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The complexity of compressing subsegments of images described by finite automata☆

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Abstract

We investigate how the compression size of the compressed version of a two-dimensional image changes when we cut off a part of it, e.g. extract a photo of one person from a photo of a group of people, when compression is considered in terms of finite automata. Denote by $c(T)$ the compression size of a square image T in terms of deterministic automata, it is the smallest size of a deterministic acyclic automaton A describing T. The corresponding alphabet of A has only four letters, corresponding to four quadrants. We consider an independent useful combinatorial interpretation of $c(T)$ in terms of *regular* subsquares of T. Denote by $\Psi(n)$ the largest compression size $c(R)$ of a square subsegment R of the image T such that $c(T)=n$. We show that there is a constant $c>0$ such that:

$$cn^{2.5} \leqslant \Psi(n) \leqslant n^{2.5}.$$

For weighted automata we show that the compression size grows only linearly, if T is described by a weighted automaton with n states and m edges then a subimage R can be described by a similar automaton having $O(n)$ states and $O(m)$ edges.

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We also show how to construct efficiently (in linear time w.r.t. the total size of the input and the produced output) the compressed representation of subsegments given the compressed representation of the whole image.

Keywords: Square images; Finite automata; Complexity

1. Introduction

The compression size of images is of crucial importance in multimedia systems and in transferring large images in WWW. Deterministic and weighted finite automata are successful tools for compressing two-dimensional images, see [2–5]. There are several software packages using this type of compression, see [3,11]. Finite automata can describe quite complicated images, for example deterministic automata can describe the Hilbert's curve with a given resolution, see [9], while weighted automata can describe even much more complicated curves, see also [2,3,5]. The objects considered are potentially exponentially compressed, so algorithms which apply decompression are theoretically not polynomial time algorithms. In practice exponential compression does not usually take place, nevertheless the compression ratio for two-dimensional images can be very high, especially compared with the one-dimensional case. This happens, for example, for images corresponding to fractals having short description. For one-dimensional words there exist polynomial-time deterministic algorithms for compressed and fully compressed pattern matching [7,8,12], despite the fact that the uncompressed size of objects could be exponential. However, these problems become much harder in the two-dimensional case. Our main result is a constructive proof of the fact that compressed size of subsegments of an automata-compressed images grows only polynomially. This contrasts with the exponential growth of compression size of subimages for compression in terms of recursive descriptions, see [1].

We consider a subsegment image R and the host image T, denote by $Compress(R)$ and $Compress(T)$ the smallest automata describing R and T, respectively.

Our main problem is the *Subsegment Compression Problem*:

Instance: $Compress(T)$ representation of a $2^k \times 2^k$ square image, a point x in T and an integer $k' < k$. Let R be a square $2^{k'} \times 2^{k'}$ subsegment of T whose left-upper corner is positioned at x.

Question: (1) Estimate the size of $Compress(R)$. (2) Construct $Compress(R)$.

For deterministic automata we define the size of the automaton as the number of its *essential* states (those which are on accepting paths). The number of edges is only linear with respect to the number of states since the alphabet is constant. Define $c(T)$, the compression size of T in terms of deterministic automata, as the smallest size of an acyclic deterministic automaton describing T. Denote by $\Psi(n)$ the largest compression size $c(R)$ of a square subsegment R of the image T such that $c(T)=n$. Our main result is the tight bound: $\Psi(n) = \Theta(n^{2.5})$.

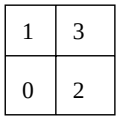

Fig. 1. Enumeration of the quadrants.

In case of weighted automata we show that the compression size of the subsegment grows only linearly. In the weighted case the size of the automaton is given by two numbers: the number n of states and the number m of edges. The weights are assumed to be of a constant size.

The algorithms for the main problem are used to improve pattern matching for images compressed in terms of deterministic automata. The similar compressed pattern-matching problem for weighted automata is *NP*-complete, see [1]. Another application is a polynomial time algorithm for the pattern checking (it consists in testing the existence of a fixed occurrence of a large compressed image). The last problem is *co-NP* complete for two-dimensional compressions in terms of recursive generations, see [1].

2. Deterministic finite automata and functions τ, ϕ

Our alphabet is $\Sigma = \{0,1,2,3\}$, the elements of which correspond to four quadrants of a square array, see Fig. 1.

A word w of length k over Σ can be interpreted, in a natural way, as a unique address of a pixel x of a $2^k \times 2^k$ image (array), we write $address(x) = w$. The length k is called the *resolution* of the image. For a language $L \subseteq \Sigma^+$ we denote by $Image_k(L)$ the $2^k \times 2^k$ black-and-white image such that the color of a given pixel x is black iff $address(x) \in L$. We consider also the weighted languages, formally they correspond to functions which associates with each word w a value $weight_L(w)$. A weighted language L over Σ and resolution k determine the gray-tone image $Image_k(L)$ such that the color of a given pixel x equals $weight_L(address(x))$. If all words in L are of the same length k then we can omit the subscript k and write $Image(L)$. Our description of the language is in terms of finite (unweighted or weighted) automaton A, cf. [2,9]. We define $Image_k(A) = Image_k(L(A))$, where $L(A)$ is the language accepted by A. We count only essential states. We call a state *essential* iff it is on a path to an accepting state.

