Knight's Tours on a Torus

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Introduction

The knight is the only piece in chess that does not move in a straight line. Instead, it moves in an "L"—two squares in either a vertical or horizontal direction and then one square in a perpendicular direction. It is the strangeness of this move that has made the *Knight's Tour Problem* one of the most intriguing in all of recreational mathematics: *Can a knight visit each square of a chessboard by a sequence of knight's moves, and finish on the same square as it began*? Since a chessboard can be represented as a graph in which each vertex corresponds to a square, and edges correspond to those pairs of squares connected by a knight's move (FIGURE 1 illustrates this for a 4×4 board), finding a knight's tour amounts to finding a Hamiltonian cycle in the corresponding graph, a notoriously difficult general problem in graph theory (see **[5]**). However, we can easily see that there is *no* knight's tour for a 4×4 board since any Hamiltonian cycle would have to include the four edges incident to the two corner vertices indicated in FIGURE 1; this is impossible since these four edges already form a cycle that includes only four vertices.

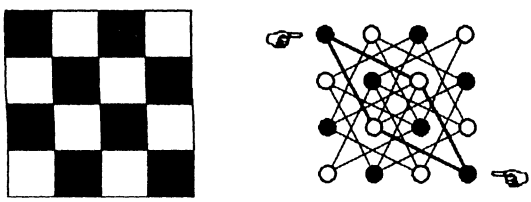


FIGURE 1
Representing a chessboard as a graph

We can also notice that the vertices in a knight's graph can be colored black and white so that *every* edge joins a black vertex and a white vertex. Such a graph is called *bipartite*. Since any cycle in a bipartite graph must have an *even* number of edges, we conclude that an $m \times n$ board with m and n odd cannot have a knight's tour, because the corresponding Hamiltonian cycle would have an odd number of edges.

There are several excellent sources for the history of this problem. We particularly recommend the discussion by W. W. Rouse Ball (**[1]**), which includes contributions by Euler as well as an ingenious method by the German mathematician H. C. Warnsdorff, dating from 1823, in which the knight is always moved to one of the squares from which it will have the fewest open moves. Combining this rule with Euler's techniques provides a remarkably efficient way to find knight's tours on various boards. Martin Gardner (**[3]**) presents several other problems involving knights, as well as giving S. W. Golomb's elegant proof that no $4 \times n$ board has a knight's tour. In 1991, Schwenk (**[4]**) answered the obvious question: *Which rectangular chessboards have a knight's tour*?

THEOREM. *An $m \times n$ chessboard with $m \leq n$ has a knight's tour unless one or more of these conditions holds*:

(1) *m and n are both odd*;
(2) $m = 1$, 2, *or* 4; *or*
(3) $m = 3$ *and* $n = 4$, 6, *or* 8.

But what if we allow the knight to move off the side of the board and then return to the board on the opposite side, as in some video games? (Such moves were used in [2] to find Hamiltonian tours for checkers.) For example, with this change it is now possible to find a knight's tour of a 5×5 board—in fact, Warnsdorff's method can be used here—since in FIGURE 2 a knight at square 25 *can* return to square 1 in a legal move by going off the bottom edge.

1	14	9	20	3
24	19	2	15	10
13	8	23	4	21
18	25	6	11	16
7	12	17	22	5

FIGURE 2
Knight's tour of a 5×5 board on a torus

This is equivalent to changing the flat chessboard into a torus (i.e., a doughnut) by gluing the top edge to the bottom edge (which creates a cylinder) and then gluing the left and right edges (which brings the two ends of the cylinder together). So we now pose the question: *Which rectangular chessboards have a knight's tour on a torus?*

Knight's Tours on a Torus

In this section we will prove that, on a torus, *every* rectangular board has a tour. First, we establish some useful notation.

A knight has eight possible moves as shown in FIGURE 3. Each move has an arithmetic description (x, y) where x indicates how many squares the knight moves to the right and y indicates how many squares down. Notice the symmetry between moves a, b, c, d and α, β, γ, δ, respectively; this will become important later.

