ELSEVIER

Discrete Applied Mathematics 74 (1997) 171–181

DISCRETE APPLIED MATHEMATICS

Bounds on the number of knight's tours

Olaf Kyek[a], Ian Parberry[b], Ingo Wegener[a,*]

[a] FB Informatik, LS II, Univ. Dortmund, 44221 Dortmund, Germany

[b] Dept. of Computer Science, Univ. of North Texas, P.O. Box 13886, Denton, TX 76203-3886, USA

Received 29 November 1994; revised 2 January 1996

Abstract

Knight's tours are a fascinating subject. New lower bounds on the number of knight's tours and structured knight's tours on $n \times n$ chessboards and even n are presented. For the natural special case $n = 8$ a new upper bound is proved.

Keywords: Hamiltonian paths; Counting; Graph theory

1. Introduction

Knight's tours on chessboards are a subject considered already by Euler [2], Legendre [4], Minding [5], Roget [7], Tait [8], Vandermonde [9], Warnsdorff [10] and many others. In graph theoretical notion a knight's tour is a Hamiltonian circuit on the graph whose vertices are the squares of an $n \times n$ chessboard and whose edges represent the legal moves of a knight. It is known that a knight's tour exists iff n is even and $n \geqslant 6$. It is also known that a Hamiltonian path from a square s to a square t exists for $n \geqslant 6$ iff n is even and s and t have different colors or n is odd and s and t are colored white (we assume that the corners are colored white). This result has been proved by Conrad et al. [1]. Their paper contains also efficient algorithms for the construction of knight's tours and Hamiltonian paths. Different approaches to construct knight's tours are discussed by Parberry [6].

A still challenging question is the problem to determine or to estimate the number of knight's tours on $n \times n$ chessboards and even n. This number equals 0 for $n = 2$ and $n = 4$ and it equals 9862 for $n = 6$. For $n = 8$ the best lower bound we have found is due to Kraitschik [3] and equals 122 802 512. In Section 4 the best known upper bound for $n = 8$, namely 3.019×10^{22}, is presented.

Perhaps one will be able to determine the number of knight's tours for $n = 8$ but it seems to be impossible to do this for general n. Hence, we investigate the asymptotic

* Corresponding author. E-mail: wegener@ls2.informatik.uni-dortmund.de.

PII S0166-218X(96)00031-5

behavior of these numbers. It follows from the results in Section 4 that the number $\mathscr{T}_n$ of knight's tours (for even n) is bounded above by 4^{n^2}. It was known that $\mathscr{T}_n \geqslant (1+\varepsilon)^{n^2}$ for some $\varepsilon > 0$ [1]. In Section 3 it is proved that $\mathscr{T}_n = \Omega(1.3535^{n^2})$. In Section 2 we estimate the number of more structured knight's tours and prove that their number $\mathscr{S}_n$ grows already exponentially, namely $\mathscr{S}_n = \Omega(1.2862^{n^2})$.

We do not claim that our bounds are optimal (in fact, we are sure that they are not optimal). But the bounds are much better than the previous known ones. Moreover, our methods might be interesting, since we combine combinatorial methods and results obtained with the help of a computer.

2. On the asymptotic number of structured knight's tours

For our first lower bound on the number of (structured) knight's tours we will simplify and extend the technique first described in Parberry [6]. Firstly, we define a closed knight's tour to be *structured* if it has the moves depicted in Fig. 1.

Theorem 1. *For all even $n \geqslant 6$ there exists a structured knight's tour on an $n \times n$ and an $n \times (n+2)$ board.*

Proof. The proof is by induction on n. The claim is easily seen to be true for $6 \leqslant n \leqslant 10$ by inspecting Fig. 2 (the knight's tours in this figure were obtained using a random walk algorithm described in Parberry [6]).

Now suppose that $n \geqslant 12$ is even and that structured knight's tours exist on $m \times m$ and $m \times (m+2)$ boards for all even $6 \leqslant m < n$. Start by dividing the $n \times n$ board into four quadrants as evenly as possible and placing knight's tours in each quadrant. More precisely, each side of length $n = 4k$ for some $k \in \mathbb{N}$ is divided into two parts of length $2k$, and each side of length $4k+2$ for some $k \in \mathbb{N}$ is divided into a part of length $2k$ and a part of length $2(k+1)$. Note that size $n \geqslant 12$ implies that $2k \geqslant 6$.

