.2 in [4]). □

1.3. **Universal integral configuration.** A symplectic manifold (M,ω) is called *integral* if $[\omega]$ is in the image of the map $H^2(M;\mathbb{Z}) \to H^2(M,\mathbb{R})$. Due to Gromov, every integral symplectic manifold embeds into complex projective space $(\mathbb{CP}^N, \omega_N)$ equipped with the standard symplectic form for N big enough (Exercise 3.4.2.(1)

in [4]). Spaces of symplectic embeddings satisfy some obvious functoriality properties and we define $\mathrm{Symp}(M, \mathbb{CP}^\infty) := \bigcup_{N\to\infty} \mathrm{Symp}(M, \mathbb{CP}^N)$. Its quotient by $\mathrm{Symp}(M, \omega)$ is denoted by B_∞ and called *the universal symplectic configuration space*. As before we have the universal principal fibration

$$\mathrm{Symp}(M, \omega) \to \mathrm{Symp}(M, \mathbb{CP}^\infty) \to \mathrm{B}_\infty$$

and the associated one

$$(M, \omega) \to M_{\mathrm{B}_\infty} \to \mathrm{B}_\infty.$$

Let $(M, \omega) \to E \to B$ be a symplectic fibration whose classifying map $\xi\colon B \to \mathrm{BSymp}(M, \omega)$ has a lift $\widehat{\xi}\colon B \to \mathrm{B}_\infty$. This lift defines a map $E \to M_{\mathrm{B}_\infty} \xrightarrow{ev} \mathbb{CP}^\infty$ which factors through $\mathbb{CP}^N$, since E is finite dimensional. Thus E is an integral configuration (see Definition 1.1), and the pull-back of the standard symplectic form ω_N on $\mathbb{CP}^N$ is a closed integral connection form on E.

1.4. **Torsion.** The space $\mathrm{Symp}(M, \mathbb{CP}^\infty)$ has the same number of connected components as the set of maps from M to $\mathbb{CP}^\infty$ such that the generator of $H^2(\mathbb{CP}^\infty, \mathbb{R})$ pulls back to $[\omega]$. The homotopy classes of maps from M to $\mathbb{CP}^\infty$ are in one-to-one correspondence with $H^2(M, \mathbb{Z})$. The kernel of the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$ consists of the torsion of $H^2(M, \mathbb{Z})$. Thus if this torsion in nontrivial then the space $\mathrm{Symp}(M, \mathbb{CP}^\infty)$ is disconnected. If M is compact then $\mathrm{Symp}(M, \mathbb{CP}^\infty)$ has finite number of connected components. Moreover, if $\mathrm{Symp}(M, \omega)$ acts nontransitively on the preimage of ω in $H^2(M, \mathbb{Z})$ then B is disconnected.

Let us observe that any (finite) Abelian group T can be the torsion of $H^2(M; \mathbb{Z})$. First notice that the torsion of $H^2(M; \mathbb{Z})$ is isomorphic to the torsion of $H_1(M; \mathbb{Z})$ (see Corollary 5.5.4 in [15]). Take finitely presented group G such that its abelianization has torsion equal to T (e.g. $G = T$). Due to Gompf [3], there exists a closed symplectic 4-manifold (M, ω) with fundamental group G. Hence the torsion of $H^2(M; \mathbb{Z})$ is isomorphic to T.

Let (M, ω) be an integral symplectic manifold and let $\widehat{\omega}$ denote a pair (ω, τ) of the symplectic form and a lift of its class to $H^2(M, \mathbb{Z})$. Define

$$\mathrm{Symp}(M, \widehat{\omega}) := \{\psi \in \mathrm{Symp}(M, \omega)\colon \psi^*(\tau) = \tau\}$$

to be an open-closed (finite index if M is compact) subgroup of $\mathrm{Symp}(M, \omega)$ consisting of the symplectomorphisms which preserve the class τ. Notice that if $H^2(M, \mathbb{Z})$ is torsion-free then $\mathrm{Symp}(M, \widehat{\omega}) = \mathrm{Symp}(M, \omega)$.

Lemma 1.4. *If $(M, \omega) \to E \to B$ is an integral symplectic configuration, then there exists a lift $\tau \in H^2(M, \mathbb{Z})$ of $[\omega]$ such that the structure group of E reduces to $\mathrm{Symp}(M, \widehat{\omega})$.*

Proof. If $f\colon E \to \mathbb{CP}^\infty$ defines a configuration structure on E then the pull-back of the generator of $H^2(\mathbb{CP}^\infty, \mathbb{Z})$ has the above property. □

Let $\mathrm{Symp}_\tau(M, \mathbb{CP}^\infty)$ denote the connected component of $\mathrm{Symp}(M, \mathbb{CP}^\infty)$ containing symplectic embeddings corresponding to $\tau \in H^2(M, \mathbb{Z})$. The quotient $\mathrm{B} = \mathrm{B}_\tau = \mathrm{Symp}_\tau(M, \mathbb{CP}^\infty)/\mathrm{Symp}(M, \widehat{\omega})$ is the component of B_∞ corresponding to τ. In the sequel, we restrict ourselves to the investigation of B. We will also call it a universal configuration space. Again we have universal fibrations:

$$\mathrm{Symp}(M, \widehat{\omega}) \to \mathrm{Symp}_\tau(M, \mathbb{CP}^\infty) \to \mathrm{B}$$

and the associated one

$$(M, \omega) \to M_{\mathrm{B}} \to \mathrm{B}.$$

A symplectic fibration whose classifying map lifts to B_τ we call an integral symplectic configuration compatible with τ. Notice that a symplectic integral configuration might be compatible with several lifts.

Since $\mathrm{Symp}(M,\widehat{\omega})$ is open-closed in $\mathrm{Symp}(M,\omega)$ then their connected components of the identity are equal. Therefore the map of classifying spaces induced by the inclusion is a covering. In particular the higher homotopy groups of these classifying spaces are equal as well as the rational (co-) homology.

1.5. **Homotopy properties of the universal configuration space.** Integral symplectic configurations constitute a big class of symplectic fibrations. To see this consider the classifying map

$$\mathcal{F}\colon \mathrm{B} \to \mathrm{BSymp}(M,\omega)$$

and the maps $\mathcal{F}_k\colon \pi_k(\mathrm{B}) \to \pi_k(\mathrm{BSymp}(M,\omega))$ induced on homotopy groups. The following theorem is proven in Section 2.2.

Theorem 1.5. *Let (M,ω) be a compact integral symplectic manifold. Then*

(1) $\pi_k(\mathrm{B}) \overset{\mathcal{F}_k}{\cong} \pi_k(\mathrm{BSymp}(M,\omega))$ *for* $k \neq 1,2$.
(2) *The following sequences are exact.*

$$0 \to H^0(M;\mathbb{Z}) \to \pi_2(\mathrm{B}) \overset{\mathcal{F}_2}{\to} \pi_2(\mathrm{BSymp}(M,\omega)) \to \Gamma_\omega \to 0$$
$$0 \to \Gamma_\omega \to H^1(M;\mathbb{Z}) \to \pi_1(\mathrm{B}) \overset{\mathcal{F}_1}{\to} \pi_1(\mathrm{BSymp}(M,\widehat{\omega})) \to 0$$

1.6. **An extension of** $\mathrm{Symp}(M,\widehat{\omega})$**.** We identify the universal symplectic configuration space B as the classifying space $B\mathcal{G}_\tau$ of certain extension $\mathcal{G}_\tau$ of $\mathrm{Symp}(M,\widehat{\omega})$. More precisely this extension is of the form

$$0 \to \mathrm{Map}(M,U(1)) \to \mathcal{G}_\tau \to \mathrm{Symp}(M,\widehat{\omega}) \to 1,$$

where $\mathrm{Map}(M,U(1))$ is the (Abelian) gauge group of continuous maps from M to $U(1)$ (see Section 2.3 for the precise definition). We shall usually omit the subscript τ if it does not lead to a confusion. The proof of the next theorem is in Section 2.3.

Theorem 1.6. *The classifying space of $\mathcal{G}$ is homotopy equivalent to the universal symplectic configuration space:*

$$B\mathcal{G} = \mathrm{B}.$$

Moreover, the classifying map $\mathcal{F}\colon \mathrm{B} = B\mathcal{G} \to \mathrm{BSymp}(M,\widehat{\omega})$ is induced by the projection $\mathcal{G} \to \mathrm{Symp}(M,\widehat{\omega})$.

1.7. **Holonomy group.** If Ω is the symplectic form on $\mathbb{CP}^\infty$ and $ev\colon M_\mathrm{B} \to \mathbb{CP}^\infty$ is the evaluation map, then $ev^*\Omega$ is the closed connection form on the universal symplectic configuration $(M,\omega) \to M_\mathrm{B} \to \mathrm{B}$. It defines a symplectic connection whose holonomy group is denoted by $\mathcal{D} = \mathcal{D}_\tau \subset \mathrm{Symp}(M,\widehat{\omega})$.

Note that $\mathcal{D}$ depends on the lift τ and is defined only up to conjugacy. In general we will omit the subscript τ except the necessary cases.

Theorem 1.7. (1) *$\mathcal{D}$ intersects every connected component of* $\mathrm{Symp}(M,\widehat{\omega})$*;*
(2) *The identity component $\mathcal{D}_0$ of $\mathcal{D}$ is equal to* $\mathrm{Ham}(M,\omega)$*;*
(3) $\mathcal{D} \cap \mathrm{Symp}_0(M,\omega) = \mathrm{Flux}^{-1}(H^1(M;\mathbb{Z})/\Gamma_\omega)$ *that is the intersection consists of symplectomorphisms with integral flux.*

Proof. (1) Since $\pi_1(\mathrm{B}) \to \pi_1(B\,\mathrm{Symp}(M,\widehat{\omega}))$ is a surjection (Theorem 1.5) then $\mathcal{D}$ intersects every connected component of $\mathrm{Symp}(M,\widehat{\omega})$.

(2) Let $(M,\omega) \to M_W \to \mathrm{B}_W$ be a symplectic configuration fibration. Since Ω_W is a closed connection 2-form then the holonomy about any contractible loop is Hamiltonian due to Theorem 6.21 in [12]. This proves that $\mathcal{D}_0 \subset \mathrm{Ham}(M,\omega)$. The converse inclusion follows from Corollary 3.4.

(3) This part will be proved in Proposition 3.7(2). □

Finally we find a connection between the holonomy subgroup $\mathcal{D}$ and the extension $\mathcal{G}$.

Theorem 1.8. *Let* $\mathrm{Map}_0(M,U(1))$ *denote the identity component of* $\mathrm{Map}(M,U(1))$*. There exists a commutative triangle of continuous homomorphisms*

$$\begin{array}{ccc} & & \mathcal{G}/\,\mathrm{Map}_0(M,U(1)) \\ & \nearrow & \downarrow \\ \mathcal{D} & \longrightarrow & \mathrm{Symp}(M,\widehat{\omega}) \end{array}$$

in which the diagonal arrow is a homotopy equivalence. In particular, the structure group of E *lifts to* $\mathcal{G}/\,\mathrm{Map}_0(M,U(1))$ *if and only if the structure group of* E *reduces to* $\mathcal{D}$*.*

The next two theorems summarize all main results of the paper.