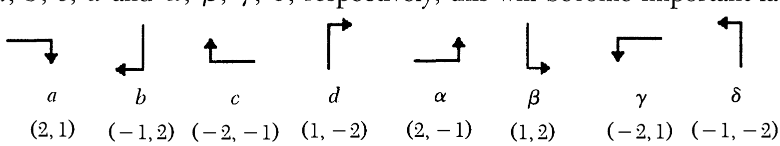


FIGURE 3
The eight knight moves

Our strategy will be to provide explicit tours for boards with a small number of rows (but any number of columns) and then to show how to 'stack' these boards together to form tours for arbitrary boards.

$1 \times n$ and $2 \times n$ boards You can easily tour any $1 \times n$ board by starting at any square and making move $\beta = (1, 2)$ n consecutive times. This is illustrated in FIGURE 4 for a 1×6 board. Similarly, you can tour any $2 \times n$ board by making move $(1, 2)$ until

1	2	3	4	5	6

1	2	3	4	5	6	7
9	8	14	13	12	11	10

FIGURE 4
Tours for 1×6 and 2×7 boards

you get stuck half way through, at which point you make move $(2,1)$ once, and then continue with move $(-1,2)$ until every square has been visited and you can take move $(-2,-1)$ back to the starting point. This is illustrated in FIGURE 4 for a 2×7 board.

$3\times n$ boards You can tour any $3\times n$ board, as long as n is not a multiple of 5, by repeating the three moves $(2,1)$, $(2,1)$, and $(1,-2)$ over and over again. If n is a multiple of 5 you can repeat the moves $(2,1)$, $(2,1)$, and $(-1,-2)$ over and over instead. These two cases are illustrated in FIGURE 5 for a 3×8 and a 3×10 board. Notice that in neither case do you ever go off the top or bottom edge.

1	16	7	22	13	4	19	10
20	11	2	17	8	23	14	5
15	6	21	12	3	18	9	24

1	22	13	4	25	16	7	28	19	10
20	11	2	23	14	5	26	17	8	29
9	30	21	12	3	24	15	6	27	18

FIGURE 5
Tours for 3×8 and 3×10 boards

$4\times n$ boards There are two cases. If n is *odd* you can alternate moves $(1,2)$ and $(1,-2)$ until you get stuck half way through (and the squares in the first and third rows have all been visited), at which point you make move $(2,-1)$, from 18 to 19 in FIGURE 6, and then continue alternating with moves $(-1,2)$ and $(-1,-2)$ until every square has been visited (at 36) and you can take move $(-2,1)$ back to the starting point. If n is *even* you again alternate moves $(1,2)$ and $(1,-2)$, but this time you get stuck a quarter of the way through, at 10 in FIGURE 6, at which point you make move $(2,1)$; then alternate $(-1,-2)$ and $(-1,2)$ until you get stuck (at 20) and make move $(-2,1)$; next, alternate $(1,-2)$ and $(1,2)$ until you get stuck (at 30) and make move $(2,1)$; finally, alternate $(-1,2)$ and $(-1,-2)$ until every square has been visited (at 40) and you can take move $(-2,1)$ back to the starting point. These two cases are illustrated in FIGURE 6 for a 4×9 and a 4×10 board. Notice in each case that *only* the last move goes off the top or bottom edge.

1	11	3	13	5	15	7	17	9
29	19	27	35	25	33	23	31	21
10	2	12	4	14	6	16	8	**18**
20	28	**36**	26	34	24	32	22	30

1	22	3	24	5	26	7	28	9	**30**
12	31	**20**	39	18	37	16	35	14	33
21	2	23	4	25	6	27	8	29	**10**
32	11	**40**	19	38	17	36	15	34	13

FIGURE 6
Tours for 4×9 and 4×10 boards

Notice that at this point we have already taken care of exceptions (2) and (3) of Schwenk's theorem. Strictly speaking, all that remains to do is the case of an odd by odd board on a torus. However, in part for the sake of completeness and in part because we like the constructions involved, we will instead consider *all* remaining boards.