In the construction of an $n \times n$ board in which $n = 4k$ for some $k \in \mathbb{N}$, the four quadrants are each $2k \times 2k$ and have tours by the induction hypothesis. Alternatively,

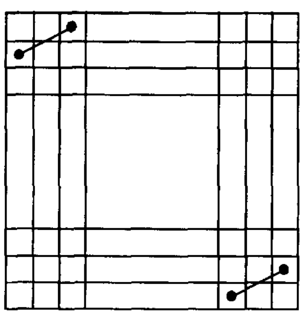

Fig. 1. Required moves for a structured knight's tour.

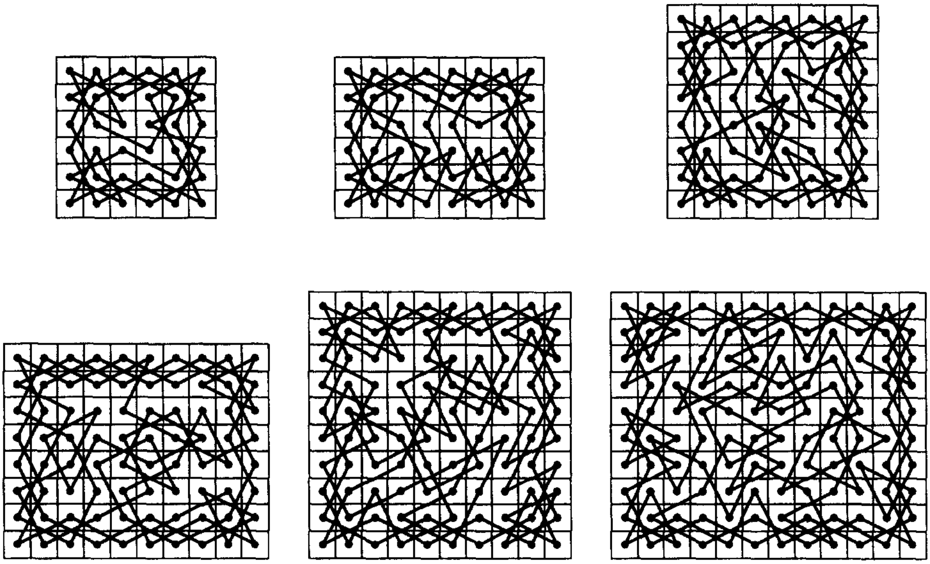

Fig. 2. Structured knight's tours for (in row-major order) 6×6, 6×8, 8×8, 8×10, 10×10, and 10×12 boards.

if $n = 4k + 2$ for some $k \in \mathbb{N}$, then the quadrants on the diagonal are $2k \times 2k$ and $2(k+1) \times 2(k+1)$, and the off-diagonal quadrants are $2k \times 2(k+1)$ and $2(k+1) \times 2k$. The former pair of quadrants must be structured, and have tours by the induction hypothesis. The latter pair of quadrants need not be structured; the $2k \times 2(k+1)$ quadrant has a tour by the induction hypothesis and the $2(k+1) \times 2k$ tour is obtained by rotating it through 90° (it is important that this quadrant does not need to be structured, since the rotation destroys the structure). The construction of an $n \times (n+2)$ board is similar and actually slightly easier, and so is left for the reader.

The moves at the inside corners of the quadrants are illustrated in Fig. 3(a). (Although the moves from the corner square were not specified in Fig. 3, note that there are no other choices for knight's moves out of a corner square.) The four tours are combined by deleting the edges A, B, C, D shown in Fig. 3(b) and replacing them with the four edges E, F, G, H shown in Fig. 3(c). Clearly, the result is a structured knight's tour. □

Fig. 4 illustrates the technique of Theorem 1 on a 12×12 board, constructed from four copies of the knight's tour on a 6×6 board in Fig. 2. Fig. 5 illustrates the technique on a 30×30 board.