Let $A = (\{0,1,2,3\}, Q, q_0, \delta)$ be a deterministic acyclic automaton of depth n where Q is a set of states, $q_0 \in Q$ is the initial state, and $\delta : Q \times \Sigma^* \to Q$ is a partial transition function. The automaton A defines the language

$$L(A) = \{w\colon \delta(q_0, w) \text{ is defined, } |w| = n \text{ and } \delta(q_0, w) \text{ is accepting}\}.$$

Note, that in the definition of an automaton we do not need to specify the single accepting state. For $q \in Q$, denote by $Image(q)$ the image which is generated by

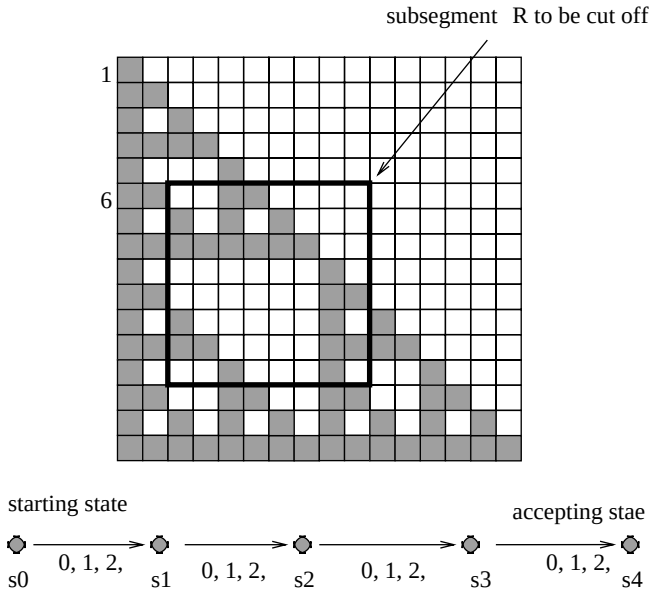

Fig. 2. The image S_4 and a smallest acyclic deterministic automaton describing it. Edges which are not on accepting paths are disregarded. $\tau(S_4) = 5,\ \phi(S_4) = 20,\ \tau(R) = 9$.

the automaton which is obtained from A by changing its initial state to q. Clearly, $Image(q_0) = Image(A)$. All considered automata are assumed to be *acyclic*. It is motivated by the fact that we consider only finite resolution images, or more precisely finite resolution approximations of infinite resolution images. We define *the depth* of the automaton as the longest path from the initial state to an accepting state. The depth corresponds to the resolution of the image. An acyclic automaton can be transformed to an equivalent automaton in which for each state q each path from the initial state to q has the same length. We say that a state q *belongs to level* t if all paths from the initial state to q are of length t. Here, the length of a path is the number of edges in the path.

A *regular block* of a $2^k \times 2^k$ image T is defined as follows. T is a regular block, and if B is a regular block then all its quadrants are also regular blocks.

We can interpret the state q as a name of a regular block. We can also consider the state as a nonterminal of a two-dimensional grammar generating the image. A *balanced two-dimensional grammar* describes recursively each regular subsquare of the image in terms of the names (nonterminals) of its quadrants. Each production of such grammar corresponds to the decomposition of a square into 4 smaller quadrants of *the same shape* (this motivates the word *balanced*). The terminal symbols are 0 and 1, they correspond to blank and black pixels, respectively. For simplicity denote by $\hat{\emptyset}$ a square blank block consisting only of white pixels, and use the same notation for all possible sizes of $\hat{\emptyset}$. For the automaton from Fig. 2 the corresponding two-dimensional grammar

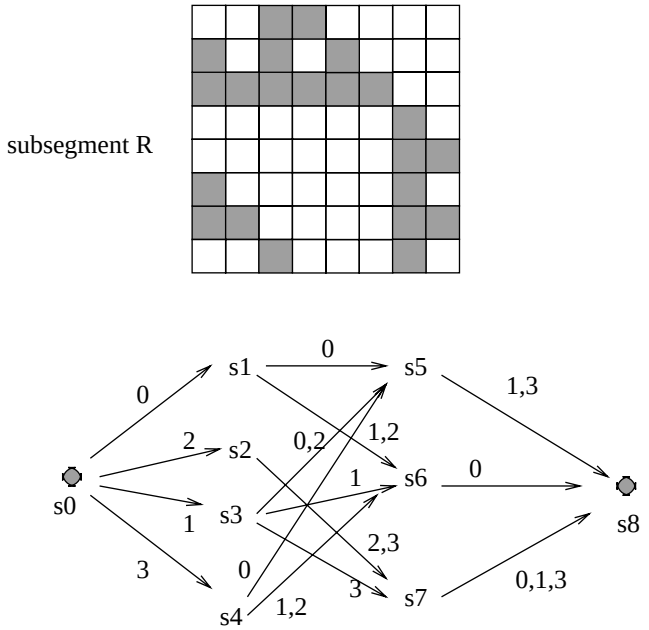

Fig. 3. The subsegment R of S_4 and an acyclic 9-state automaton describing R.

looks as follows:

$$s_0 \to \begin{bmatrix} s_1 & \hat{\emptyset} \\ s_1 & s_1 \end{bmatrix}, \quad s_1 \to \begin{bmatrix} s_2 & \hat{\emptyset} \\ s_2 & s_2 \end{bmatrix}, \quad s_2 \to \begin{bmatrix} s_3 & \hat{\emptyset} \\ s_3 & s_3 \end{bmatrix}, \quad s_3 \to \begin{bmatrix} s_4 & \hat{\emptyset} \\ s_4 & s_4 \end{bmatrix}, \quad s_4 \to [1].$$

Essential nonterminals are the ones which generate nonblank texts. Observe that essential states of a minimal acyclic deterministic automaton describing an image T correspond to essential nonterminals in a minimal two-dimensional grammar describing T.