Even $\times$ odd boards In order to show that any board with an even number of rows and an odd number of columns can be toured, we will simply stack together an appropriate number of boards each having two rows. However, a difficulty arises since the tour of a $2\times n$ board shown in FIGURE 4 uses the top and bottom edge of the board many times. Fortunately, if a $2\times n$ board has an *odd* number of columns, there is a tour that does not use the top and bottom edge: you simply alternate moves $(2,1)$ and $(2,-1)$. This is illustrated in FIGURE 7 for a 2×7 board.

1	5	9	13	3	7	11
8	12	2	6	10	14	4

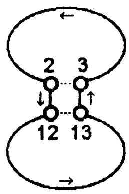
FIGURE 7
Alternate tour for a 2×7 board

It is easy to stack any number of these boards on top of one another. We illustrate this by creating a tour for a 4×7 board from the tours of two 2×7 boards. It is perhaps best to think in terms of the corresponding graphs. The idea—used by Euler—is to remove two edges, one from each Hamiltonian cycle, and then add two edges that join the two pieces into a single cycle. The only trick is to make sure the edges you add correspond to legal knight moves.

In FIGURE 8 we remove edge 2–3 from the top board and edge 12–13 from the bottom board, and then add edge 2–12 and edge 3–13, *both of which correspond to legal knight moves*. Still thinking in terms of the graph, it is now routine to do a knight's tour by beginning at square 1 on the top board, going to square 2, then to the bottom board at square 12, at which point we travel *backwards* on the bottom board until we reach square 13 from which we return to the top board at square 3 and finish the tour on the top board by taking the squares in order. The result, with the appropriate renumbering, is shown in FIGURE 8.

1	5	9	13	3	7	11
8	12	2	6	10	14	4

1	5	9	13	3	7	11
8	12	2	6	10	14	4



1	19	23	27	17	21	25
22	26	2	20	24	28	18
14	10	6	16	12	8	4
7	3	13	9	5	15	11

FIGURE 8
Stacking two 2×7 boards

It is clear that this process can be continued indefinitely; for example, we can stack another 2×7 board on top of the 4×7 board by again removing edge 2–3 from the top board and removing edge 26–27 from the bottom board. In this way we can construct a knight's tour for any board with an even number of rows and an odd number of columns (and, by symmetry, any board with an odd number of rows and an even number of columns).

Odd $\times$ odd boards We can now take care of exception (1) in Schwenk's theorem. In order to do a board with an odd number of rows (and an odd number of columns) we simply stack a board with 3 rows on top of a board with an even number of rows as done above. We illustrate this for a 7×7 board in FIGURE 9 using edge 5–6 from the top board and edge 19–20 from the bottom board.

1	10	19	7	16	4	13
5	14	2	11	20	8	17
9	18	6	15	3	12	21

1	19	23	27	17	21	25
22	26	2	20	24	28	18
14	10	6	16	12	8	4
7	3	13	9	5	15	11

1	38	47	35	44	4	41
5	42	2	39	48	36	45
37	46	34	43	3	40	49
24	6	30	26	8	32	28
31	27	23	33	29	25	7
11	15	19	9	13	17	21
18	22	12	16	20	10	14

FIGURE 9
A 7×7 board

Even $\times$ even boards We consider two cases. First, if the *number of rows is divisible by* 4, then we can stack multiple copies of $4 \times n$ boards. We illustrate this in FIGURE 10 for an 8×6 board.

1	14	3	16	5	18
8	19	12	23	10	21
13	2	15	4	17	6
20	7	24	11	22	9

1	14	3	16	5	18
8	19	12	23	10	21
13	2	15	4	17	6
20	7	24	11	22	9

1	14	3	16	5	18
8	19	12	23	10	21
13	2	15	4	17	6
20	7	24	11	22	9
25	38	27	40	29	42
32	43	36	47	34	45
37	26	39	28	41	30
44	31	48	35	46	33

FIGURE 10
An 8×6 board

We remove edge 1–24 from each board—remember this was the only move that used the top and bottom edge of the board—and join the 24 at the top to the 1 at the bottom and the 24 at the bottom to the 1 at the top. By noticing the position of the 1 and the 48 in the 8×6 board, we see that we can repeat this procedure as many times as we like, simply adding four rows at a time.