Theorem 1.9. *Let* (M,ω) *be a closed integral symplectic manifold and let* τ *be an integral lift of* $[\omega]$*. Let* $(M,\omega) \to E \to B$ *be a symplectic fibration. The following conditions are equivalent:*

1. *E admits a closed integral connection 2-form Ω compatible with τ, i.e. there exists $\mathcal{T} \in H^2(E,\mathbb{Z})$ such that it lifts Ω and extends τ;*
2. *E is an integral symplectic configuration compatible with τ;*
3. *the structure group of E lifts to $\mathcal{G}_\tau$.*

Proof. **(1) ⇒ (2)** is proved in Section 2.4,
(1) ⇒ (3) is proved in Section 2.5,
(2) ⇒ (1) follows from Lemma 1.3,
(2) ⇔ (3) follows form Theorem 1.6, □

Clearly, if E is an integral symplectic configuration with respect to the lift τ then its structure group reduces to the holonomy group $\mathcal{D}_\tau$. In other words, if the structure group lifts to $\mathcal{G}_\tau$ then it lifts to $\mathcal{G}_\tau/\,\mathrm{Map}_0(M,U(1))$. The folowing example due to McDuff shows that the converse in not true.

Example 1.10. Let $S^2 \to BSO(2) \xrightarrow{\pi} BSO(3)$ be the universal fibration associated with the action of $SO(3)$ on S^2. Let τ be the generator of $H^2(S^2,\mathbb{Z})$. We claim that τ does not admit an integral extension although the fibration is Hamiltonian. Indeed, $H^3(BSO(2);\mathbb{Z}) = 0$ therefore $H^3(BSO(3),\mathbb{Z}) = \mathbb{Z}/2$ is in the kernel of π^*. Hence the generator of the latter group is killed by the generator $\tau \in H^2(S^2,\mathbb{Z})$ in the spectral sequence. Thus τ is not in the image of $\pi^* : H^2(BSO(2)) \to H^2(S^2,\mathbb{Z})$ i.e. it does not admit an integral extension.

The proof of the following theorem will be given in Section 3.4.

Theorem 1.11. *Suppose that the structure group of a symplectic fibration $M \to E \to B$ reduces to $\mathcal{D}_\tau \subset \mathrm{Symp}(M, \widehat{\omega})$. If $H_2(B, \mathbb{Z})$ is torsion-free then E is an integral configuration.*

Remark 1.12. We do not prove that E is a configuration compatible with τ. We show that $d_2(\tau) = 0$ and $d_3(\tau) \in E_3^{3,0}$ is torsion in the spectral sequence assocoated with $M \to E \to B$. Thus *a priori* τ might not extend. Notice that, although $E_2^{3,0} = H^3(B, \mathbb{Z})$ is (by the assumption) torsion free, $E_3^{3,0} = E_2^{3,0}/d_2(E_2^{1,1})$ might have nontrivial torision. However, we don't know any example in which E is a configuration not compatible with τ.

1.8. **Other results.**

(1) Section 3 is devoted to the geometry of symplectic configurations. More precisely, we define a principal connection on fibrations $\mathrm{Symp}(M, W) \to \mathrm{B}_W$ and investigate its properties.
(2) In Section 4, we investigate the group cohomology relations between groups $\mathcal{D}$ and $\mathcal{G}$. We also compare these groups with certain subgroups $\mathrm{Ham}^{s\mathbb{Z}}(M, \omega)$ of $\mathrm{Symp}(M, \omega)$ recently discovered by McDuff [11].
(3) In Section 5, we investigate characteristic classes of integral symplectic configurations and relate them to other known characteristic classes.

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2. Topology of symplectic configurations

2.1. **Gromov's h-principle.** Let $\mathrm{Map}(M, \mathbb{CP}^\infty)$ (respectively $\mathrm{Map}^\infty(M, \mathbb{CP}^\infty)$) denotes the space of all continuous (respectively smooth) maps from M to $\mathbb{CP}^\infty$ equipped with compact-open (respectively C^∞) topology. The space of smooth maps is defined as $\bigcup_n C^\infty(M, \mathbb{CP}^n)$. It is well known that the inclusion $\mathrm{Map}^\infty(M, \mathbb{CP}^\infty) \to \mathrm{Map}(M, \mathbb{CP}^\infty)$. induces a (weak) homotopy equivalence.

Theorem 2.1. *Let $\tau \in H^2(M, \mathbb{Z})$ be a lift of $[\omega] \in H^2(M, \mathbb{R})$. Let $\mathrm{Symp}_\tau(M, \mathbb{CP}^\infty)$ and $\mathrm{Map}_\tau(M, \mathbb{CP}^\infty)$ denote the connected components corresponding to the class τ. The inclusion $i\colon \mathrm{Symp}_\tau(M, \mathbb{CP}^\infty) \to \mathrm{Map}_\tau(M, \mathbb{CP}^\infty)$ induces an isomorphisms on homotopy groups.*

Remark 2.2. The above theorem easily follows from the parametric version of *h-principle for symplectic embeddings*. Unfortunately, it seems that there is no proof of it in the literature yet. That is why we prove the above theorem appealing only to the well known Gromov's *h-principle* for symplectic immersions and its parametric version (Theorem (A) and Exercise (2) in Section 3.4.2 of [4]).

Proof of Theorem 2.1. Since the spaces of smooth and continuous maps are weakly homotopy equivalent we can assume that all maps are smooth.

Injectivity: Suppose that $s \in \ker i_*$ that is there exist a commutative diagram

$$\begin{array}{ccc} S^k \times M & \xrightarrow{f} & \mathbb{CP}^m \\ \downarrow & \nearrow_{F} & \\ D^{k+1} \times M & & \end{array}$$

where f is symplectic embedding on each fiber $s \times M$ and F is smooth embedding on fibers and equals f over the boundary. The argument consists of several steps.

We first deform F to a fiberwise symplectic immersion F' so that it equals f over the boundary. This can be done according to the parametric h-principle for symplectic immersions (Theorem 16.4.3 in [2]). We want to improve it so that it will be fiberwise symplectic embedding.

Let $j : M \to \mathbb{R}^N \subset \mathbb{C}^N$ be the composition of an embedding if M into $\mathbb{R}^N$ and the standard inclusion of Lagrangian $\mathbb{R}^N$ into $\mathbb{C}^N$. Then j is an isotropic embedding. Moreover, let $\varphi : \mathbb{CP}^m \times D^{2N} \to \mathbb{CP}^l$ be a symplectic embedding that is linear on $\mathbb{CP}^m \times 0$. Take

$$D^{k+1} \times M \to \mathbb{CP}^m \times D^{2N} \to \mathbb{CP}^l$$

defined by

$$(d, m) \mapsto \varphi(F'(d, m), \alpha(d)j(m)),$$

where $\alpha : D^{k+1} \to \mathbb{R}$ is a scaling function such that it equals 0 exactly on the boundary sphere. Clearly, it is an embedding because j is. It is symplectic because F' is fiberwise symplectic and j is isotropic.

Surjectivity: We shall show that every fiberwise smooth embedding $f : S^k \times M \to \mathbb{CP}^m$ (fiberwise in $\mathrm{Map}_\tau(M, \mathbb{CP}^m)$) is homotopic to a fiberwise symplectic embedding. First take the composition $F : T^*S^k \times M \to \mathbb{CP}^m$ of f and the projection $T^*S^k \times M \to S^k \times M$. According to the standard h-principle there exist arbitrarily small deformation of F that is a symplectic embedding (with respect to the product symplectic form). The restriction of it to the image of the "zero section" $S^k \times M \subset T^*S^k \times M$ gives a symplectic embedding. The above homotopy might not preserve the basepoint but this is not a problem due to the injectivity of i_*.

□

2.2. **Proof of Theorem 1.5.**

Lemma 2.3. *$\pi_k \mathrm{Map}(M, \mathbb{CP}^\infty) = H^{2-k}(M, \mathbb{Z})$, where we set $H^m(M, \mathbb{Z}) = 0$ for $m < 0$.*

Proof. If follows from the basic algebraic topology (e.g. [15]). Let $s \in S^k$ and $f \in \mathrm{Map}(M, X)$ be a base point, the constant map $m \mapsto x \in X$. Moreover let M_+ denote M with artificially added base point. Then we calculate

$$\begin{aligned} \pi_k(\mathrm{Map}(M, X)) &= [(S^k, s), (\mathrm{Map}(M, X), f)] \\ &= [M_+ \wedge S^k, (X, x)] \\ &= [M_+, \Omega^k(X, x)] \end{aligned}$$

Since $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$ we have that $\pi_k(\mathrm{Map}(M, \mathbb{CP}^\infty)) = [M, K(\mathbb{Z}, 2-k)] = H^{2-k}(M, \mathbb{Z})$.

It is easy to see a composition of this isomorphism with the projection $H^{2-k}(M, \mathbb{Z}) \to \mathrm{Hom}(H_{2-k}(M, \mathbb{Z}), \mathbb{Z})$ (which is an isomorphism for $k \in \{1, 2\}$) as follows. Let $[\xi] \in \pi_k(\mathrm{Map}(M, \mathbb{CP}^\infty))$ be represented by a map $\xi \colon S^k \times M \to \mathbb{CP}^\infty$. The value of

the resulting cohomology class on $(2-k)-$cycle a is equal to $\left\langle [S^k \times a], \xi^* \omega_\infty \right\rangle$, where $\omega_\infty \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$ is the generator.

□

Corollary 2.4. $\pi_k \operatorname{Symp}_\tau(M, \mathbb{CP}^\infty) = \begin{cases} H^{2-k}(M; \mathbb{Z}) & \text{for } k \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$

and hence $\pi_k(\text{Б}) \cong \pi_k(\text{BSymp}(M, \widehat{\omega}))$ *for* $k > 3$.

Proof. Notice that $\operatorname{Map}(M, \mathbb{CP}^\infty)$ is an H-space (since $\mathbb{CP}^\infty$ is). Therefore all its connected components are homotopy equivalent. The first statement follows immediately from Theorem 2.1 and Lemma 2.3. The second is the direct application of the long exact sequence of homotopy group for the fibration $\operatorname{Symp}(M, \widehat{\omega}) \to \operatorname{Symp}_\tau(M, \mathbb{CP}^\infty) \to \text{Б}$. □

Example 2.5. Let Σ_g be a surface of genus $g > 0$. Let $i_0, i_1 \in \operatorname{Symp}(\Sigma_g, \mathbb{CP}^\infty)$ be two embeddings such that $i_0 = i_1 \circ f$, where $f \colon \Sigma_g \to \Sigma_g$ is a Dehn twist. One can construct a symplectic Lefschetz fibration $(M, \omega) \to S^2$ with the generic fiber Σ_g such that the monodromy about some critical value is exactly f and (M, ω) is an integral symplectic manifold [1]. Embedding (M, ω) into $\mathbb{CP}^n$ we also get a family of symplectic embeddings $i_t \colon \Sigma_g \to \mathbb{CP}^N$ for $t \in [0, 1]$. Since $\pi_0(\operatorname{Symp}(\Sigma_g, area))$ is generated by Dehn twists, then we obtain that the map $\pi_0(\operatorname{Symp}(\Sigma_g, area)) \to \pi_0(\operatorname{Symp}(\Sigma_g, \mathbb{CP}^\infty))$ is trivial.