The crucial notion is that of a *basic block* of T, defined as a *regular* block or a subsegment consisting of 4 adjacent *regular* blocks of T. Define:

$\tau(T)$, $\phi(T)$ to be the number of distinct nonblank regular (basic) blocks of T;

The basic properties (to be proved later) of these functions are:

(1) $\tau(T)$ corresponds to the automata and grammar compression size of T;
(2) $\tau(R) \leqslant \phi(T)$ for each square subsegment R of T;
(3) $\phi(T) \leqslant \tau(T)^{2.5}$.

Example. For images S_4, R in Figs. 2 and 3, respectively, we have

$$\tau(S_4) = 5, \quad \phi(S_4) = 20, \quad \tau(R) = 9.$$

Recall that $c(T)$ was defined as the minimal number of essential states of a deterministic automaton describing T.

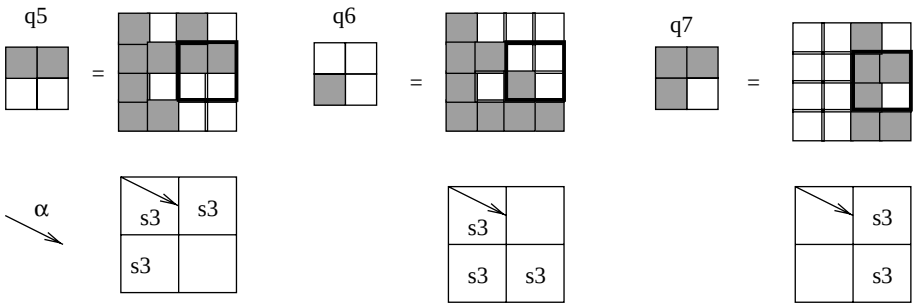

Fig. 4. Regular blocks of R (in bold) are parts of basic blocks of T.

Lemma 2.1. $c(T) = \tau(T)$.

Proof. Let d be the number of nonblank regular blocks of T. First, we prove that there is an automaton A describing T with d states. Each state q of A corresponds to one regular block $Image(q)$, and different states corresponds to different regular blocks. There is an edge from state q_1 to q_2 labeled by i if $Image(q_2)$ is in the ith quadrant of $Image(q_1)$. Assume that there is an automaton A describing T and having $s < d$ states. Consider the set $\{Image(q)\colon q \in States(A)\}$. By pigeon hole principle there is a regular block of T which does not correspond to any state of A so there is a regular block which is not generated by states of A. This means that no state of A corresponds to T. □

Example. $Image(\{0,1,2\}^k) = S_k$ is the $2^k \times 2^k$ black-and-white square part of Sierpinski's triangle, see Fig. 2 for the case $k = 4$. The corresponding smallest acyclic deterministic automaton accepting all paths describing black pixels has 5 states. The automaton for its subsegment R is illustrated in Fig. 3. The states at depth 2 of the automaton from Fig. 3 correspond to the blocks of the subsegment R in the way shown in Fig. 4.

3. $\Psi(n) \leqslant n^{2.5}$

We can use the combinatorial interpretation of $\tau(T)$, which is especially more convenient for the upper bound (as defined in terms independent of automata). A square subsegment of the shape $2^t \times 2^t$ is said to be of *rank* t.

For a subblock X of a block X define the position of X in Y ($pos_X(Y)$) as the position of left-upper corner of Y inside X.

For a block X of rank $t+1$ and vector α define $Sub_t(\alpha, X)$ to be the subblock Y of X such that $pos_X(Y) = \alpha$.

Example. Fig. 4 illustrates 3 nonblank regular blocks of rank 1 in segment R (see Fig. 3) which correspond to the states of the automaton for R, each of these blocks

is a part of a pseudo-regular block of rank 2 of T with the same vector α. We have $q5 = Sub(\alpha, s3, s3, \emptyset, s3)$, $q6 = Sub(\alpha, s3, s3, s3, \emptyset)$, $q7 = Sub(\alpha, \emptyset, \emptyset, s3, s3)$.

Lemma 3.1. *Assume R is a subsquare of T, then* (1) $\tau(R) \leqslant \phi(T)$. (2) *For each t there is a vector α_t such that each regular block of rank t in the subimage R equals $Sub(\alpha_t, (A, B, C, D))$ where (A, B, C, D) is a basic block of T of rank $t+1$.*

Proof. It is enough to show the point (2). We proceed by induction on t. If the rank of R is k, then the lemma is clearly true. We suppose that it is true for some $t \leqslant k$ and prove it is true for $t-1$. Clearly, $\alpha_t < [2^t, 2^t]$. We assume that $\alpha_t \leqslant [2^{t-1}, 2^{t-1}]$, the other cases being treated similarly. Take any regular block B of rank t of R. By the induction hypothesis it equals $Sub(\alpha_t, (A, B, C, D))$ for some basic block (A, B, C, D) of T of rank $t+1$. Assume

$$A \to \begin{bmatrix} A_1 & A_3 \\ A_0 & A_2 \end{bmatrix}, \quad B \to \begin{bmatrix} B_1 & B_3 \\ B_0 & B_2 \end{bmatrix}, \quad C \to \begin{bmatrix} C_1 & C_3 \\ C_0 & C_2 \end{bmatrix}, \quad D \to \begin{bmatrix} D_1 & D_3 \\ D_0 & D_2 \end{bmatrix}.$$

Then the basic block (A, B, C, D) has the following structure:

$$\begin{bmatrix} B_1 & B_3 & D_1 & D_3 \\ B_0 & B_2 & D_0 & D_2 \\ A_1 & A_3 & C_1 & C_3 \\ A_0 & A_2 & C_0 & C_1 \end{bmatrix}.$$