Let $\mathcal{T}_{n,m}$ and $\mathcal{S}_{n,m}$ respectively denote the number of distinct closed knight's tours, and the number of distinct structured knight's tours on an $n \times m$ board. We will use $\mathcal{T}_n$ and $\mathcal{S}_n$ respectively as abbreviations for $\mathcal{T}_{n,n}$ and $\mathcal{S}_{n,n}$. Lower bounds for $\mathcal{S}_{n,m}$ for small values of n and m can be found using simple search techniques described in Parberry [6]. The exact values for $\mathcal{S}_6$ and $\mathcal{T}_6 = 9862$ can easily be obtained using a

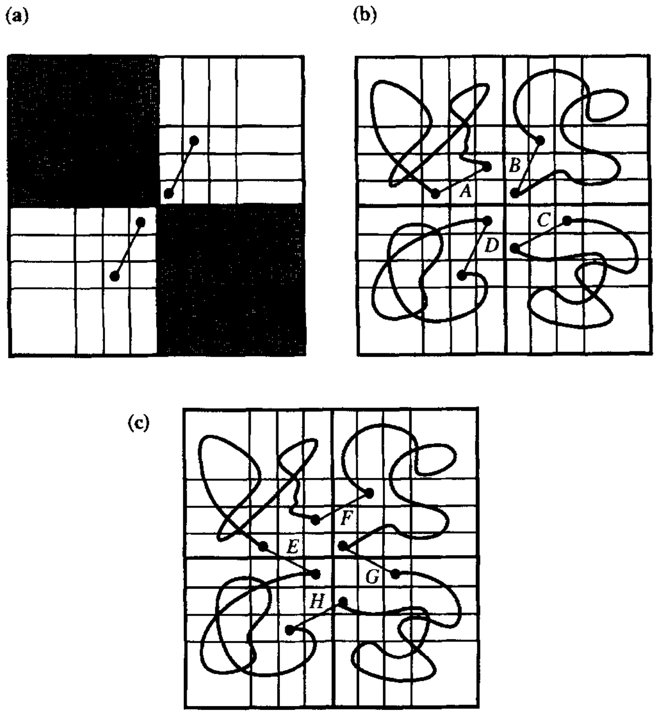

Fig. 3. How to combine two structured knight's tours and two arbitrary closed knight's tours into one: (a) the moves at the inside corners (the structured subtours are shaded), (b) the edges A, B, C, D to be deleted, and (c) the replacement edges E, F, G, H.

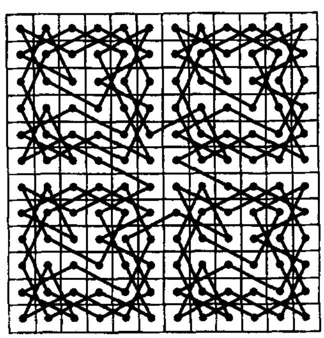

Fig. 4. A 12×12 knight's tour constructed from the 6×6 knight's tour in Fig. 2 using the technique of Theorem 1.

straightforward backtracking algorithm. The results are shown in Table 1. We can deduce very quickly from the recursive construction described above that $\mathcal{T}_{12} \geqslant \mathcal{S}_{12} \geqslant \mathcal{S}_6^2 \cdot \mathcal{T}_6^2 = 7532^2 \cdot 9862^2 > 5.5 \times 10^{17}$. A similar argument can also be used to show a general lower bound on $\mathcal{S}_n$, as follows.