Lemma 2.6. *The map* $\pi_2(\operatorname{Symp}(M, \omega)) \to \pi_2(\operatorname{Symp}(M, \mathbb{CP}^\infty))$ *is trivial. In particular,* $\pi_3(\text{Б}) \cong \pi_3(\text{BSymp}(M, \omega))$.

Proof. Let $\xi \in \pi_2(\operatorname{Symp}(M, \omega))$. Its image in $\pi_2(\operatorname{Map}(M, \mathbb{CP}^\infty)) = \mathbb{Z}$ is equal to $\langle [\omega], ev_*(\xi) \rangle$, where $ev \colon \operatorname{Symp}(M, \omega) \to M$ is the evaluation at the base point $pt \in M$ (cf. proof of Lemma 2.3). More precisely, consider the following commutative diagram.

$$\begin{array}{ccc} \operatorname{Symp}(M, \omega) & \xrightarrow{i} & \operatorname{Symp}(M, \mathbb{CP}^\infty) \\ \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} \\ M & \xrightarrow{f} & \mathbb{CP}^\infty \end{array}$$

The isomorphism $\pi_2(\operatorname{Map}(M, \mathbb{CP}^\infty)) = \mathbb{Z}$ is given by the integration of the generator $[\omega_\infty] \in H^2(\mathbb{CP}^\infty)$ over the sphere $S^2 \to \operatorname{Map}(M, \mathbb{CP}^\infty) \xrightarrow{ev} \mathbb{CP}^\infty$. Hence we calculate that $\langle [\omega_\infty], ev_* i_*(\xi) \rangle = \langle [\omega_\infty], f_* ev_*(\xi) \rangle = \langle f^*[\omega_\infty], ev_*(\xi) \rangle = \langle [\omega], ev_*(\xi) \rangle$ as required.

We claim that this number is equal to zero. Indeed, if $[\omega]$ is a sum of products of 1-dimensional classes then $\langle [\omega], ev_*(\xi) \rangle = \langle \xi^*(ev^*[\omega]), [S^2] \rangle = 0$. Suppose that $[\omega]$ is not a sum of products. Recall that, according to Milnor and Moore [13], the cohomology ring of $\operatorname{Symp}(M, \omega)$ (which is an H-space) is free graded commutative algebra. If $ev^*[\omega]$ were nonzero then it would be of infinite order, which is impossible since $[\omega]^{n+1} = 0$. □

Let $\operatorname{Flux} : \pi_1(\operatorname{Symp}(M, \omega)) \to H^1(M; \mathbb{R})$ be the *flux homomorphism* defined as follows. $\langle \operatorname{Flux} \xi, \gamma \rangle = \langle [\omega], \xi_*(\gamma) \rangle$, where $\xi_*(\gamma) \colon \mathbb{T}^2 \to M$ sends $(t, s) \to \xi_t(\gamma(s))$ (see Chapter 9 in [12] for details). Its image is called the flux group and is denoted by Γ_ω. Notice that in the case of integral symplectic structure Γ_ω is a subgroup of the

integral cohomology $H^1(M,\mathbb{Z})$. The following lemma is an immediate consequence of the above definition.

Lemma 2.7. *The following diagram is commutative.*

$$\begin{array}{ccc} \pi_1(\mathrm{Symp}(M,\omega)) & \longrightarrow & \pi_1(\mathrm{Symp}(M,\mathbb{CP}^\infty)) \\ {\scriptstyle \mathrm{Flux}}\downarrow & & \cong\downarrow \\ \Gamma_\omega & \longrightarrow & H^1(M,\mathbb{Z}) = \pi_1(\mathrm{Map}(M,\mathbb{CP}^\infty)) \end{array}$$

□

Proof of Theorem 1.5.

(1) This part follows from Corollary 2.4 and Lemma 2.6.
(2) To prove that the first sequence is exact one applies Lemma 2.6 and 2.7 to the exact sequence associated with the fibration $\mathrm{Symp}(M,\widehat{\omega}) \to \mathrm{Symp}_\tau(M,\mathbb{CP}^\infty) \to B$. Similarly, Corollary 2.4 and Lemma 2.7 applied to the long exact sequence of the the same fibration proves the exactness of the second sequence.

□

2.3. **Proof of Theorem 1.6.** Recall that G-principle bundles over M are classified by $H^1(M,G)$ and $H^1(M,U(1))$ is canonically isomorphic to $H^2(M,\mathbb{Z})$. The bundle corresponding to $\tau \in H^2(M,\mathbb{Z})$ will be denoted L_τ, and the class of a bundle L in $H^2(M,\mathbb{Z})$ we will be denoted by $[L]$.

Assume that (M,ω) and (W,ω_W) are integral symplectic manifolds. Choose $\tau \in H^2(M,\mathbb{Z})$ and $\tau_W \in H^2(W,\mathbb{Z})$ to be some lifts of ω and ω_W respectively. Let

$$\mathrm{Map}^{U(1)}_{Symp}(L_\tau, L_{\tau_W})$$

be the space of maps from L_τ to L_{τ_W} which are $U(1)$-equivariant and cover a symplectomorphism $(M,\omega) \to (W,\omega_W)$:

$$\begin{array}{ccc} L_\tau & \xrightarrow{\hat{\psi}} & L_{\tau_W} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\psi} & W. \end{array}$$

Observe that if $\psi\colon M \to W$ is a continuous map, then $\psi^*(L_\tau) = L_{\psi^*\tau}$, thus there exists a covering $U(1)$-equivariant map $\widehat{f}\colon L_{\psi^*\tau} \to L_\tau$.

In particular, for $(W,\omega_W) = (M,\omega)$ we get a group

$$\mathcal{G} := \mathrm{Map}^{U(1)}_{Symp}(L_\tau, L_\tau).$$

Lemma 2.8. *There is an extension*

$$1 \to \mathrm{Map}(M,U(1)) \to \mathcal{G} \to \mathrm{Symp}(M,\widehat{\omega}) \to 1$$

Proof. The kernel of the map that assigns to $\tilde{\psi}$ the underlying symplectomorphism ψ consists of the automorphisms which cover the identity. Over each point of M such automorphism is given by an element of $U(1)$. □

Proposition 2.9. *Let ω_n denote the standard symplectic form on $\mathbb{CP}^n$, and let $[L_n] = [\omega_n] \in H^2(\mathbb{CP}^n, \mathbb{Z})$. Then the space*

$$\mathrm{Map}^{U(1)}_{Symp}(L_\tau, L_\infty) := \bigcup_n \mathrm{Map}^{U(1)}_{Symp}(L_\tau, L_n)$$

is contractible.

Proof. Let $\mathrm{Map}^{U(1)}(L_\tau, L_\infty)$ be a space of all $U(1)$-equivariant maps $L_\tau \to L_\infty$. Then we have a bundle:

$$\mathrm{Map}(M, U(1)) \to \mathrm{Map}^{U(1)}(L_\tau, L_\infty) \to \mathrm{Map}_\tau(M, \mathbb{CP}^\infty),$$

Consider the following two exact sequences of homotopy groups

$$\begin{array}{ccccc}
\ldots \pi_k(\mathrm{Map}(M,U(1))) & \longrightarrow & \pi_k(\mathrm{Map}^{U(1)}(L_\tau,L_\infty)) & \longrightarrow & \pi_k(\mathrm{Map}_\tau(M,\mathbb{CP}^\infty)) \ldots \\
= \uparrow & & \uparrow & & \uparrow \\
\ldots \pi_k(\mathrm{Map}(M,U(1))) & \longrightarrow & \pi_k(\mathrm{Map}^{U(1)}_{Symp}(L_\tau,L_\infty)) & \longrightarrow & \pi_k(\mathrm{Symp}_\tau(M,\mathbb{CP}^\infty)) \ldots
\end{array}$$

where the vertical arrows are induced by inclusions. According to Theorem 2.1, the map

$$\pi_k(\mathrm{Symp}_\tau(M, \mathbb{CP}^\infty)) \to \pi_k(\mathrm{Map}_\tau(M, \mathbb{CP}^\infty))$$

is an isomorphism. By the five lemma the same is true for

$$\pi_k(\mathrm{Map}^{U(1)}_{Symp}(L_\tau, L_\infty)) \to \pi_k(\mathrm{Map}^{U(1)}(L_\tau, L_\infty)).$$

We are left to show that $\mathrm{Map}^{U(1)}(L_\tau, L_\infty)$ is contractible. Since $L_\infty = EU(1)$, then the following lemma finishes the proof. □

Lemma 2.10. *Let E be a free G space. Then $\mathrm{Map}^G(E, EG)$ is contractible.*

Proof. It is a standard fact (about universal principal bundles) that $\mathrm{Map}^G(E, EG)$ is connected (see Theorem 8.12 on page 58 in [16]). We need to show that for all k the space $\mathrm{Map}(S^k, \mathrm{Map}^G(E, EG)) = \mathrm{Map}^G(E \times S^k, EG)$ is connected. But $S^k \times E$ is again a free G space and we apply the observation we began with. □

Proof of Theorem 1.6. Consider the following two fibrations

$$\mathrm{Map}(M, U(1)) \to \mathrm{Map}^{U(1)}_{Symp}(L_\tau, L_\infty) \xrightarrow{p_1} \mathrm{Symp}_\tau(M, \mathbb{CP}^\infty)$$

and

$$\mathrm{Symp}(M, \widehat{\omega}) \to \mathrm{Symp}_\tau(M, \mathbb{CP}^\infty) \xrightarrow{p_2} \mathrm{B}_\tau,$$

where the base B_τ is defined to be this quotient. Observe that the composition $p_2 \circ p_1$ defines a fibration

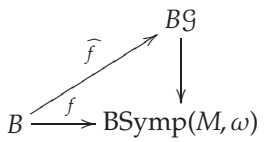
$$\mathcal{G} \to \mathrm{Map}^{U(1)}_{Symp}(L_\tau, L_\infty) \xrightarrow{p_2 \circ p_1} \mathrm{B}_\tau$$

with a contractible total space. □

2.4. **Proof of Theorem 1.9** ($1 \Rightarrow 2$)**.** By assumption the connection form Ω has a lift $\mathcal{T} \in H^2(E, \mathbb{Z}) = [E, K(\mathbb{Z}, 2)]$. Since $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$ then we get an embedding $f : E \to \mathbb{CP}^N$ such that $\mathcal{T}$ is the pull-back of the generator. According to to the original Gromov's h-principle (Theorem 3.4.2 A in [4]) we can deform f to an immersion preserving forms. That is an immersion $f' : E \to \mathbb{CP}^N$ so that $f^*(\omega_N) = \Omega$. At the price of increasing N we can deform it further to get a embedding as we did in the proof of Theorem 2.1. □

2.5. **Proof of Theorem 1.9** ($1 \Rightarrow 3$)**.** A class $\mathcal{T}$ defines a line bundle $L_{\mathcal{T}} \to E$. Notice that the pull-back bundle under the inclusion of the fiber $i\colon M \to E$ is the line bundle $L_{i^*\mathcal{T}} \to M$. Thus $i^*\mathcal{T}$ is a lift of $[\omega]$ to $H^2(M, \mathbb{Z})$.