The second case—where the *number of rows is even but not divisible by* 4—is a good bit harder. This will be done by showing how to stack a board with 6 rows on a board with $4k$ rows. First we show how to join two $3 \times n$ boards to get the $6 \times n$ board which we need. From the top $3 \times n$ board remove the edge that joins the next to last square in the second row to the first square in the last row—for example, edge 14–15 in Figure 11. From the bottom $3 \times n$ board remove the edge that joins the next to last square in the first row to the second from last square in the last row—that is, edge 19–18 in Figure 11. We can now add two edges in the obvious way to create a Hamiltonian cycle—namely, edges 15–19 and 14–18 in Figure 11. Notice that in the resulting tour of the board with 6 rows, the next to the last square in the second row is connected to the second from last square in the last row—that is, 14–15 in Figure 11. It is this edge that we will remove in the next step.

1	16	7	22	13	4	19	10
20	11	2	17	8	23	14	5
15	6	21	12	3	18	9	24

1	16	7	22	13	4	19	10
20	11	2	17	8	23	14	5
15	6	21	12	3	18	9	24

1	40	7	46	13	4	43	10
44	11	2	41	8	47	14	5
39	6	45	12	3	42	9	48
32	17	26	35	20	29	38	23
37	22	31	16	25	34	19	28
18	27	36	21	30	15	24	33

FIGURE 11
A 6×8 board

Next, we get a tour for the board with $4k$ rows exactly as we did previously *except* that we begin the tour in the top row *five* squares from the right-hand edge of the board rather than in the upper left-hand corner as we usually do—notice the placement of the 1 in the 4×8 board in Figure 12. This is so that we will end up in the bottom row *three* squares from the right-hand edge of the board (at 32 in Figure 12), and we can then join the two boards with two legal knight moves—namely, 14–32 and 1–15 in Figure 12.

1	40	7	46	13	4	43	10
44	11	2	41	8	47	14	5
39	6	45	12	3	42	9	48
32	17	26	35	20	29	38	23
37	22	31	16	25	34	19	28
18	27	36	21	30	15	24	33

22	7	24	1	18	3	20	5
29	12	27	10	25	16	31	14
6	23	8	17	2	19	4	21
13	28	11	26	9	32	15	30

1	72	7	78	13	4	75	10
76	11	2	73	8	79	14	5
71	6	77	12	3	74	9	80
64	49	58	67	52	61	70	55
69	54	63	48	57	66	51	60
50	59	68	53	62	47	56	65
25	40	23	46	29	44	27	42
18	35	20	37	22	31	16	33
41	24	39	30	45	28	43	26
34	19	36	21	38	15	32	17

FIGURE 12
A 10×8 board

This completes the proof of the following theorem.

THEOREM. *On a torus, every rectangular chessboard has a knight's tour.*

Tours on Square Boards

In particular, all square boards have knight's tours on a torus. In this section we shall see that tours on square boards can have patterns that are far nicer than those offered by the foregoing inductive procedure. Moreover, we shall see that the attractiveness of these patterns is due to an underlying algebraic structure. Interestingly, a Fulani astronomer and mathematician, Muhammad Ibn Muhammad, used similar knight's patterns in his native northern Nigeria to produce magic squares at just about the same time that Euler was working on knight's tours in Europe (see **[7]**, 137–151). We will deal with $n \times n$ boards in three cases.

Case 1: $n \neq 5k$ Simply repeat the move $(2,1)$ $n-1$ times—we call this a *stroll.* Then use the move $(1,-2)$—a *shift*—once, and resume the stroll, shifting every n moves, until you return to the starting point. FIGURE 13 shows the resulting tour for a 7×7 board. (Notice that the result is a magic square; in fact, this procedure produces a magic square for all n not a multiple of 2, 3, or 5; see **[6]**.)