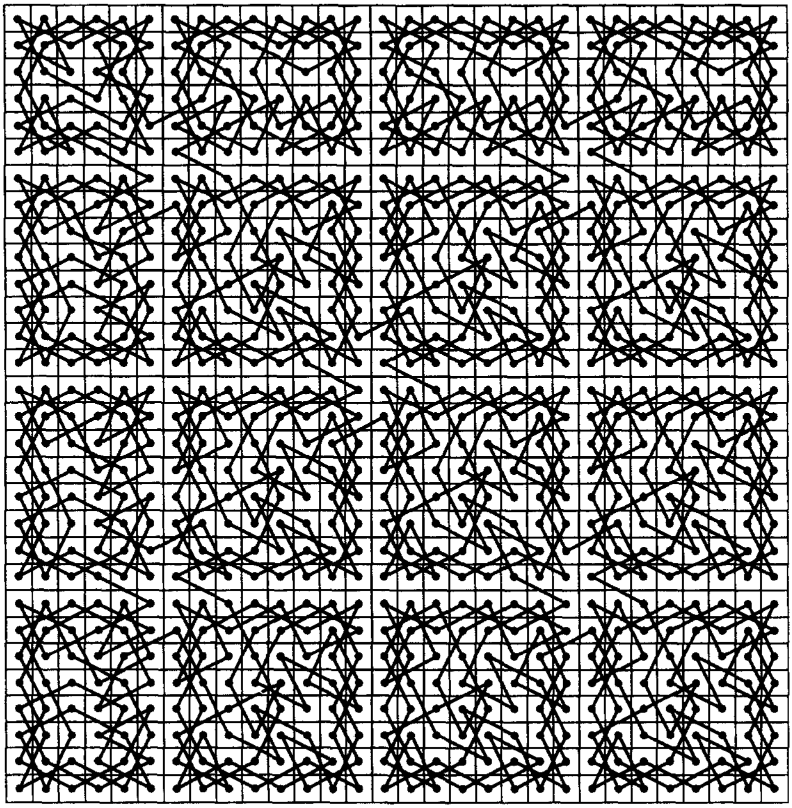

Fig. 5. A 30×30 knight's tour constructed from the smaller knight's tours in Fig. 2 using the technique of Theorem 1.

Table 1
Lower bounds for $\mathscr{S}_{n,m}$, the number of structured knight's tours on an $n \times m$ board. The value of $\mathscr{S}_{6,6}$ is exact

n	m	$\mathscr{S}_{n,m}$
6	6	7 532
6	8	58 012
8	8	930 753
8	10	2 607 905
10	10	5 944 191
10	12	6 587 711

Theorem 2. *For all* $n \geqslant 12$, $\mathscr{S}_n \geqslant 1.1646^{n^2}$.

Proof. The algorithm described in the proof of Theorem 1 constructs a tour on an $n \times n$ board from copies of tours with dimensions drawn from the base set: $6 \times 6, 6 \times 8, 8 \times 8, 8 \times 10, 10 \times 10$, and 10×12. That is, both sides of the board are divided into segments of length i and $i+2$ for some $i \in \{6,8,10\}$. Suppose that there are $\alpha_i n$ segments of length i, for some $0 \leqslant \alpha_i \leqslant 1$. Then, there are

$n^2\alpha_i^2$ subtours of dimension $i \times i$,

$2n^2\alpha_i(1-\alpha_i i)/(i+2)$ subtours of dimension $i \times (i+2)$, and

$n^2(1-\alpha_i i)^2/(i+2)^2$ subtours of dimension $(i+2) \times (i+2)$.

Suppose we know some lower bound $\mathscr{B}_{i,j}$ for all $\mathscr{S}_{i,j}$ with $j \in \{i, i+2\}$, $i \in \{6,8,10\}$ (such as those contained in Table 1). Then, the total number of tours is at least

$$(\mathscr{B}_{i,i}^{\alpha_i^2}\mathscr{B}_{i,i+2}^{2\alpha_i(1-\alpha_i i)/(i+2)}\mathscr{B}_{i+2,i+2}^{(1-\alpha_i i)^2/(i+2)^2})^{n^2}.$$

This is minimized when the function

$$f(\alpha_i)=\alpha_i^2(\ln\mathscr{B}_{i,i})+2\alpha_i(1-\alpha_i i)(\ln\mathscr{B}_{i,i+2})/(i+2)+(1-\alpha_i i)^2(\ln\mathscr{B}_{i+2,i+2})/(i+2)^2$$

is minimized, under the conditions $0 \leqslant \alpha_i \leqslant 1$. The unconditional minimum of f occurs when $\alpha_i = \hat{\alpha}_i$, where

$$\hat{\alpha}_i = \frac{2i(\ln\mathscr{B}_{i+2,i+2}) - (i+2)(\ln\mathscr{B}_{i,i+2})}{2((i+2)^2(\ln\mathscr{B}_{i,i}) - i(i+2)(\ln\mathscr{B}_{i,i+2}) + i^2(\ln\mathscr{B}_{i+2,i+2}))}.$$

Let $\beta_i = f(\hat{\alpha}_i)$. Then, $\mathscr{S}_{n,n} \geqslant e^{\beta_i n^2}$, where e is the base of the natural logarithm. A simple case analysis shows that $e^{\beta_6} > e^{\beta_8} > e^{\beta_{10}} > 1.1646$, and hence $\mathscr{S}_{n,n} \geqslant 1.1646^{n^2}$. □

A stronger asymptotic lower bound can be proved as follows. We say that $f(n) = \Omega(g(n))$ if there exist a positive constant c and infinitely many values of n such that $f(n) \geqslant c \cdot g(n)$.

Theorem 3. $\mathscr{S}_n = \Omega(1.2862^{n^2})$.