Consider the composition $L_{\mathcal{T}} \to E \to B$ as a bundle over B with fiber $U(1)$. The structure group of this fibration is contained in $\mathcal{G}$. Thus we get a classifying map $\widehat{f}\colon B \to B\mathcal{G}$ such that the following diagram commutes up to homotopy (here f classifies E):



□

Recall that according to Theorem 1.6 $B\mathcal{G} \simeq \mathrm{B}$. The classifying map $\widehat{f}\colon B \to B\mathcal{G} = \mathrm{B}$ is the one constructed in the previous section.

3. Geometry of symplectic configurations

3.1. **A principal connection on** Symp(M, W)**.** The main reference for connections is Kobayashi-Nomizu [9]. The tangent space to Symp(M, W) at point f is equal to

$$\{\mathcal{X} \in \Gamma(f^*TW) : \mathcal{X}(p) = \frac{d}{dt} f_t(p) \quad f_t \in \mathrm{Symp}(M, W)\}.$$

We define the horizontal space $\mathcal{H}_f$ to be the space of the sections $f^*\omega_W$-orthogonal to M:

$$\mathcal{H}_f := \{\mathcal{X} \in T_f \mathrm{Symp}(M, W) : f^*\omega_W(\mathcal{X}, Y) = 0 \quad \forall Y \in \Gamma(TM)\}.$$

Consider the following 1-form θ on Symp(M, W) with values in closed 1-forms on M which is identified with the Lie algebra of Symp(M, ω)

$$\theta_f(\mathcal{X}) \colon = f^*(\iota_{\mathcal{X}_f}\omega_W)$$

for $\mathcal{X} \in T_f \mathrm{Symp}(M, W)$. More precisely

$$(\theta_f(\mathcal{X}))(p)(Y) = \omega_W(\mathcal{X}_f(p), f_*Y(p)),$$

where Y is a vector field on M and $\mathcal{X}_f$ is a vector field on W defined on $f(M)$.

Lemma 3.1. *The 1-form θ is a connection form, that is it satisfies*

1. $\theta_{f\circ\psi} = Ad_\psi\theta = \psi^* \circ \theta$,
2. $\theta_f(\underline{X}) = X$, *where $X \in \Omega^1(M)$ is a closed 1-form and $\underline{X}$ denotes the fundamental vector field.*

Proof. (1) This is an immediate calculation:

$$\begin{aligned}(\theta_{f\circ\psi}(\mathcal{X}))(p)(Y) &= \omega_W(\mathcal{X}_{f\circ\psi}(p), (f\circ\psi)_*Y(p)\\ &= \omega_W(\mathcal{X}_f(\psi(p)), f_*(\psi_*Y(p))\\ &= (\theta_f(\mathcal{X}_f))(\psi(p))(\psi_*Y(p))\\ &= (\psi^*\circ\theta_f)(\mathcal{X}_f)(p)(Y).\end{aligned}$$

(2) If X is a closed one form on M then we denote by $X^\sharp$ the ω-corresponding vector field and by ψ_t its flow. That is $\iota_{X^\sharp}\omega = X$. Thus $\underline{X}_f = X^\sharp$ under the identification between M and $f(M) \in W$ given by f. Hence we get

$$\theta_f(X^\sharp) = f^*(\iota_X^\sharp\omega_W) = X.$$

□

Proposition 3.2. *The connection associated to θ on M_W and connection defined by the 2-form $\Omega_W := ev^*(\omega_W)$ coincide.*

Proof. Both distributions consist of vectors $\mathcal{X} \in T_{[f,p]}M_W$ satisfying the condition $ev_*(\mathcal{X}) \in T_{f(p)}W$ is ω_W-orthogonal to $f_*(T_pM)$. □

3.2. **The curvature of θ.** Let's calculate the curvature 2-form of the connection θ. By definition it is

$$\Theta(\mathcal{X}, \mathcal{Y}) := d\theta(\mathcal{X}^h, \mathcal{Y}^h) = -\theta([\mathcal{X}^h, \mathcal{Y}^h]),$$

where $\mathcal{X}^h, \mathcal{Y}^h \in \mathcal{H}$ are horizontal parts of tangent vectors. Calculating it further we get

$$\begin{aligned}-\Theta_f(\mathcal{X}^h, \mathcal{Y}^h) &= \theta_f[\mathcal{X}^h, \mathcal{Y}^h]\\ &= f^*(\iota_{[\mathcal{X}^h,\mathcal{Y}^h]}\omega_W)\\ &= f^*(\{\alpha_\mathcal{X}, \alpha_\mathcal{Y}\})\end{aligned}$$

Here $\alpha_\mathcal{X}$, $\alpha_\mathcal{Y}$ are 1-forms associated with extensions of $\mathcal{X}^h$, $\mathcal{Y}^h \in \Gamma(f^*TW)$ to vector field on some neighborhood of $f(M) \subset W$ and $\{\alpha_\mathcal{X}, \alpha_\mathcal{Y}\}$ is the Poisson bracket. Since $\mathcal{X}^h$ and $\mathcal{Y}^h$ are horizontal then the 1-forms $\alpha_\mathcal{X}$, $\alpha_\mathcal{Y}$ vanish on M. Therefore their local extensions might be chosen to be exact and we get

$$-\Theta_f(\mathcal{X}^h, \mathcal{Y}^h) = d\left[\, f^*\{H_\mathcal{X}, H_\mathcal{Y}\}\,\right],$$

where $H_\mathcal{X}$, $H_\mathcal{Y} : W \to \mathbb{R}$ are Hamiltonians, i.e. $dH_\mathcal{X} = \alpha_\mathcal{X}$ and $dH_\mathcal{Y} = \alpha_\mathcal{Y}$.

In other words the curvature of the connection θ at the embedding $f : (M, \omega) \to (W, \omega_W)$ is measured by the subspace of functions on M consisting of restrictions of Poisson brackets of functions on W which are constant on $f(M)$. We will use this curvature form to construct characteristic classes in Section 5.4.

Lemma 3.3. *Let $f : (M, \omega) \to (M, \omega) \times (\Sigma, \omega_\Sigma)$ be a symplectic embedding given by $f(p) = (p, s_0)$, where $s_0 \in \Sigma$ is a base point on the surface. The image of the curvature 2-form*

$$\Theta_f : \mathcal{H}_f \times \mathcal{H}_f \to \mathit{Lie}\,(\mathrm{Symp}(M, \omega))$$

contains the subspace of all exact 1-forms.

Proof. What in fact we need to show is that given a function $H\colon M \to \mathbb{R}$ there are functions $F, G\colon M \times \Sigma \to \mathbb{R}$ constant on $f(M)$ such that $\{F, G\} \circ f = H$.

Let $F', G'\colon \Sigma \to \mathbb{R}$ be smooth functions such that $F'(s_0) = 0$ and $\{F', G'\}(s_0) = 1$. Define $F(p, s) := F'(s)$, $G(p, s) = G'(s)$ and $\widehat{H}(p, s) = H(p)$. The following calculation follows from standard properties of the Poisson bracket.

$$\{\widehat{H}F, G\} = \widehat{H}\{F, G\} + F\{\widehat{H}, G\} = \widehat{H}\{F, G\}$$

Hence, $\{\widehat{H}F, G\} \circ f = \widehat{H} \circ f = H$. Note that the functions $\widehat{H}F$ and G are constant on $f(M)$ as required. □

Corollary 3.4. *The image of the curvature 2-form*

$$\Theta_{iof}\colon \mathcal{H}_{iof} \times \mathcal{H}_{iof} \to \mathit{Lie}\,(\mathrm{Symp}(M, \omega))$$

of the connection in the universal configuration fibration contains the subspace of all exact 1-forms for every $f\colon (M, \omega) \to \mathbb{CP}^\infty$. □

3.3. **The holonomy group.** Let $(M, \omega) \xrightarrow{i} E \xrightarrow{\pi} B$ be a symplectic fibration. Let $K := \ker[\,\pi_* : H_2(E, \mathbb{Z}) \to H_2(B, \mathbb{Z})]$. The following exact sequence:

$$0 \to H_2(F, \mathbb{Z}) \to K \to H_1(B; H_1(F, \mathbb{Z})) \to 0$$

is derived from the Leray-Serre spectral sequence associated with E. An element of $H_1(B, H_1(F, \mathbb{Z}))$ may be described as a (class of a) loop γ in B and a γ-invariant cycle $[\ell] \in H_1(F, \mathbb{Z})^\gamma$ in the fiber F over a base point of γ.

Let Ω be a closed extension of ω. We construct certain lift $\Sigma_{\gamma,\ell} \in H_2(E, \mathbb{Z})$ of $(\gamma, \ell) \in H_1(B; H_1(M, \mathbb{Z}))$ to K. It is a union of two chains $f(S^1 \times I)$ and C. The map f is such that $f(\{s\} \times I)$ is a horizontal lift of γ for each s and $f(S^1 \times \{0\}) = \ell$. The chain C is a chain in F such that $\partial C = \gamma_*\ell - \ell$, where γ_* is the holonomy of γ with respect to Ω. Observe that $\Omega(f_*(\frac{d}{ds}), f_*(\frac{d}{dt})) = 0$ because the kernel of Ω is the horizontal subspace. Thus $\int_{\Sigma_{\gamma,\ell}} \Omega = \int_C \omega$.

Let $\psi \in \mathrm{Symp}(M, \omega)$ and $H_1(M, \mathbb{Z})^\psi = \{\ell \in H_1(M, \mathbb{Z})\colon \psi_*(\ell) = \ell\}$ be the ψ-invariant part of the first cohomology. Define a flux-like homomorphism

$$\mathrm{Flux}^\psi : H^1(M, \mathbb{Z})^\psi \to \mathbb{R}/\omega$$

as follows (here $\mathbb{R}/\omega$ denotes the quotient of $\mathbb{R}$ by the group of periods of the symplectic form). If $[\ell] \in H_1(M, \mathbb{Z})^\psi$ then there exists a 2-chain C such that $\partial C = \ell - \psi_*\ell$. Then

$$\mathrm{Flux}^\psi(\ell) := \int_C \omega \mod (\text{periods of } \omega).$$

The integral is defined up to the period of ω and does not depend on the choice of representative ℓ in the homology class.