1	24	47	21	37	11	34
12	35	2	25	48	15	38
16	39	13	29	3	26	49
27	43	17	40	14	30	4
31	5	28	44	18	41	8
42	9	32	6	22	45	19
46	20	36	10	33	7	23

1	12	23	9	20
10	16	2	13	24
14	25	6	17	3
18	4	15	21	7
22	8	19	5	11

FIGURE 13
Tours for 7×7 and 5×5 boards

Case 2: $n \neq 3k$ The reason the previous pattern does not work when n is a multiple of 5 is that eventually the shift can't be made. Obviously, the thing to do is to make a different shift. So we use the shift $(-1,-2)$ instead. This works for all n not a multiple of 3. We illustrate this in FIGURE 13 for a 5×5 board. (Notice that the result is a semi-magic square; in fact, this procedure produces a semi-magic square for all n not a multiple of 2 or 3; moreover, it is only the main diagonal whose sum fails to be correct in each case; see [6].)

Case 3: $n = 15k$ Unfortunately, this still leaves us having to deal with square boards where n is a multiple of 15. Our approach in this case will be very similar to the previous two cases, but the actual details turn out to be considerably more involved. Therefore, we delay our discussion of this case until the Appendix, and turn now to an alternate approach.

An Algebraic Approach

Anyone familiar with the concept of a group will have sensed that there is an underlying algebraic structure for these tours. For example, it is clear that if $\gcd(m, n) = 1$ where, without loss, we take n to be odd, then there is a tour of the $m \times n$ board using only the move (2, 1). This, of course, is because the element (2, 1) generates the group $\mathbb{Z}_n \times \mathbb{Z}_m$.

Similarly, we see that in the tour of the 7×7 board in FIGURE 13, the first stroll, which uses (2, 1) six times, yields the subgroup of $\mathbb{Z}_7 \times \mathbb{Z}_7$ generated by the element (2, 1)—namely, $\{(0,0),(2,1),(4,2),(6,3),(1,4),(3,5),(5,6)\}$. The shift $(1, -2)$ then moves us to a different coset of this subgroup, where the stroll now takes us through this new coset. In this way, we tour the entire group, one coset at a time.

In order to see how this works in general for a square board, we label the $n \times n$ board by the elements of $\mathbb{Z}_n \times \mathbb{Z}_n$ viewed as vectors (a, b), $a, b \in \mathbb{Z}_n$. In particular, the upper left-hand corner is labeled (0, 0). We can then make a *change of coordinates*, such as

$$(a,b) = c(2,1) + d(1,-2).$$

So, for example, a knight at position $(a, b) = (2, 1)$ in the original coordinates would be at $(c, d) = (1, 0)$ under the change of coordinates, or a knight at (4, 2) would now be at (2, 0). In this way, the 7×7 knight's tour in FIGURE 13, under the change of coordinates, becomes the 1-step rook's tour in FIGURE 14. (Such tours are discussed in [2].)

1	2	3	4	5	6	7
9	10	11	12	13	14	8
17	18	19	20	21	15	16
25	26	27	28	22	23	24
33	34	35	29	30	31	32
41	42	36	37	38	39	40
49	43	44	45	46	47	48

FIGURE 14
A 1-step rook's tour

Thus, we can turn the knight's tour problem into an obviously simpler rook's tour problem, a process that worked in this case because (2, 1) and $(1, -2)$ form a basis for $\mathbb{Z}_7 \times \mathbb{Z}_7$. This happened, in turn, because $\det\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ is a unit in the ring $\mathbb{Z}_7$. This particular change of coordinates, therefore, will work as long as 5 does not divide n.

On the other hand, since $\det\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = 4$, the change of coordinates given by $(a, b) = c(2, 1) + d(-2, 1)$ will work as long as n is odd. It is worth noting, however, that the two tour problems are not equivalent. For example, the knight's tour for the 5×5 board in FIGURE 13 does not become a rook's tour under this particular change of variables. This is not at all surprising since a knight has more moves than a 1-step rook. Similarly, the change of variables given by $(a, b) = c(2,1) + d(1,2)$ works as long as 3 does not divide n. There are three additional changes of variables that are possible, but they are equivalent to the three already mentioned. This method, therefore, handles any $n \times n$ board where n is not divisible by 30.