Proof. Suppose n has the form $6 \cdot 2^k$ for some $k \in \mathbb{N}$. It can be proved by induction that the base of the recursive construction of Theorem 2 consists of only the 6×6 tour. Furthermore, it can also be proved by induction that exactly half of the $n^2/36 = 4^k$ copies of the base tour need be structured, and the rest can be arbitrary tours. Since there are $\mathscr{S}_6 = 7532$ structured 6×6 tours and $\mathscr{T}_6 = 9862$ arbitrary 6×6 tours, we conclude that for infinitely many values of n,

$$\mathscr{S}_n \geqslant 9862^{n^2/72} \cdot 7532^{n^2/72} \geqslant 1.2862^{n^2}. \quad \square$$

3. On the asymptotic number of knight's tours

We again use a divide-and-conquer approach for the construction of different knight's tours. The $n\times n$ chessboard is divided into boards of size $6\times6, 6\times8, 8\times6$ and 8×8. Two subboards either have no common border or they share one total border of their boards. For some $\alpha \in [0, 1/6]$ chosen later we obtain $\alpha^2 n^2$ boards of size 6×6, $\frac{1}{8}\alpha(1 - 6\alpha)n^2$ boards of size 6×8 and also of size 8×6, and $\frac{1}{64}(1 - 6\alpha)^2 n^2$ boards of size 8×8.

Our tours start in the upper right square of the upper left subboard. We take a Hamiltonian path through this board such that we can then jump to the board below. Then we run through all the boards using in each board a Hamiltonian path. The last board has to be the board right to the starting board. In this board we have to use a Hamiltonian path ending at a square such that we can finish the tour by jumping back to the first square.

We have some choices to obtain different knight's tours:

- We may choose different values for α. Being precise, α has to be chosen in a way that $(1 - 6\alpha)n$ is divisible by 8. Hence, the number of different α is $\Theta(n)$. Since we are not interested in polynomial factors, it is sufficient to consider that α which gives the best lower bound.
- We may choose different arrangements of the subboards of different size, i.e. we may partition the rows (and columns) in different ways into segments of length 6 and 8. This gives a factor bounded above by 2^n. Since we prove lower bounds whose exponents are of size n^2, it is sufficient to consider a fixed arrangement.
- We may choose different orders of the subboards.
- We may choose different Hamiltonian paths within the subboards.

Now we consider the third problem. Let $r := \alpha n + (1/8)(1 - 6\alpha)n$ be the parameter such that the board is partitioned to r^2 subboards. A knight can reach from one subboard directly only one of the neighbored boards. With respect to the subboards the knight moves like a king on a usual board. Hence, we look at king's tours on an $r \times r$ board. In order to make possible more Hamiltonian paths within the subboards we consider only king's tours consisting of horizontal and vertical moves and not of diagonal moves. For this purpose we partition the $r \times r$ board into $r^2/49$ boards of size 7×7. With the help of the computer we have proved there are at least $L = 11\,600$ king's tours on a 7×7 board starting at some given square in the first row and ending at an arbitrary square in the last row. The same holds, if we like to end at an arbitrary square in the first (last) column. Using different king's tours with respect to the subboards of size 7×7 we obtain an additional factor of $11\,600^{r^2/49}$ for the number of knight's tours on the $n \times n$ board. (Since we are interested in asymptotic bounds we do not care whether r is a multiple of 7 and the special situation between the first and last two subboards.)

The more important part is the number of different Hamiltonian paths within a subboard. Let us consider an 8×8 board (see Fig. 6) which is reached at one of the positions b8, d8 or f8. If we want to reach the corresponding positions in the neighbored board below, we can choose any Hamiltonian path ending in one of the positions a2, b1, c2, d1, e2, f1, g2 and h1. If we like to leave the board to the left (right), it is sufficient that the Hamiltonian path ends at one of the 8 white squares in the first (last) two columns of the board. With the help of a computer it has been shown that for each of the squares b8, d8 and f8 and each of the choices of the next board left neighbor,