Let $\overline{\mathcal{D}}$ be a subset of $\mathrm{Symp}(M, \widehat{\omega})$ consisting of those $\psi \in \mathit{Symp}(M, \widehat{\omega})$ for which $\mathit{Flux}^\psi = 0$. Note that $\overline{\mathcal{D}}$ is conjugacy invariant, but it is not a subgroup of $\mathrm{Symp}(M, \widehat{\omega})$.

Proposition 3.5. *Let $M \to E \to B$ be a symplectic fibration. Let Ω be an extension of the symplectic form ω and let $K := \ker H_2(E, \mathbb{Z}) \to H_2(B, \mathbb{Z})$. Then $\langle \Omega, K \rangle \subset \mathbb{Z}$ if and only if the holonomy $\pi_1(B) \to \mathrm{Symp}(M, \widehat{\omega})$ is contained in $\overline{\mathcal{D}}$*

Proof. If $C' \in H_2(M,\mathbb{Z})$ then $\int_{i_*(C')} \Omega = \int_C' \omega$ since Ω extends ω. If $(\gamma,\ell) \in H_1(B;H_1(M,\mathbb{Z}))$ then

$$\int_{\Sigma_{\gamma,\ell}} \Omega = \int_C \omega = \mathrm{Flux}^{\psi}(\ell) \mod (\text{periods of } \omega),$$

where ψ is the monodromy around loop γ. □

Recall that we define $\mathcal{D}$ to be the holonomy group of the universal connection θ. Notice that $\mathcal{D}$ is not well defined as a subgroup of $\mathrm{Symp}(M,\omega)$ since it depends of the reference point $f\colon (M,\omega) \to \mathbb{CP}^\infty$. However, according to standard theory of connections, different holonomy subgroups are conjugate.

Corollary 3.6. *The set $\overline{\mathcal{D}}$ is the sum of all subgroups conjugated to $\mathcal{D}$.*

Proof. Let $E = M \times \mathbb{R}/\sim$, where $(m, t+k) \sim (\psi^k(m), t)$ for $k \in \mathbb{Z}$. The form $\Omega = \pi_M^*\omega$ on $M \times \mathbb{R}$ descends to a closed connection form on E with a holonomy ψ. Since $H_2(S^1,\mathbb{Z}) = 0$ we deduce from 3.5 that if $\psi \in \overline{\mathcal{D}}$ then Ω has integral periods.

To prove the converse inclusion notice that it is enough to show that $\mathcal{D} \subset \overline{\mathcal{D}}$, according to the conjugacy invariance. Since $\mathcal{D}$ is the universal holonomy group, for any $\psi \in \mathcal{D}$ there exists a bundle $M \to E \to S^1$ with a closed integral connection form Ω such that ψ is a holonomy of that form. Thus $\psi \in \overline{\mathcal{D}}$ again by 3.5. □

Proposition 3.7.

(1) *Let $H \subset \mathrm{Symp}(M,\widehat{\omega})$ be any subgroup. If $H \subset \overline{\mathcal{D}}$ then $H < \mathcal{D}$ (up to conjugacy).*
(2) $\mathcal{D} \cap \mathrm{Symp}_0(M,\widehat{\omega}) = \overline{\mathcal{D}} \cap \mathrm{Symp}_0(M,\widehat{\omega})$,
(3) *$\mathcal{D}$ is a closed subgroup of* $\mathrm{Symp}(M,\widehat{\omega})$

Proof. First we will prove (1) with the additional assumption that H is countable. The we will use it to prove (2) and (3). Finally we will conclude general case of (1) with the use of (3).

(1) We shall construct a fibration over a noncompact surface (possibly of infinite genus) with closed connection form Ω and holonomy equal to H. Since $H < \overline{\mathcal{D}}$ then, according to Proposition 3.5 $\langle \Omega, K\rangle \subset \mathbb{Z}$. Because the base is a noncompact surface then the latter is equivalent to the integrality of Ω. To see this notice that $K = H_2(E,\mathbb{Z})$ as $H_2(B,\mathbb{Z})$ is trivial. By the universality of $\mathcal{D}$ we will conclude that, up to a conjugation, $H < \mathcal{D}$.

Take $M_{\psi_i} \times S^1 \to T^2$ for each $\psi_i \in H$, $i = 1,2,\ldots$ and form a countable fiber connected sum $(M_{\psi_1} \times S^1)\#_M(M_{\psi_2} \times S^1)\#_M \ldots$. It is a bundle over Σ_∞ the surface of infinite genus and admits a closed connection form. Indeed, every $M_\psi \times S^1$ admits a closed connection form as in the proof of Corollary 3.6. One can easily extend it over the fiber connected sum.

Every loop in the connected sum is a composition of loops in the summands, thus the holonomy of the bundle is contained in H. Since all the generators of H are contained in the holonomy, H equals the holonomy group of that bundle.

(2) Let $\psi \in \overline{\mathcal{D}} \cap \mathrm{Symp}_0(M,\omega)$. Observe that $\mathrm{Flux}^{\psi} : H^1(M,\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ is equal to the value of the usual flux on ψ. This shows that $\overline{\mathcal{D}} \cap \mathrm{Symp}_0(M,\omega) \subset \mathrm{Flux}^{-1}(H^1(M,\mathbb{Z}))$. In fact, the equality holds, because $\mathcal{D}_0 = \mathrm{Ham}(M,\omega)$, according to Theorem 1.7 (2).

Now take one element ψ from each component of $\overline{\mathcal{D}} \cap \mathrm{Symp}_0(M,\omega)$ and generate subgroup $H \subset \mathrm{Symp}_0(M,\omega)$. It is clearly countable, hence it is contained in $\mathcal{D} \cap \mathrm{Symp}_0(M,\omega)$, due to the part (1). Since $\mathcal{D}_0 = \mathrm{Ham}(M,\omega)$ then $\overline{\mathcal{D}} \cap \mathrm{Symp}_0(M,\omega) = H \cdot \mathrm{Ham}(M,\omega) \subset \mathcal{D} \cap \mathrm{Symp}_0(M,\omega)$.

(3) Since $\overline{\mathcal{D}} \cap \mathrm{Symp}_0(M,\widehat{\omega}) = \mathrm{Flux}^{-1}(H^1(M,\mathbb{Z}))$ is a closed subgroup of $\mathrm{Symp}_0(M,\omega)$ then (3) follows.

To finish the proof we need to show that (1) holds in full generality. Let H' be a countable dense subgroup of H. Such a group exists because $\mathrm{Symp}(M,\omega)$ is a separable group modeled on the space of closed 1-forms. Then $gH'g^{-1} \subset \mathcal{D}$ by already proven part of (1) and $gHg^{-1} \subset \mathcal{D}$ by (3). □

3.4. **Proof of Theorem 1.11.** Let $\tau \in H^2(M,\mathbb{Z})$ be a preimage of the class $[\omega] \in H^2(M,\mathbb{R})$ of the symplectic form. Let $(M,\omega) \to E \to B$ be a symplectic fibration with the classifying map $f : B \to B\,\mathrm{Symp}(M,\widehat{\omega})$.

Lemma 3.8. *There exists a lift $\widehat{f} : B \to B\left(\mathcal{G}/\mathrm{Map}_0(M,U(1))\right)$ if and only if $d_2(\tau) = 0$. Here d_2 is the differential in the spectral sequence associated to E.*

Proof. Recall that $\mathrm{Map}(M,U(1))/\mathrm{Map}_0(M,U(1)) = H^1(M,\mathbb{Z})$ and we have a fibration

$$0 \to BH^1(M,\mathbb{Z}) \to B(\mathcal{G}/\mathrm{Map}_0(M,(U(1)))) \to B\,\mathrm{Symp}(M,\widehat{\omega}) \to 0$$

induced by the extension $0 \to H^1(M,\mathbb{Z}) \to \mathcal{G}/\mathrm{Map}_0(M,(U(1)) \to \mathrm{Symp}(M,\widehat{\omega}) \to 0$. Observe that the only obstruction to the existence of the lift is a class $\kappa \in H^2(B,\pi_1(BH^1(M,\mathbb{Z}))) = H^2(B,H^1(M,\mathbb{Z}))$. We shall show that κ vanishes if and only if $d_2([\tau])$ vanishes.

Suppose that $d_2(\tau) = 0$. This implies that for any map $h : \Sigma \to B$ from a surface the pull-back bundle $h^*E \to \Sigma$ admits an integral extension of τ. Hence there exists a lift $(\widehat{f \circ h}) \colon \Sigma \to B(\mathcal{G}/\mathrm{Map}_0(M,U(1)))$. More precisely, the lift is the composition of the lift $\Sigma \to B\mathcal{G}$ (guaranteed by Theorem 1.9 (1⇔3)) and the map induced by the projection $\mathcal{G} \to \mathcal{G}/\mathrm{Map}_0(M,U(1))$. Therefore κ evaluated on $h_*[\Sigma]$ equals zero. Since h was arbitrary the class κ vanishes.

Now suppose that there exists a lift $\widehat{f} : B \to B\mathcal{G}/\mathrm{Map}_0(M,U(1))$. The only obstruction for lifting it further to $B\mathcal{G}$ is an element of

$$H^3(B(\mathcal{G}/\mathrm{Map}_0(M,U(1)));\pi_2(B\,\mathrm{Map}_0(M,U(1)))).$$

Hence it vanishes on two dimensional cycles. This implies that any map $h : \Sigma \to B$ can be lifted to $B\mathcal{G}$. Thus the pull-back bundle admits an integral extension of the class τ (according to Theorem 1.9 (1⇔3)) which means that $h^*(d_2(\tau)) = d_2(h^*\tau) = 0$. Since h is arbitrary then we have $d_2(\tau) = 0$ over B. □

Lemma 3.9 ([8])**.** *If $d_2(\tau) = 0$ then $d_3(\tau) \in E_3^{3,0}$ is torsion.*

Proof. Recall that $\dim M = 2n$ and that $\pi_1(\mathrm{BSymp}(M,\widehat{\omega}))$ acts trivially on $H^{2n}(M,\mathbb{R})$. Then we have that

$$\begin{aligned} 0 &= d_3[\tau^{n+1}] \\ &= (n+1)d_3[\tau]\otimes[\tau^n] \in H^3(\mathrm{BSymp}(M,\omega);H^{2n}(M,\mathbb{Z})). \end{aligned}$$

It means that $d_3[\tau]$ is torsion. □

Proof of Theorem 1.11. Assume that there is a lift $\widehat{f} : B \to B\mathcal{G}/\operatorname{Map}_0(M, U(1))$. Lemma 3.8 and 3.9 show that $d_2(\tau) = 0$ and $d_3(\tau)$ is torsion.