Open Questions

There are several directions for further study. Since our proof for the torus only rarely makes use of *both* the top to bottom and the left to right identifications, the most obvious question is to ask which rectangular boards have tours on a cylinder. In addition, there are always projective planes and Klein bottles on which to put chessboards. Finally, the algebraic approach could be applied to rectangular boards.

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Appendix

We now return to our discussion of Case 3 for square boards where n is a multiple of 15. In order to better understand our solution for the general $15k \times 15k$ board, it is worth looking at the 15×15 board in some detail. Let us begin with the move $(2,1)$ as a stroll. After 14 moves we make a shift using $(1, -2)$. All goes well in this fashion until exactly $1/5$ of the squares have been visited and we are unable to use our shift at square 45, as we see in FIGURE 15.

What we notice, however, is that the 45 squares that have already been visited form a perfectly arranged lattice on the board. Furthermore, from any of these squares, any of the knight moves a, b, c, d—see FIGURE 3—takes you to another of these squares; whereas, any of the knight moves $\alpha, \beta, \gamma, \delta$ takes you to a *previously unvisited* square. Now, it is clear what to do: use move a as a stroll and use move d as a shift (every 15 moves) until you reach 45, then use, say γ, once before resuming the strolling and shifting with a and d, using γ at 45, 90, 135, 180, and 225.

Since it is far less confusing if one uses colored pens when doing this by hand—red for squares 1–45, green for 46–90, and so on—we call a move such as γ a *color change*. In this way, γ acts as a translation of a lattice of one color to an identical lattice of another color. The five disjoint, but identical, lattices comprise the board.

This strategy certainly allows a knight to visit every square on a $15k \times 15k$ board, but does not always produce a closed tour. In fact, using a, d, and γ in this same way on a 30×30 board leaves an exhausted knight stranded after 900 moves in the 16th row and 16th column. In order to find a *closed* tour we use a little algebra.

1					19					37				
		2					20					38		
				3					21					39
	40					4					22			
			41					5					23	
24					42					6				
		25					43					7		
				26					44					8
	9					27					45			
			10					28					31	
32					11					29				
		33					12					30		
				34					13					16
	17					35					14			
			18					36					15	

FIGURE 15
Start of a 15×15 tour

Let us examine the case $n = 15$ more closely. We consider three variables, s, t, and ω, representing the stroll, the shift, and the color change, respectively. In a tour of a 15×15 board, we stroll for 14 moves and then shift, and repeat for a total of 3 strolls and 2 shifts for each of 5 colors—that is, we repeat the sequence $3(14s) + 2t + \omega$ five times, once for each color, and end up back where we started. Substituting a, d, and γ for s, t, and ω, and multiplying by 5, we get $15(14s) + 10t + 5\omega = 210\cdot(2,1) + 10\cdot(1,-2) + 5\cdot(-2,1) = (420,195) \equiv (0,0) \pmod{15}$ which explains precisely why this pattern returns us to the starting point. (A similar computation for the case $n = 30$ also shows why the knight ends up stuck in the 16th row and 16th column.)

Let us now turn to the general case $n = 15k$. It is necessary to allow the stroll and shift to vary from color to color, and to use different color changes as well. We thus have *fifteen* variables s_i, t_i, and ω_i, for $i = 1, \dots, 5$. Since we stroll for $15k - 1$ moves and then shift, and repeat for a total of $3k$ strolls and $3k - 1$ shifts for each of 5 colors, the result of all the moves is given by

$$\sum_{i=1}^{5} 3k(15k-1)s_i + (3k-1)t_i + \omega_i \equiv \sum_{i=1}^{5} (3k-1)t_i - 3ks_i + \omega_i \pmod{15k}.$$

Thus, we are looking to solve the following congruence

$$(**) \quad (3k-1)(t_1 + \cdots + t_5) - 3k(s_1 + \cdots + s_5) + (\omega_1 + \cdots + \omega_5) \equiv 0 \pmod{15k},$$

where $s_i, t_i \in \{a, b, c, d\}$ and $\omega_i \in \{\alpha, \beta, \gamma, \delta\}$ for each i. Moreover, we obviously require that $s_i \neq \pm t_i$ for any i.