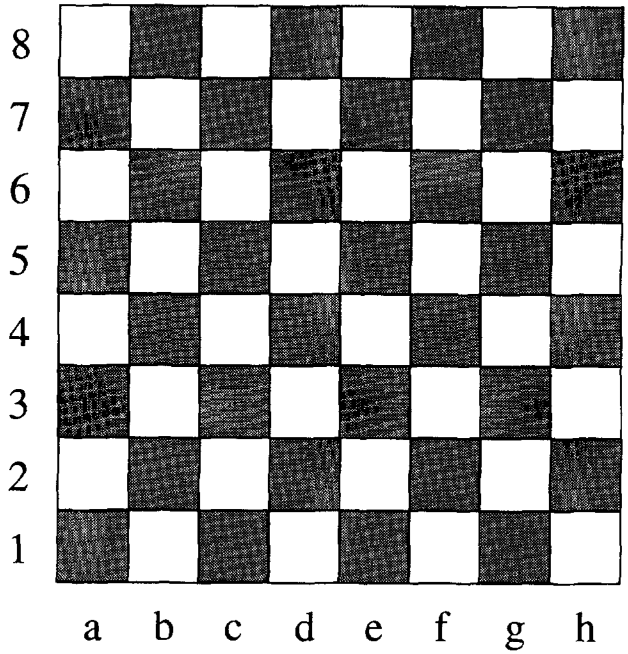

Fig. 6. 8×8 chessboard

lower neighbor and right neighbor there exist at least $M_{8,8} = 19\,610\,000$ Hamiltonian paths with the desired properties. Similar arguments for $6 \times 6, 6 \times 8$ and 8×6 boards lead to $M_{6,6} = 44\,670$ and $M_{8,6} = M_{6,8} = 1\,800\,000$.

Our lower bound for the number of knight's tours equals asymptotically

$$L^{(1/49)(\alpha n+(1/8)(1-6\alpha)n)^2} M_{6,6}^{\alpha^2 n^2} M_{6,8}^{(1/4)\alpha(1-6\alpha)n^2} M_{8,8}^{(1/64)(1-6\alpha)^2 n^2}.$$

It turns out that this bound takes its maximal value for $\alpha = \frac{1}{6}$ (remember that $0 \leqslant \alpha \leqslant \frac{1}{6}$). The partition into 6×6 boards gives the best lower bound, since $M_{6,6}$ is almost the precise value for 6×6 boards while the bounds $M_{6,8}$ and, in particular, $M_{8,8}$ are estimates which may be improved using much more CPU time. The lower bound for $\alpha = \frac{1}{6}$ equals asymptotically

$$(L^{1/1764} M_{6,6}^{1/36})^{n^2}.$$

Theorem 4. *The number of knight's tours on $n \times n$ boards is $\Omega(1.3535^{n^2})$.*

4. On the number of knight's tours on 8×8 chessboards

The number of knight's tours on 8×8 chessboards is a well-defined number. It can be computed in principle by enumerating all knight's tours. But this approach is not practical, since most probably the number of knight's tours is too large. Moreover, we cannot avoid to run in too many deadlocks. Already the best known lower bound of Kraitschik [3] is proved by concatenating Hamiltonian paths on smaller boards. This leads to the conjecture that the number of knight's tours is much larger than 10^9.

Our upper bounds are based on the following lemma.

Lemma 5. *Let G be an undirected graph on n vertices with m edges. Let $k \in \mathbb{N}$ be chosen such that*

$$(n-1)\left\lfloor \frac{m}{n-1} \right\rfloor + k = m.$$

Then the number of Hamiltonian paths starting at some vertex v is bounded above by

$$\left\lfloor \frac{m}{n-1} \right\rfloor^{n-1-k} \left(\left\lfloor \frac{m}{n-1} \right\rfloor + 1 \right)^k.$$

Proof. Hamiltonian paths are constructed starting at v. If we have reached vertex w with degree $d(w)$, the number of possible successors is only $d^*(w)$, the number of adjacent vertices which have not been visited yet. Hence, each edge is only once a candidate to be chosen. The number of edges is m and the number of edges on the Hamiltonian path equals $n-1$. We obtain an upper bound on the number of Hamiltonian paths by solving the following integer optimization problem.

$$\prod_{1 \leqslant j \leqslant n-1} d_j \to \max$$

under the conditions

$$\sum_{1 \leqslant j \leqslant n-1} d_j = m, \quad d_j \in \mathbb{N}.$$

The solution of this optimization problem is trivial and leads to the stated upper bound. □

The number of undirected Hamiltonian circuits is at most half the number of Hamiltonian paths starting at some vertex. The graph for the knight on the 8×8 chessboard has 168 edges. Hence, we obtain the upper bound

$$\frac{1}{2} 2^{21} 3^{42} \leqslant 1.148 \cdot 10^{26}$$

for the number of knight's tours on 8×8 chessboards.