Since all differentials of $[\omega]$ vanish, there exists a closed extension Ω' of ω. We shall define a new closed connection form $\Omega := \Omega' + \pi^*\beta$ so that it is integral (cf. Section 3 in [11]). Since $H_2(B, \mathbb{Z})$ is torsion-free then $H_2(E, \mathbb{Z}) = \pi_*(H_2(E, \mathbb{Z})) \oplus K$, where $K := \ker \pi_*$ as in Section 3.3. It follows from Proposition 3.5 that Ω' has integral periods on the cycles from K. We then define $\beta \in \Omega^2(B)$ to be any two form satisfying $\int_{\pi_*(C_i)} \beta = -\int_{C_i} \Omega'$, where $\{\pi_*(C_i)\}$ form a basis of the image $\pi_*(H_2(E, \mathbb{R}))$.

Finally, since E admits closed integral connection form it is an integral symplectic configuration, due to Theorem 1.9. Notice that the class of the above integral connection form might not restrict to τ, i.e. $\tau \neq i^*[\Omega] \in H^2(M, \mathbb{Z})$.

□

3.5. **Other results.** We conclude this section with several geometric results. The proofs are straightforward and are left to the reader.

Proposition 3.10.

(1) *The action of* $\operatorname{Symp}(W, \omega_W)$ *on* $\operatorname{Symp}(M, W)$ *given by the composition* $\varphi \cdot f := \varphi \circ f$ *preserves the connection* θ.
(2) *The induced action on the associated bundle* $(M, \omega) \to M_W \to \mathrm{B}_W$ *is given by* $\varphi \cdot [f, p] := [\varphi \circ f, p]$. *The evaluation map* $ev\colon M_W \to W$ *is* $\operatorname{Symp}(W, \omega_W)$*-equivariant.*

The general idea of this paper is to fix the first variable (M, ω) and investigate the functor $\mathrm{B}(M, \cdot)$ in order to get various informations about (M, ω). One can do the other way around and fix the second variable. This approach was taken by Haller-Vizman in [6].

They write explicitly a moment map for the action and express B_W as a coadjoint orbit in the dual Lie algebra of the group $\operatorname{Ham}(W, \omega_W)$.

Proposition 3.11 ([6])**.**

(1) *The fiber integral* $p_!(ev^*(\omega_W^{n+1})) \in \Omega^2(\mathrm{B}_W)$ *is a symplectic form on* B_W.
(2) *The action induced on* B_W *preserves the symplectic form*

$$p_!(ev^*(\omega_W^{n+1})).$$

4. Group cohomology and the group of McDuff

4.1. **Preparation on crossed homomorphisms.** Let G be a topological group and M be a topological G-module. All the modules we consider are right modules. A continuous map $\phi\colon G \to M$ is called a *continuous crossed homomorphism* or *continuous 1-cocycle* if $\phi(gh) = \phi(h) + \phi(g) \cdot h$.

Let S be a topological group, R be a topological S-module, and Z its submodule. Let $\phi\colon S \to R/Z$ be a crossed homomorphism. Define G to be the pull-back in the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & R \\ {\scriptstyle p}\downarrow & & \downarrow \\ S & \xrightarrow{\phi} & R/Z \end{array}$$

where vertical arrows are homomorphisms and Φ is a crossed homomorphism with respect to the action of S. In other words G is a subgroup of the semidirect product $R \rtimes S$ consisting of the pairs (s, r) such that $\phi(s) = r + Z$.

Lemma 4.1. *Let H be a kernel of ϕ. Then there exist a section $\sigma\colon H \to G$ (i.e. continuous homomorphism, such that $p \circ \sigma = id_H$).*

Proof. Consider a pull-back:

$$\begin{array}{ccc} G' & \xrightarrow{\Phi'} & R \\ {\scriptstyle p_H}\downarrow & & \downarrow \\ H & \xrightarrow{\phi_{|H}} & R/Z \end{array}$$

One observes that since $\phi_{|H}$ is trivial then G' is a semidirect product $Z \rtimes H$ and, in particular, there is a splitting $H \to G'$. Clearly, G' is a subgroup of G. Hence composing the splitting $H \to G'$ with the inclusion $G' < G$ we get the required section. □

Note that $\ker p = Z$ and consider the two (topological) fibrations:

$$Z \to G \xrightarrow{p} S,$$

and

$$H \to S \xrightarrow{\phi} R/Z.$$

Lemma 4.2. *The corresponding long exact sequences of homotopy groups are matched up as in the following commutative diagram:*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_{k+1}(S) & \longrightarrow & \pi_{k+1}(R/Z) & \longrightarrow & \pi_k(H) & \longrightarrow & \cdots \\ & & \| & & {\scriptstyle \partial_R}\downarrow & & {\scriptstyle \pi_k(\sigma)}\downarrow & & \\ \cdots & \longrightarrow & \pi_{k+1}(S) & \longrightarrow & \pi_k(Z) & \longrightarrow & \pi_k(G) & \longrightarrow & \cdots \end{array}$$

where ∂_R is the connecting homomorphism in the long exact sequence of the homotopy groups of the fibration $Z \to R \to R/Z$.

Proof. This is straightforward to check using standard diagram-chasing. □

Corollary 4.3. *If R is contractible then σ constructed above is a homotopy equivalence.*

Proof. If R is contractible then ∂_R is an isomorphism. Hence, by the five lemma, $\pi_k(\sigma)$ is also an isomorphism and the statement follows. □

4.2. **The cocycles and group of McDuff.** Let ω be a closed (possibly singular) 2-form on a manifold W. Let $SH_1(W; \mathbb{Z})$ be space of integral 1-cycles quotient out by boundaries of chains of zero ω-area. There is a split extension

$$0 \to \mathbb{R}/\omega \to SH_1(W, \mathbb{Z}) \xrightarrow{p} H_1(W, \mathbb{Z}) \to 0.$$

The existence of a section $s\colon H_1(W, \mathbb{Z}) \to SH_1(W, \mathbb{Z})$ follows since $\mathbb{R}/\omega$ is divisible. Precisely such a section may be constructed generator-by-generator. Assume that the section is defined on a submodule $V \subset H_1(W, \mathbb{Z})$, $v \in H_1(W, \mathbb{Z})$ and n is the least integer such that $nv \in V$. Let $\tilde{v}$ by any lift of v in $SH_1(W, \mathbb{Z})$. If $n = \infty$ we set $s(v) = \tilde{v}$. In the other case we see that $s(nv) - n\tilde{v} = [r] \in \mathbb{R}/\omega$. If $\gamma_{\frac{r}{n}}$ is a contractible loop bounding a disk of area r/n we set $s(v) = \tilde{v} + \gamma_{\frac{r}{n}}$.

Clearly, any group G that fixes ω acts on $SH_1(W,\mathbb{Z})$. In [11] McDuff defines a continuous crossed homomorphism $F_s\colon G \to H^1(M;\mathbb{R}/\omega)$ as follows

$$F_s(g)(\gamma) \colon = s(g_*(\gamma)) - g_*(s(\gamma)) \in \mathbb{R}/\omega = \ker\left(SH_1(W,\mathbb{Z}) \to H_1(W,\mathbb{Z})\right).$$

Straightforward computation shows that F_s satisfies

$$F_s(gh)(\gamma) = g_*(F_s(h)(\gamma)) + F_s(g)(h_*(\gamma)).$$

Since G acts trivially on $\mathbb{R}/\omega$ then $g_*(F_s(h)(\gamma)) = F_s(h)(\gamma)$. Hence

$$F_s(gh) = F_s(g) + (F_s(g)) \cdot h,$$

that is F_s is indeed a crossed-homomorphism.

We shall apply this construction in the following situation:

(1) $(W_1,\omega_1) = (L_\tau, \pi^*\omega)$ where L_τ is a prequantum $U(1)$-bundle with $[L] = \tau \in H^2(M,\mathbb{Z})$ and $G_1 = \mathcal{G}$.
(2) $(W_2,\omega_2) = (M,\omega)$ and $G_2 = \mathrm{Symp}(M,\widehat{\omega})$.

Lemma 4.4.

(1) *Since $\pi^*\omega$ is exact on L_τ then the group of periods is trivial i.e. $H^1(L_\tau,\mathbb{R}/\pi^*\omega) = H^1(M,\mathbb{R})$.*
(2) *One can choose sections $\tilde{s}\colon H_1(L_\tau,\mathbb{Z}) \to SH_1(L_\tau,\mathbb{Z})$ and $s\colon H_1(M,\mathbb{Z}) \to SH_1(M,\mathbb{Z})$ such that the following diagram commutes.*

$$\begin{array}{ccccc}
\mathbb{R} & \longrightarrow & SH_1(L_\tau,\mathbb{Z}) & \underset{}{\overset{\tilde{s}}{\rightleftarrows}} & H_1(L_\tau,\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow = \\
\mathbb{R}/\omega & \longrightarrow & SH_1(M,\mathbb{Z}) & \underset{}{\overset{s}{\rightleftarrows}} & H_1(M,\mathbb{Z})
\end{array}$$

(3) *If $\phi\colon \mathcal{G} \to \mathrm{Symp}(M,\widehat{\omega})$ assigns to a bundle map its underlying symplectomorphism then the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{G} & \xrightarrow{F_{\tilde{s}}} & H^1(M;\mathbb{R}) \\
{\scriptstyle\phi}\downarrow & & \downarrow \\
\mathrm{Symp}(M,\widehat{\omega}) & \xrightarrow{F_s} & H^1(M;\mathbb{R}/\omega)
\end{array}$$

(4) *The restriction of $F_{\tilde{s}}$ to $\mathrm{Map}_0(M,U(1))$ is trivial, and thus descends to a crossed homomorphism $\widehat{F_s}\colon \mathcal{G}/\mathrm{Map}_0(M,U(1)) \to H^1(M;\mathbb{R})$.*
(5) *The following diagram*

$$\begin{array}{ccc}
\mathcal{G}/\mathrm{Map}_0(M,U(1)) & \xrightarrow{\widehat{F_{\tilde{s}}}} & H^1(M;\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathrm{Symp}(M,\omega) & \xrightarrow{F_s} & H^1(M;\mathbb{R}/\mathbb{Z})
\end{array}$$

is a pull-back.

Proof. Most of the statements are trivial.
(2) First we have to define $\tilde{s}$ on $K = \ker H_1(L_\tau,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ in such a way that its image lives in $\ker SH_1(L_\tau,\mathbb{Z}) \to SH_1(M,\mathbb{Z})$. The generator of K is represented by a loop in a fiber, and the value of $\tilde{s}$ may be also be taken to be a loop in the fiber.

Then we extend $\tilde{s}$ to whole $H_1(L_\tau, \mathbb{Z})$ as explained above. It clearly descends to the map $s\colon H_1(M, \mathbb{Z}) \to SH_1(M, \mathbb{Z})$.
(4) This is obvious since $Map_0(M, U(1))$ is connected and the fiber of the map $H^1(M, \mathbb{R}) \to H^1(M, \mathbb{R}/\mathbb{Z})$ is discrete. □

Remark 4.5 (about Lemma 4.4 (2)). McDuff [11] shows that for each s one can choose a lift τ of $[\omega]$ and a section $\tilde{s}$ such that s is constructed as above.