Since $\alpha_t < [2^{t-1}, 2^{t-1}]$ the left-upper corner of B is in B_1. Now we consider quadrants of B. The upper left one is

$$Sub(\alpha_t, (B_0, B_1, B_2, B_3)),$$

the upper right one is

$$Sub(\alpha_t, (B_2, B_3, D_0, D_1)),$$

the lower left one is

$$Sub(\alpha_t, (A_1, B_0, A_3, B_2))$$

and the lower right one is

$$Sub(\alpha_t, (A_3, B_2, C_1, D_0)).$$

Observe that the vector α_t is common to all pseudo-regular blocks and is independent of the choice of a regular subblock B of R (Fig. 5). This completes the proof. □

Lemma 3.2. *Let ϕ_i be the number of basic blocks of rank i for $1 \leqslant i \leqslant r$, and k_i be the number of regular blocks of rank i for $0 \leqslant i \leqslant r$. Then*:

$$\phi_i \leqslant \min(k_{i-1}^4, n^2).$$

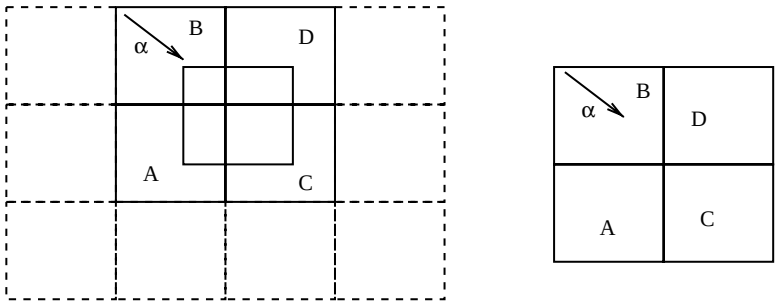

Fig. 5. A regular block of the subimage is determined by the 5-tuple (α, A, B, C, D). The 4-tuple (A, B, C, D) corresponds to a basic block.

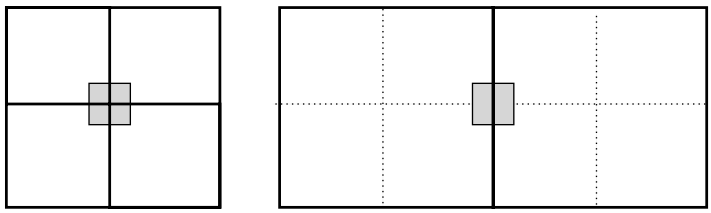

Fig. 6. Illustration of the claim.

Proof. The proof is based on the following fact, see Fig. 6.

Claim. *Each basic block of rank i is a central block of a regular block of rank k, or a central block of* 2 (*horizontally or vertically*) *adjacent regular blocks of rank k, where* $k \geqslant i$.

Proof. Consider a basic block B in T. Let

$$T \to \begin{bmatrix} A_1 & A_3 \\ A_0 & A_2 \end{bmatrix},$$

where A_0, A_1, A_2 and A_3 are regular in T. There are several possibilities.

Case 1: B is in the center of T. Then we are done.

Case 2: B is inside A_0 or A_1 or A_2 or A_3. Then we proceed recursively. Eventually, we come to one of cases 1 and 3.

Case 3: B is in the common border of A_1 and A_3 or A_0 and A_2 or A_1 and A_0 or A_3 and A_2. Now either B is in the center of two adjacent regular blocks or we proceed recursively. Eventually, B will be a central block of two adjacent regular blocks, not necessarily of the same rank as B. Fig. 7 illustrates a possible situation when B is in the border of A_1 and A_3. It is essential that B is a basic block, and it consists of four regular blocks. This completes the proof of the claim. □

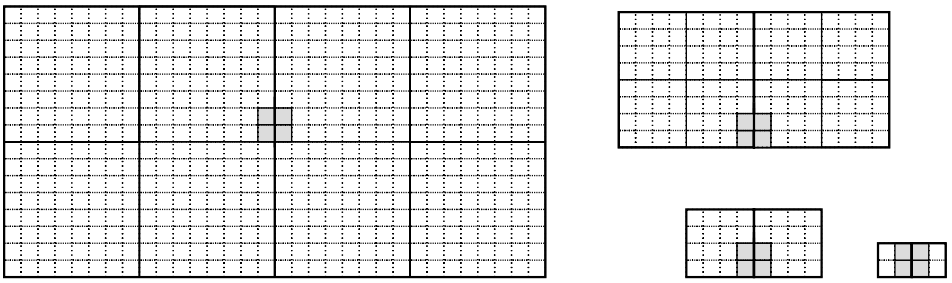

Fig. 7. Case 3: a situation when B (shaded basic block) is in the border of A_1, A_3 but not in the center. Then B is in the border of two regular blocks of smaller ranks.

Proof of Lemma 3.2 (Conclusion).

Due to the claim and inequality $k_i + k_{i+1} + \cdots + k_r \leqslant n$ we have

$$\phi_i \leqslant k_i^2 + k_{i+1}^2 + \cdots + k_r^2 + k_i + k_{i+1} + \cdots + k_r \leqslant n^2.$$

On the other hand each basic block of rank i is composed of 4 regular blocks of rank $i-1$, hence $\phi_i \leqslant k_{i-1}^4$. Consequently $\phi_i \leqslant \min(k_{i-1}^4 n^2)$. □

We omit the proof of the following simple lemma which says that the maximum of the sum of 4th powers of nonnegative numbers bounded by a certain number δ is the largest when all these number are equal δ except possibly one number.