Remark 6. For general n the number of edges is a little less than $4n^2$ leading to a 4^{n^2} upper bound on the number of knight's tours.

For the special case $n = 8$ we can improve the upper bound considerably. The solution of the integer optimization problem becomes smaller, if we can force some factors d_j to be large and some other factors to be small. We can start at d5 (see Fig. 6). Then we have 8 choices for the first step. If we have reached a corner like a8, we have found a deadlock or only one choice. If we reach the neighbor c7 of a8 before b6, we may have several choices but if we do not choose a8 we will not find a Hamiltonian path which can be closed to a Hamiltonian circuit. Hence, we know that 8 factors equal 1 and 1 factor equals 8. The number of edges for the remaining 54 choices is in the worst case $168 - 8 \cdot 1 - 1 \cdot 8 = 152$. This gives an upper bound on the number of knight's tours of

$$\frac{1}{2} 1^8 8^1 2^{10} 3^{44} \leqslant 4.034 \cdot 10^{24}.$$

We also can distinguish the different 8 choices for the first step.

Case 1: The first step reaches c7 or b6. Here we have 5 choices but only one, namely a8, may lead to a knight's tour. The number of edges for the remaining 54 choices is in the worst case 148. The number of knight's tours containing the step (d5, c7) or (d5, b6) is bounded by

$$\frac{1}{2} 1^8 2^1 2^{14} 3^{40} \leqslant 5.312 \cdot 10^{23}.$$

Case 2: The first step reaches e7 or b4. Then we have 5 choices. The number of edges for the remaining 53 choices is in the worst case 147. The number of knight's tours containing the step (d5, e7) or (d5, b4) is bounded by

$$\frac{1}{2} 1^8 2^1 5^1 2^{12} 3^{41} \leqslant 7.470 \cdot 10^{23}.$$

Case 3: The first step reaches f6, f4, c3 or e3. Then we have in any case 7 choices for the second step. The number of edges for the remaining 53 choices is in the worst case 145. The number of knight's tours containing one of the steps (d5, f6), (d5, f4), (d5, c3) and (d5, e3) is bounded by

$$\frac{1}{2} 1^8 4^1 7^1 2^{14} 3^{39} \leqslant 9.296 \cdot 10^{23}.$$

Hence, the number of knight's tours is bounded by $2.208 \cdot 10^{24}$. We have improved the previous bound by a factor of approximately 1.83. Hence, we should distinguish all possible paths of length $2, 3, \ldots$. We have used the computer to consider all 1 231 367 690 paths of length 14. Many of them are detected as deadlocks (one vertex which is not a neighbor of d5 has degree 1 or two vertices have degree 1). For the other 175 417 016 paths we have estimated the number of knight's tours containing this path. The upper bounds for the different paths of length 14 differ between $2.097 \cdot 10^6$ and $1.333 \cdot 10^{18}$. Altogether we obtain the following result.

Theorem 7. *The number of knight's tours on* 8×8 *chessboards is at most* $3.019 \cdot 10^{22}$.

Acknowledgements

We would like to thank Stefan Blauel for discussions on upper bounds on the number of knight's tours on 8×8 boards.

References

[1] A. Conrad, T. Hindrichs, H. Morsy and I. Wegener, Solution of the knight's Hamiltonian path problem on a chessboard, Discrete Appl. Math. 50 (1994) 125–134.
[2] L. Euler, Solution d'une question curieuse qui ne parait soumise à aucune analyse, Mémoire de l'Académie Royale des Sciences XV, Berlin (1759) 310–337.
[3] M. Kraitschik, Mathematical Recreations (Ch. 11. The problem of the knight) (Dover Publications, New York, 1953).
[4] A.M. Legendre, Théorie des Nombres (Paris, 1830).
[5] Minding, Cambridge Dublin Math. J. 7 (1852) 147–156.
[6] I. Parberry, Algorithms for touring knights, Technical Report University of North Texas (1994).
[7] P.M. Roget, Philos. Mag. 3 XVI (1840) 305–309.
[8] P.G. Tait, Listing's topology, Philos. Mag. 17 (1884) 30–46.
[9] A.T. Vandermonde, L'Histoire de l'Académie des Sciences 1771, Paris (1774) 566–574.
[10] H.C. Warnsdorff, Des Rösselsprunges einfachste und allgemeinste Lösung, Schmalkalden (1823).