McDuff proves that the restriction of F_s to $\mathrm{Symp}_0(M, \omega)$ is equal to the flux homomorphism. She defines $\mathrm{Ham}^{s\mathbb{Z}}(M, \omega) := \ker F_s$. Clearly, the component of the identity of $\mathrm{Ham}^{s\mathbb{Z}}(M, \omega)$ is equal to $\mathrm{Ham}(M, \omega)$.

4.3. **The groups $\mathrm{Ham}^{s\mathbb{Z}}$ and $\mathcal{D}$ are equal.** Let G be a topological group and $F\colon G_0 \to A$ a G-equivariant homomorphism to a right G-module A. That is it satisfies $F(f^{-1} g f) = F(g) \cdot f$. Let $H < G$ be a subgroup intersecting every connected component of G and such that $H \cap G_0 = \ker F$. Let $\sigma\colon G/G_0 = \pi_0(G) \to H$ be a set theoretic section of the natural projection $p\colon G \to G/G_0$. Define 1-cocycle $F^\sigma\colon G \to A$ by

$$F^\sigma(g) := F(\sigma(p(g))^{-1} g).$$

The following lemma that is a reformulation of Proposition 1.8(ii) from [11].

Lemma 4.6. *The 1-cocycle F^σ extends F and $H = \ker F^\sigma$.*

Proof. It is straightforward that F^σ extends F. To prove that F^σ is a 1-cocycle choose $f, g \in G$. Since $\sigma(G) \subset H$ then $h = \delta(\sigma \circ p)(f, g) = \sigma(p(fg))\sigma(p(g))^{-1}\sigma(p(f))^{-1} \in H$. We calculate

$$\begin{aligned} G_0 \ni \sigma(fg)^{-1} fg &= h\sigma(p(g))^{-1}\sigma(p(f))^{-1} fg \\ &= h\sigma(p(g))^{-1} g g^{-1}\sigma(p(f))^{-1} fg \\ &= h\left(\sigma(p(g))^{-1} g\right)\left(g^{-1}\sigma(p(f))^{-1} fg\right) \in hG_0. \end{aligned}$$

Thus $h \in G_0$ and finally $h \in \ker F = G_0 \cap H$. Then

$$\begin{aligned} F^\sigma(fg) &= F\left(\sigma(p(fg))^{-1} fg\right) \\ &= F(h) + F\left(\sigma(p(g))^{-1} g\right) + F\left(g^{-1}\sigma(p(f))^{-1} fg\right) \\ &= F^\sigma(g) + F^\sigma(f) \cdot g \end{aligned}$$

The proof that $H = \ker F^\sigma$ is also immediate and is left to the reader. □

Theorem 4.7.

(1) *There exists a continuous group homomorphism* $\mathrm{Ham}^{s\mathbb{Z}}(M, \omega) \to \mathcal{G}/\mathrm{Map}_0(M, U(1))$ *that is a homotopy equivalence.*
(2) *The group* $\mathrm{Ham}^{s\mathbb{Z}}(M, \omega)$ *and the holonomy group $\mathcal{D}$ are isomorphic.*

Proof. **(1)** Since $H^1(M; \mathbb{R})$ is contractible then the statement follows from Corollary 4.3.
(2) *Step 1.* We will show that $\mathrm{Ham}^{s\mathbb{Z}}(M, \omega) \subset \overline{\mathcal{D}}$. We have to observe that if $[l] \in H_1(M, \mathbb{Z})^\psi$ then $F_s(\psi)(l) = \mathrm{Flux}^\psi(\ell)$.
Step 2. By Proposition 3.7 (1) we conclude that $\mathrm{Ham}^{s\mathbb{Z}}(M, \omega) \subset \mathcal{D}$ (up to conjugacy).
Step 3. Let F be the following composition

$$\mathrm{Symp}_0(M, \widehat{\omega}) \overset{\mathrm{Flux}}{\to} H^1(M, \mathbb{R})/\Gamma_\omega \to H^1(M, \mathbb{R}/\mathbb{Z}).$$

From the definition $\mathrm{Ham}^{s\mathbb{Z}}(M,\omega)\cap\mathrm{Symp}_0(M,\widehat{\omega}) = \ker F$. Let σ be a section $\pi_0(\mathrm{Symp}(M,\widehat{\omega}))\to\mathrm{Ham}^{s\mathbb{Z}}(M,\omega)$. Then by Lemma 4.6 $\mathrm{Ham}^{s\mathbb{Z}}(M,\omega)=\ker F^\sigma$. By Theorem 1.7(3) $\mathcal{D}=\mathrm{Ham}(M,\omega)\cap\mathrm{Symp}_0(M,\widehat{\omega})=\ker F$. Since $\sigma(\pi_0(\mathrm{Symp}(M,\widehat{\omega})))\subset\mathrm{Ham}^{s\mathbb{Z}}(M,\omega)\subset\mathcal{D}$, by Lemma 4.6 we conclude that $\mathcal{D}=\ker F^\sigma$. □

4.4. **An obstruction associated to a crossed homomorphism.** Let V be an S-module, i.e. there is a homomorphism $\rho\colon S\to\mathrm{Aut}(V)=:A$. A crossed homomorphism $S\to V$ may be understood as a deformation of ρ. Precisely the map

$$(\psi,\rho)\colon S\to V\rtimes A=\widetilde{A}$$

is a homomorphism if and only if ψ is a crossed homomorphism.

A (homomorphic) section $s\colon A\to\widetilde{A}$ of the extension

$$0\to V\to\widetilde{A}\xrightarrow{p}A\to 0$$

defines an universal cocycle $\Psi^s\colon\widetilde{A}\to V$ by the formula $\Psi^s(a)=(sp(a))^{-1}a$. It is universal in the following sense. Given a homomorphism $h\colon S\to\widetilde{A}$ one defines a crossed homomorphism $\psi^s_h\colon S\to V$ by the composition $\psi^s_h=\Psi^s\circ h$ associated to the action $p\circ h\colon S\to A$. Different choice of a section $s\colon A\to\widetilde{A}$ changes Ψ^s by a coboundary.

The kernel $H:=\ker\psi$ is a pull-back of the following diagram:

$$\begin{array}{ccc} H & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle s} \\ S & \xrightarrow{h} & \widetilde{A} \end{array}$$

Where $h=(\psi,\rho)$. Thus BH is a homotopy pull-back of the following diagram

$$\begin{array}{ccc} BH & \longrightarrow & BA \\ \downarrow & & \downarrow{\scriptstyle Bs} \\ BS & \xrightarrow{Bh} & B\widetilde{A} \end{array}$$

The homotopy kernel of the map $BA\to B\widetilde{A}$ equals to V, moreover the fibration

$$V\to BA=V\rtimes_{\widetilde{A}}E\widetilde{A}\to B\widetilde{A}$$

is the universal V-bundle over $B\widetilde{A}$. Note that V is an affine $\widetilde{A}$-space and it is an A-torsor.

In particular the obstruction to the existence of a section $BS\to BH$ is a pull-back of the obstruction to the existence of a section $B\widetilde{A}\to BA$.

In our set-up $S=\mathrm{Symp}(M,\widehat{\omega})$ and $V=H^1(M,\mathbb{R})/H^1(M,\mathbb{Z})$ and, in particular, $V=K(H^1(M,\mathbb{Z}),1)$. Thus the total obstruction class $\mathcal{O}_V$ to the existence of the lift $B\widetilde{A}\to BA$ is the element of $H^2(B\widetilde{A},\pi_1(V))=H^2(B\widetilde{A},H^1(M,\mathbb{Z}))$.

Theorem 4.8 (McDuff)**.** *There exists a class $\mathcal{O}_M\in H^2(B\,\mathrm{Symp}(M,\widehat{\omega}),H^1(M,\mathbb{Z}))$ with the following property. A $\mathrm{Symp}(M,\widehat{\omega})$-bundle $M\to P\to B$ admits a reduction to $\mathcal{D}_\tau$ if and only if $f^*\mathcal{O}=0\in H^2(B,H^1(M,\mathbb{Z}))$, where $f\colon B\to B\,\mathrm{Symp}(M,\widehat{\omega})$ is the classifying map for the bundle P.*

Proof. We set $\mathcal{O}_M=(Bh)^*\mathcal{O}_V$ where $h=(\rho,F^s)$, ρ is an action of $\mathrm{Symp}(M,\widehat{\omega})$ on $H^1(M,\mathbb{Z})$ and F^s is the cocycle of McDuff. □

In 4.1 we have described the Bockstein map β which, for given short exact sequence of S-modules

$$0 \to Z \to R \to R/Z \to 0,$$

to every (cohomology class of a) crossed homomorphism $S \to R/Z$ associates an extension $0 \to Z \to G \to S \to 0$. If Z is discrete, then such an extension is classified by an element $\sigma_G \in H^2(BS, Z)$ (cf. [14]).

Corollary 4.9. $\mathcal{O}_M = \sigma_{\mathcal{G}/\operatorname{Map}_0(M,U(1))} = \beta([F^s])$.

Proof. Recall that Ham^s and $\mathcal{G}/\operatorname{Map}_0(M, U(1))$ are homotopy equivariant, thus the obstruction to the existence of a section $\operatorname{Ham}^s(M,\omega) \to \operatorname{Symp}(M, \widehat{\omega})$ equals the the obstruction to the existence of a section $\mathcal{G}/\operatorname{Map}_0(M, U(1)) \to \operatorname{Symp}(M, \widehat{\omega})$.

The second equality follows directly from the definition of β. □

5. Cohomology ring of symplectic configurations

5.1. **Characteristic classes of configurations.** We define a characteristic class of configurations of (M, ω) in (W, ω_W) to be an element of the cohomology ring $H^*(\mathrm{B}_W)$. If M compact then certain characteristic classes can be obtained as fiber integrals

$$\chi_k^W := p_!(ev^*([\omega_W^{n+k}])).$$

These classes are natural in the sense that if $\psi\colon (W, \omega_W) \to (V, \omega_V)$ is a symplectic embedding then $\chi_k^W = \Psi^* \chi_k^V$, where $\Psi\colon \mathrm{B}_W \to \mathrm{B}_V$ is the map induced by ψ.

Since any integral symplectic manifold embeds into $\mathbb{CP}^N$ then we define universal characteristic classes with respect to $\mathbb{CP}^\infty$. We denote the above fiber integrals by $\chi_k \in H^{2k}(\mathrm{B})$. In this case those are usual characteristic classes of a fibration with the structure group $\mathcal{G}$.

Fundamental question is then if these classes are *symplectic* that is if they come from $H^*(B\operatorname{Symp})$ via the canonical map $\mathrm{B} \to B\operatorname{Symp}$. In general the answer is negative.