Lemma 3.3. *Assume* $0 < \delta < n$, $0 < k$, $0 \leqslant \alpha_i \leqslant \delta$ *for* $1 \leqslant i \leqslant k$ *and* $\sum_{i=1}^{k} \alpha_i = n$. *Then*

$$\sum_{i=1}^{k} \alpha_i^4 \leqslant \frac{n}{\delta}\delta^4.$$

Theorem 3.4. $\Psi(n) \leqslant n^{2.5}$.

Proof. Due to Lemma 3.1 it is enough to show $\phi(T) \leqslant \tau(T)^{2.5}$. The conclusion of the theorem is now a consequence of Lemma 3.2 and of the following fact.

Claim. *If* $n_1 + n_2 + \cdots + n_k = n$, *then* $S = \sum_{i=1}^{k} \min\{n_i^4, n^2\} \leqslant n^{2.5}$.

Proof. Let $\delta = \sqrt{n}$, $A = \{i\colon n_i \leqslant \delta\}$, and $B = \{i\colon n_i > \delta\}$. Then:

$$\sum_{i=1}^{k} \min\{n_i^4, n^2\} \leqslant \sum_{i\in A} n_i^4 + \sum_{i\in B} n^2.$$

Denote $x = \sum_{i=1}^{k} n_i$. Then:

$$\sum_{i \in A} n_i^4 \leqslant \frac{x}{\delta}\,\delta^4$$

due to Lemma 3.3 and

$$\sum_{i \in B} n^2 \leqslant |B| n^2 \leqslant \frac{n-x}{\delta} n^2.$$

Putting these inequalities together we have:

$$S \leqslant \frac{x}{\sqrt{n}}\sqrt{n}^4 + \frac{n-x}{\sqrt{n}} n^2 \leqslant \frac{n}{\sqrt{n}} n^2 \leqslant n^{2.5}.$$

This completes the proof. □

4. $\Psi(n) \geqslant cn^{2.5}$

In the proof of the lower bound we first need to generate two objects of sizes $O(n)$ which composed together give an object whose compression size is $\Omega(n^2)$. The construction is based on morphic representations of strings. Assume that the alphabet consists of nonnegative integers and we have morphisms h and g defined by

$$h(i) = 2i\ 2i\ 2i+1\ 2i+1 \quad \text{and} \quad g(i) = 2i\ 2i+1\ 2i\ 2i+1, \quad \text{for } i = 0, 1, \ldots .$$

Example. The generation of

$$h^2(0) = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 2\ 2\ 3\ 3\ 2\ 2\ 3\ 3,$$

$$g^2(0) = 0\ 1\ 0\ 1\ 2\ 3\ 2\ 3\ 0\ 1\ 0\ 1\ 2\ 3\ 2\ 3$$

is illustrated in Fig. 8.

Observe that the set of pairs $(h^2(0)[i], g^2(0)[i])$ for $i \leqslant 16$ is exactly the set

$$\{(s,t)\colon 0 \leqslant s \leqslant 3, 0 \leqslant t \leqslant 3\}.$$

For a one-dimensional string w whose length is a power of two, define 1-*regular blocks* similarly as (two-dimensional) regular blocks for two-dimensional texts. Both halves of w are regular, and recursively both halves of them are also. Denote by $\tau_1(w)$ the number of distinct 1-regular blocks of a one-dimensional string.

Lemma 4.1 (Key lemma). *Assume $n = 2^k$ for a natural k. Then there are two strings u_n, w_n of size n^2 over the alphabet $\{0, 1, \ldots, n-1\}$ such that*:
(1) *All n^2 pairs $(u_n[i], w_n[i])$ are different for $1 \leqslant i \leqslant n^2$.*
(2) $\tau_1(w_n), \tau_1(u_n) \leqslant 6n$.

Proof. Take $u_n = h^k(0)$ and $w_n = g^k(0)$.

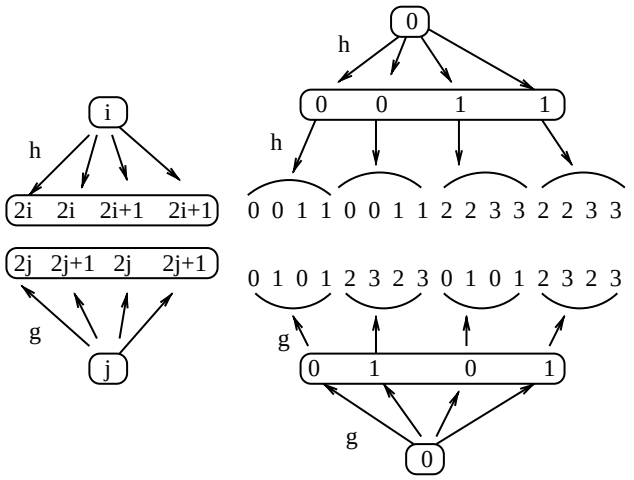

Fig. 8. The generation of $h^2(0)$ and $g^2(0)$.