One further restriction applies to the color changes, since not every sequence of 5 colors changes will cycle you through all 5 colors. It is easy to find appropriate sequences by constructing a directed graph with 5 vertices, one for each color, and joining each ordered pair of vertices with an arc labeled with the color change α, β, γ, or δ which takes you between the corresponding colors. We can thus see that there are 24 allowable sequences. Since we are only concerned with the arithmetic at present, these can be grouped into the following 8 classes where, in each case, we give

1	132	211	161	89	44	108	202	175	46	30	99	193	147	75
146	74	2	91	185	160	88	45	124	218	174	60	16	115	209
131	225	145	73	3	107	201	159	87	31	98	192	173	59	17
58	18	105	184	144	72	4	123	217	158	86	32	114	208	172
224	171	57	19	106	200	143	71	5	97	191	157	85	33	130
34	104	183	170	56	20	122	216	142	70	6	113	207	156	84
155	83	35	120	199	169	55	21	96	190	141	69	7	129	223
103	182	154	82	36	121	215	168	54	22	112	206	140	68	8
67	9	119	198	153	81	37	95	189	167	53	23	128	222	139
181	138	66	10	135	214	152	80	38	111	205	166	52	24	102
25	118	197	137	65	11	94	188	151	79	39	127	221	180	51
179	50	26	134	213	136	64	12	110	204	165	78	40	101	195
117	196	178	49	27	93	187	150	63	13	126	220	164	77	41
76	42	133	212	177	48	28	109	203	149	62	14	100	194	163
210	162	90	43	92	186	176	47	29	125	219	148	61	15	116

FIGURE 16
Knight's tour of a 15×15 board

the total effect of the five moves:

$$\alpha^2\beta^2\delta = (5,0) \quad \alpha^2\gamma\delta^2 = (0,-5) \quad \beta\gamma^2\delta^2 = (-5,0) \quad \alpha\beta^2\gamma^2 = (0,5)$$
$$\alpha^5 = (10,-5) \quad \beta^5 = (5,10) \quad \gamma^5 = (-10,5) \quad \delta^5 = (-5,-10)$$

We are now ready to present a solution of the Knight's Tour Problem for a $15k \times 15k$ chessboard! In fact, the following moves provide a solution that works for all $15k \times 15k$ boards.

$$\begin{array}{lll} s_1 = a = (2,1) & t_1 = b = (-1,2) & \omega_1 = \alpha = (2,-1) \\ s_2 = c = (-2,-1) & t_2 = d = (1,-2) & \omega_2 = \beta = (1,2) \\ s_3 = d = (1,-2) & t_3 = a = (2,1) & \omega_3 = \beta = (1,2) \\ s_4 = c = (-2,-1) & t_4 = d = (1,-2) & \omega_4 = \alpha = (2,-1) \\ s_5 = d = (1,-2) & t_5 = a = (2,1) & \omega_5 = \delta = (-1,-2) \end{array}$$

In order to see that this does yield a knight's tour, note that

$$s_1 + s_2 + s_3 + s_4 + s_5 = (0,-5)$$
$$t_1 + t_2 + t_3 + t_4 + t_5 = (5,0)$$
$$\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 = (5,0)$$

so that congruence $(**)$ becomes

$$(3k-1)\cdot(5,0) - 3k\cdot(0,-5) + (5,0) = (15k,15k) \equiv (0,0) \pmod{15k}$$

which shows that our wandering knight does indeed return to the original square. You might notice that the key was to make the sum of the five shifts and the sum of the five color changes equal; thus, other solutions are possible. FIGURE 15 shows the tour produced by this particular solution for a 15×15 board. We encourage you to grab five colored pens, a 30×30 grid, and have at it!