Lemma 5.1. *Let $\sigma \in \pi_2(\mathrm{B})$ be an element coming from $H^0(M; \mathbb{Z})$ (see Theorem 1.5). Then $\langle \chi_1, \sigma \rangle \neq 0$. In particular, χ_1 is not symplectic class. Moreover, χ_1 is of infinite order in $H^*(\mathrm{B})$.*

Proof. Recall that σ is represented by a configuration $M \times S^2 \to \mathbb{CP}^\infty$ such that the S^2 summand is represents the positive generator of $\pi_2(\mathbb{CP}^\infty)$. Clearly, $ev^*(\omega_\infty) = \omega \oplus \omega_{S^2}$ and hence $\pi_!(ev^*(\omega_\infty^{n+1})) = (n+1)\,\omega_{S^2}$. Since the configuration is a trivial fibration then no symplectic characteristic class can be nontrivial.

In order to prove the last statement, consider a configuration $P := M \times (S^2 \times \cdots \times S^2) \to \mathbb{CP}^\infty$ such that every S^2 is as above. Clearly, $\chi_1(P)$ is a sum of generators and its top power is nonzero. Since it works for any number of factors S^2, then it follows that χ_1 is of infinite order. □

Let $(M, \omega) \xrightarrow{i} M_{\operatorname{Ham}(M,\omega)} \xrightarrow{p} \operatorname{BHam}(M, \omega)$ be the universal Hamiltonian fibration. Recall that the *coupling class* $\Omega \in H^2(M_{\operatorname{Ham}(M,\omega)})$ is defined by the following two conditions:

(1) $i^*(\Omega) = [\omega]$
(2) $p_!(\Omega^{n+1}) = 0$

One can define characteristic classes by $\mu_k := p_!(\Omega^{n+k}) \in H^{2k}(\mathrm{BHam}(M,\omega))$ [7, 8].

Let $\widetilde{\mathrm{B}} := \mathrm{Symp}(M, \mathbb{CP}^\infty)/\mathrm{Ham}(M,\omega)$ and let $(M,\omega) \xrightarrow{i} M_{\widetilde{\mathrm{B}}} \xrightarrow{p} \widetilde{\mathrm{B}}$ be the associated bundle. Notice that $\widetilde{\mathrm{B}} \to \mathrm{B}$ is the universal cover. We have the following commutative diagram of classifying spaces:

$$\begin{array}{ccc} \widetilde{\mathrm{B}} & \longrightarrow & \mathrm{B} \\ \downarrow & & \downarrow \\ \mathrm{BHam}(M,\omega) & \longrightarrow & \mathrm{BSymp}(M,\omega) \end{array}$$

We want to compare the pull-backs of characteristic classes χ_k and μ_k in $H^*(\widetilde{\mathrm{B}})$. To avoid clumsy notation we will denote these pull-backs by the same symbols as the original classes. The guiding idea is that the χ_k classes might be easier to calculate than μ_k's.

Lemma 5.2. *Let $(M,\omega) \xrightarrow{i} M_{\widetilde{\mathrm{B}}} \xrightarrow{p} \widetilde{\mathrm{B}}$ be the above associated bundle. Then $\chi_k \cong \mu_k$ modulo an ideal in $H^*(\widetilde{\mathrm{B}})$ generated by χ_1.*

Proof. First observe that the coupling class $\Omega = ev^*(\omega_0) - \frac{1}{n+1}p^*(\chi_1)$. Calculating the appropriate fiber integral we get the statement.

$$\begin{aligned} \chi_k &= p_!(ev^*(\omega_0^{n+k})) \\ &= p_!\left((\Omega + \frac{1}{n+1}p^*(\chi_1))^{n+k}\right) \\ &= p_!\left(\sum_{i=0}^{n+k}\binom{n+k}{i}\Omega^{n+k-i}\left(\frac{1}{n+1}p^*(\chi_1)\right)^i\right) \\ &= \sum_{i=0}^{n+k}\binom{n+k}{i}\left(\frac{1}{n+1}\right)^i p_!(\Omega^{n+k-i})\ (\chi_1)^i \\ &= \sum_{i=0}^{k}\binom{n+k}{i}\left(\frac{1}{n+1}\right)^i \mu_{k-i}\ (\chi_1)^i \end{aligned}$$

□

Corollary 5.3. *Let $(M,\omega) \to P \to B$ be a configuration whose classifying map admits a lift to $\widetilde{\mathrm{B}}$. If $H^2(B) = 0$ then $\chi_k(P) = \mu_k(P)$. In particular, if $\chi_k(P)$ is nonzero then so does $\mu_k(P)$.* □

5.2. **Calculating χ-classes.** It is important to know if there are relations between characteristic classes. If $(M,\omega) \to P \to S^{2k}$ is a symplectic configuration with nontrivial $\chi_k(P)$ then χ_k is not a product in $H^*(\mathrm{B})$. Indeed, if it were a product then its pull-back $\chi_k(P) \in H^{2k}(S^{2k})$ would vanish.

Theorem 5.4. *Let $(M,\omega) \to P \to S^{2k}$ be a symplectic configuration, $k > 1$. The following conditions are equivalent.*

(1) $\chi_k(P) \neq 0$;
(2) $\mu_k(P) \neq 0$;
(3) $ev_*[P] \neq 0$ *in* $H_{2n+2k}(\mathbb{CP}^\infty;\mathbb{Z})$;

(4) *There exists* $a \in H^2(P)$ *extending* $[\omega]$ *such that* $a^{n+k} \neq 0$*; in particular* P *is c-symplectic.*

Moreover, any of these conditions implies that $k \leq n+1$.

Proof. The first two conditions are equivalent, according to Corollary 5.3. Each of them is equivalent to the third by the basic properties of fiber integration:

$$\left\langle \chi_k(P), [S^{2k}] \right\rangle = \left\langle ev^*(\omega_\infty^{n+k}), [P] \right\rangle = \left\langle \omega_\infty^{n+k}, ev_*[P] \right\rangle.$$

The third implies the fourth since the c-symplectic class is given by $[ev^*(\omega_\infty)]$. The converse implication follows from Theorem 1.9.

For the last statement, observe that $H^{2n+2}(P) = 0$ if $k > n+1$. Hence P cannot be c-symplectic. □

Remark 5.5. Consider the question of finding a large $k \in \mathbb{Z}$ such that the Hurewicz map $\pi_{2k}(\mathrm{BSymp}(M,\omega)) \to H_{2k}(\mathrm{BSymp}(M,\omega);\mathbb{Q})$ is nontrivial. The characteristic classes constructed as fiber integrals give quite restricted method for answering this question. As mentioned in the above theorem one can detect homologically nontrivial spheres up to dimension $\dim M + 2$.

5.3. **Examples.**

(1) Consider the symplectic fibration $\mathbb{CP}^1 \to \mathbb{CP}^3 \to S^4 = \mathbb{HP}^1$. The total space is symplectic hence, due to Theorem 5.4, we get that $\mu_2 = \chi_2 \neq 0$ in $H^4(\mathrm{B}(S^2))$.

(2) Moreover, take a symplectic fibration $F \to P \to \mathbb{CP}^3$ such that P admits a compatible symplectic structure. Composing with the above we get another symplectic fibration

$$(M,\omega) \to P \to S^4,$$

whose fiber is the total space of $F \to (M,\omega) \to \mathbb{CP}^1$. Again, we obtain that $\chi_2 = \mu_2 \neq 0$ in $H^4(\mathrm{B}(M,\omega))$.

(3) Similarly, consider the symplectic fibration

$$\mathbb{CP}^3 \to \mathbb{CP}^7 \to S^8 = \mathbb{C}\mathrm{a}\mathbb{P}.$$

Here, we get that χ_4 and μ_4 are nonzero for $\mathbb{CP}^3$.

(4) Taking a symplectic fibration over $\mathbb{CP}^7$ admitting a compatible symplectic structure we get a result analogous to the one in (2).

(5) Let $(M,\omega) := U(m)/U(m_1) \times \cdots \times U(m_l)$, $m = m_1 + \cdots + m_l$ and $m_1 \geq m_2 \geq \cdots \geq m_l$ be a generalized flag manifold equipped with the homogeneous symplectic form. It is proved in Kędra-McDuff [8] that the classes $\mu_k \in H^{2k}(\mathrm{BSymp}_0(M,\omega))$ are nonzero for $1 < k \leq m$. Moreover, it is shown that they are nontrivial for certain fibrations over spheres, say $(M,\omega) \to P \to S^{2k}$, with the structure group $SU(m)$. According to Theorem 5.4, P is c-symplectic, however, it is not clear if it admits any symplectic form in general.

As in the previous examples we consider a symplectic fibration $F \to E \to P$ and compose it with $P \to S^{2k}$. Suppose that this composition is a symplectic fibration with fiber $F \to Q \to M$. Then the classes $\mu_k(E) \in H^{2k}(\mathrm{BSymp}_0(Q))$ are nonzero.

5.4. **Chern-Weil approach.** Recall from Section 3.2 that the universal configuration fibration admits a connection with curvature form Θ. It is an *ad*-invariant two form on the total space $\mathrm{Symp}(M, \mathbb{CP}^\infty)$ of the universal principal fibration with values in the Lie algebra of $\mathrm{Symp}(M, \omega)$ which is identified with closed 1-forms on M. Since the reduced holonomy group is Hamiltonian (Theorem 1.7 (1)) then the curvature form has values in the Lie algebra of $\mathrm{Ham}(M, \omega)$ which is identified with $C_0^\infty(M)$, the space of functions on M with zero mean value.

Let $\mathcal{P}\colon C_0^\infty(M) \to \mathbb{R}$ be an invariant polynomial. It means that $\mathcal{P}(h_1 \circ \psi, ..., h_k \circ \psi) = \mathcal{P}(h_1, ..., h_k)$, where $\psi \in \mathrm{Symp}(M, \omega)$ and $h_i \in C_0^\infty(M)$. Then

$$\chi^{\mathcal{P}}_{[f]}(X_1, Y_1, \dots, X_k, Y_k) \colon = \mathcal{P}(\Theta_f(X_1^h, Y_1^h), \dots, \Theta_f(X_k^h, Y_k^h))$$

defines a $2k$-form on the base B. The definition does not depend on the choice of f in the fiber since the polynomial $\mathcal{P}$ is invariant. It is a standard fact that these forms are closed [10]. We define universal characteristic classes to be cohomology classes of the forms $\chi^{\mathcal{P}}$.

The following formula gives a sequence of invariant polynomials.

$$\mathcal{P}_k(h_1, \dots, h_k) \colon = \int_M h_1 \dots h_k \, \omega^n$$

The proof of the next lemma is analogous to the proof of Lemma 3.9 in Kędra-McDuff [8].

Lemma 5.6. $\left[\chi^{\mathcal{P}_k}\right] = \mathrm{const}\ \chi_k$ *for* $k > 1$. □

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