Proof of (1). We proceed by induction on k. For $k=0$ this is clear. Suppose the lemma is true for $k>0$. We will prove that it holds for $k+1$. Take any pair (a,b) such that $0 \leqslant a,\ b < 2^{k+1}$. The numbers $a,\ b$ can be uniquely written in the form $a = 2a_1 + a_2$ and $b = 2b_1 + b_2$, where $0 \leqslant a_1,\ b_1 < 2^k$ and $0 \leqslant a_2,\ b_2 \leqslant 1$. By inductive hypothesis $a_1 = h^k(0)[i]$ and $b_1 = g^k(0)[i]$ for some $0 \leqslant i < 2^k$. Then,

$$
\begin{aligned}
&h^{k+1}(0)[4i] = 2a_1 + 0, \quad g^{k+1}(0)[4i] = 2b_1 + 0,\\
&h^{k+1}(0)[4i+1] = 2a_1 + 0, \quad g^{k+1}(0)[4i+1] = 2b_1 + 1,\\
&h^{k+1}(0)[4i+2] = 2a_1 + 1, \quad g^{k+1}(0)[4i+2] = 2b_1 + 0,\\
&h^{k+1}(0)[4i+3] = 2a_1 + 1, \quad g^{k+1}(0)[4i+3] = 2b_1 + 1.
\end{aligned}
$$

Hence, among pairs $(h^{k+1}(0)[j], g^{k+1}(0)[j])$, for $4i \leqslant j < 4i+4$, there is a pair (a,b). Since the length of the word $h^{k+1}(0)$ is $(2^{k+1})^2$ each pair (a,b) with $0 \leqslant a,\ b < 2^{k+1}$ occurs among the pairs $(h^{k+1}(0)[i], g^{k+1}[i])$ exactly once.

Proof of (2). It can be easily seen that each 1-regular block of u_n is of the form

$$(*) \quad h^s(a) \text{ for } 0 \leqslant a < 2^{k-s},$$

or it is one of the halves of a word of this type.

There are 2^{k-s} words of type $(^*)$. Summing over $0 \leqslant s \leqslant k$ we have at most $2n$ such words. Together with their halves we have an upper bound of $2n + 2 * 2n = 6n$ on the number of 1-regular blocks. Similarly we proceed for the string w_k. This complete the proof. □

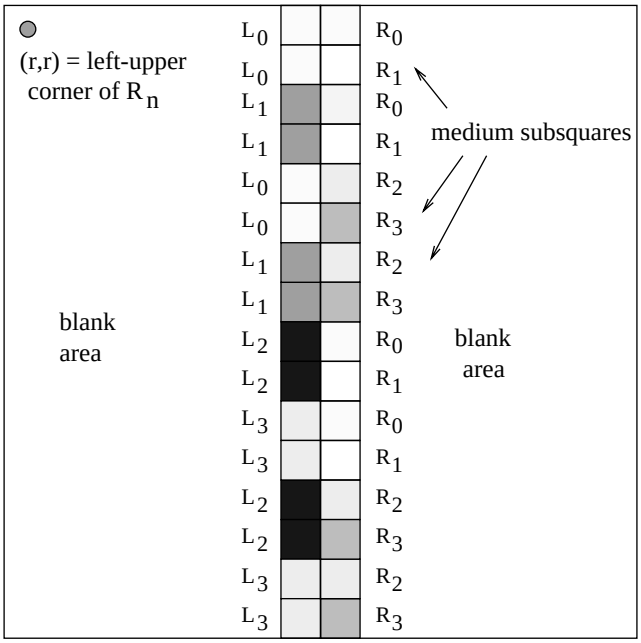

Fig. 9. The structure of the image T_n, for $n = 4$, of shape $(s2n^2) \times (s2n^2)$.

4.1. Construction of the image T_n

For each n, which is a power of two, we construct an image T_n as illustrated in Fig. 9. It consists of two vertical central columns being sequences of *medium* subsquares

$$L_{i_1}, L_{i_2}, \ldots, L_{i_m} \quad \text{and} \quad R_{j_1}, R_{j_2}, \ldots, R_{j_m},$$

where

$$i_1 i_2 \ldots i_m = u_n \quad \text{and} \quad j_1 j_2 \ldots j_m = w_n.$$

Everything outside medium subsquares is blank, also most of medium subsquares is blank, but the size of a *large* blank area will matter later.

Let $r = \log(\sqrt{n})$, assume it is a power of two, otherwise we take the closest (from above) power of two. Each *medium* subsquare L_i is a $2^{\sqrt{n}}r \times 2^{\sqrt{n}}r$ square which is blank except two small subsquares $Small(p)$, $Small(q)$, see Fig. 10, where $(p,q) = code(i)$, and *code* is any injective function mapping the interval $[1 \ldots n]$ into the set of pairs $[1 \ldots \sqrt{n}] \times [1 \ldots \sqrt{n}]$. Each *small* subsquare $Small(p)$ is a $r \times r$ black-white square whose first row encodes the number p and other rows are blank. In fact any family of small subsquares encoding natural numbers not exceeding $\sqrt{n}$ will do.

These small subsquares ($Small(p)$, $Small(q)$ subsquares) are called *code blocks*. Two code blocks of $L_i(R_i)$ encode together the number i.

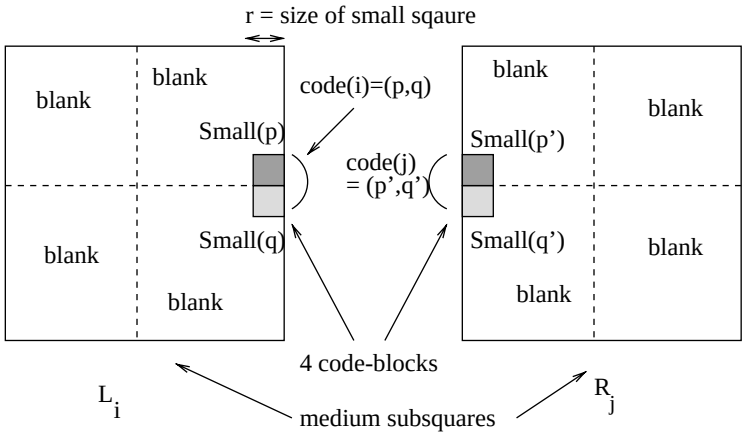

Fig. 10. The structure of the medium $s \times s$ subsquares L_i, R_j, where $s = r2^{\sqrt{n}}$. They are totally blank except the indicated $r \times r$ code blocks.

Lemma 4.2. $\tau(T_n) = \mathrm{O}(n)$.

Proof. Let s, r be the lengths of the sides of medium blocks (one of L_i, R_j) and code blocks.

Denote by $\tau_{\mathrm{small}}(n), \tau_{\mathrm{large}}(n), \tau_{\mathrm{others}}(n)$ the number of different regular nonblank blocks of T_n whose sides have length at most r, greater than s and other, respectively. Observe that all pixels of T_n are blank except of the ones contained in code blocks.

Each small code block is of a polylogarithmic size, hence:

$$\tau_{\mathrm{small}}(n) = \mathrm{o}(n). \tag{1}$$

Each nonblank regular block whose side length is between r and s has side of length at most $r2^{\sqrt{n}}$ which is a power of two. There are $\mathrm{O}(\sqrt{n})$ possible lengths. Each such block is blank except one corner containing a code block. There are $\mathrm{O}(\sqrt{n})$ such subsquares. Hence:

$$\tau_{\mathrm{others}}(n) = \mathrm{O}(n). \tag{2}$$

Each regular block of T_n whose side length is greater than s consists of a blank area and a segment of vertical column of medium subsquares which corresponds to a 1-regular block of u_n or w_n. Consequently, due to Lemma 4.1 we have:

$$\tau_{\mathrm{large}}(n) = \mathrm{O}(n). \tag{3}$$

Due to Eqs. (1)–(3) we have

$$\tau(T_n) = \tau_{\mathrm{small}}(n) + \tau_{\mathrm{large}}(n) + \tau_{\mathrm{others}}(n) = \mathrm{O}(n).$$

This completes the proof. □

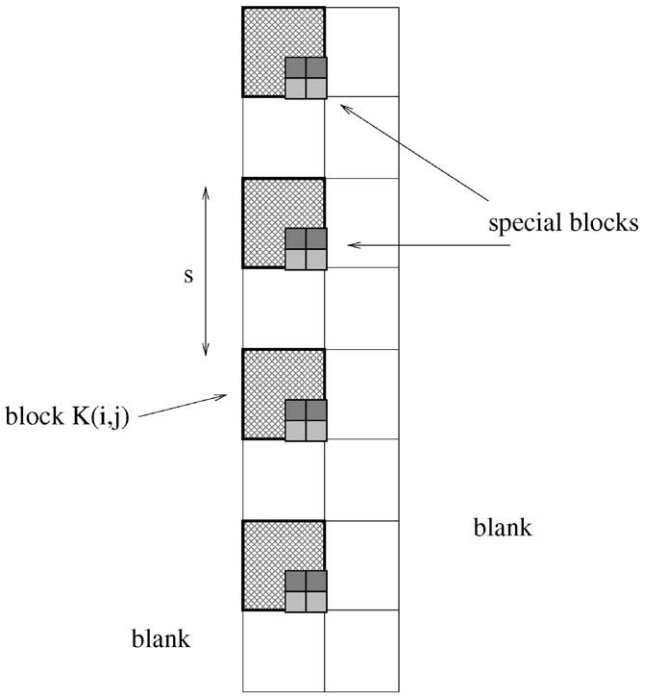

Fig. 11. The structure of the subsegment $\mathscr{R}_n$. All elements are blank except special blocks, each of them is composed of four small *code blocks*.

4.2. Construction of the subimage $\mathscr{R}_n$

Let $\mathscr{R}_n$ be the square subsegment of T_n whose side is twice smaller than the side of T_n and which starts (with its left-upper corner) in T_n at the position (r,r).

Lemma 4.3. $\tau(\mathscr{R}_n) = \Omega(n^{2.5})$.

Proof. After placing the subimage with the origin at point (r,r) the grid of regular blocks is shifted by the vector (r,r), see Figs. 11 and 12. Let us define *special* blocks as $2r \times 2r$ basic blocks of T_n composed of 4 adjacent code blocks. The first half of the vertical sequence of special blocks of T_n become now regular blocks of $\mathscr{R}_n$. After shifting the grid each special block is a right-bottom *corner block* of a regular $(s/2) \times (s/2)$ block $K(i,j)$, see Fig. 12. The block $K(i,j)$ overlaps adjacent blocks L_i and R_j in T_n, it is of the same shape as each of them. $K(i,j)$ can be viewed as a shift by the vector (r,r) of top-right quadrant of L_i, see Fig. 12. Denote by $\mathscr{K}$ the set of all such blocks $K(i,j)$. Due to Lemma 4.1 there are $\Omega(n^2)$ special blocks and they are pairwise different, so $\mathscr{K}$ satisfies:

(1) $|\mathscr{K}| = \Omega(n^2)$,

(2) each block in $\mathscr{K}$ is of the shape $\frac{1}{2}r2^{\sqrt{n}} \times \frac{1}{2}r2^{\sqrt{n}}$,

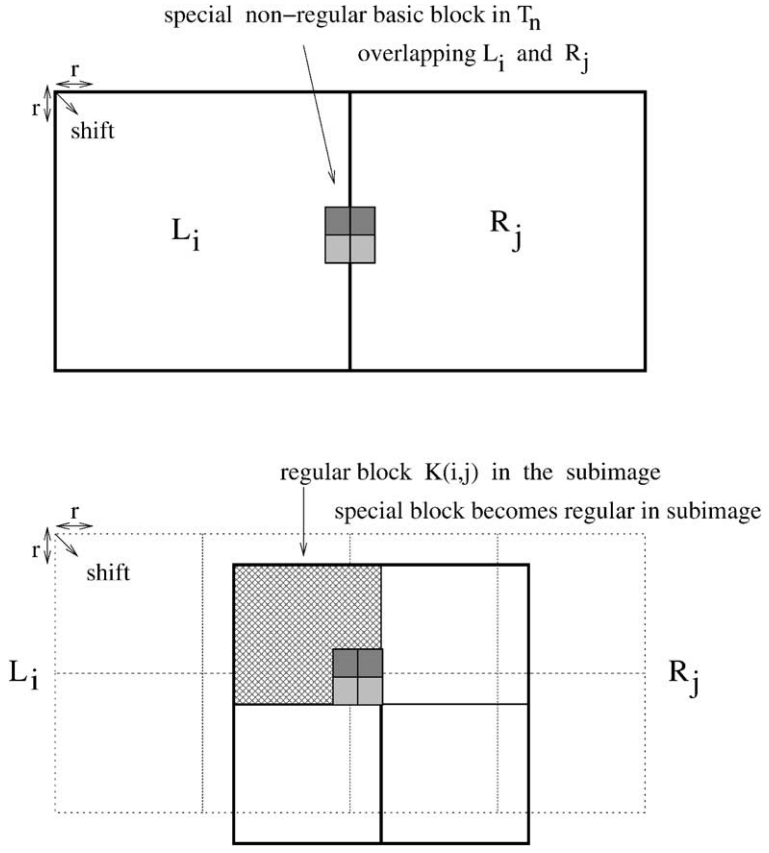

Fig. 12. The regular blocks $K(i,j)$ in $\mathscr{R}_n$ result by shifting the structure of regular blocks in T_n by the vector (r,r) over a pair of adjacent medium blocks L_i, R_j.

(3) the $2r \times 2r$ right-bottom corners of blocks in $\mathscr{K}$ are pairwise different.
(4) Each block of $\mathscr{K}$ is a regular block in $\mathscr{R}_n$.

In this moment we have shown that we have a set $\mathscr{K}$ of $\Omega(n^2)$ different regular blocks, now we show that we have in fact $\Omega(n^{2.5})$ such blocks.

For each regular block $K(i,j) \in \mathscr{K}$ we consider also all regular subblocks of $K(i,j)$ which contain the special block of $K(i,j)$. Denote the set of these blocks by $\mathscr{K}'$. More formally we can define K' as the smallest set of regular blocks of $\mathscr{R}_n$ satisfying: all blocks in $\mathscr{K}$ are in $\mathscr{K}'$, and for every block B in $\mathscr{K}'$ its bottom-right quadrant is also in $\mathscr{K}'$, if it contains (or equals) the special block of B. The basic argument is that the size of each $K(i,j)$ is $\Omega(\sqrt{n})$ larger than the size of a special block. We have $\Omega(\sqrt{n})$ different sizes of regular subblocks in $\mathscr{K}'$. Each block in $\mathscr{K}'$ is regular and these blocks are pairwise different because they differ with respect to the special block (which they contain) or with respect to the size.

We have $\Omega(n^2)$ different special blocks and $\Omega(\sqrt{n})$ sizes. Altogether $|\mathscr{K}'|=\Omega(\sqrt{n}n^2)$. Hence $\tau(\mathscr{R}_n)=\Omega(n^{2.5})$. This completes the proof. □

The following theorem is a direct consequence of the last two lemmas.

Theorem 4.4. $\Psi(n) \geqslant cn^{2.5}$.

5. Construction of compressed representation of subimages for deterministic automata

Each basic block X is identified by a composite name

$$(name(X_1), name(X_2), name(X_3), name(X_4)),$$

where $X_1,\ldots,X_4$ are regular blocks which are quadrants of X, listed in the order corresponding to Fig. 1.

The compressed size of the subsegment can grow since the number of basic blocks could be much larger than the number of regular blocks in the same image. For example, there are 2 regular blocks of rank 1 in T in Fig. 2, but 8 different basic blocks of rank 1 (including blank ones), which have the following composite names:

$$\{(s4,s4,s4,\hat{\emptyset}),\quad (s4,\hat{\emptyset},\hat{\emptyset},\hat{\emptyset}),\quad (\hat{\emptyset},\hat{\emptyset},\hat{\emptyset},\hat{\emptyset}),\quad (s4,\hat{\emptyset},s4,s4),$$
$$(s4,s4,\hat{\emptyset},s4),\quad (\hat{\emptyset},s4,\hat{\emptyset},s4)\quad (\hat{\emptyset},s4,s4,\hat{\emptyset})\quad (\hat{\emptyset},\hat{\emptyset},s4,s4)\}.$$

Theorem 5.1. *If the compression is in terms of deterministic automata, then the compressed representation of a square subsegment R of T can be computed in* $O(|Compress(T)|^{2.5})$ *time.*

Proof. Due to Lemma 3.1 for each rank t there is a vector α_t such that each regular block of rank t in the subimage R equals $Sub(\alpha_t,(A,B,C,D))$ where (A,B,C,D) is a basic block of T of rank $t+1$.

The states of the automaton A' for the subsegment are tuples $(\alpha_k,(A,B,C,D))$ where (A,B,C,D) are basic blocks of T of rank $k+1$, and α_k is a vector from Lemma 3.1. Due to Theorem 3.4 the number of basic blocks is $O(n^{2.5})$, where $n=|Compress(T)|$.

We compute names of basic blocks in a *top-down* way. This means that we construct tuples (α_t,A,B,C,D) on the basis of the tuples (α_{t+1},A,B,C,D).

We consider only these basic blocks which contain a nonblank block of the subsegment. If we know the names of basic blocks of rank $t+1$ then each basic block of rank t consists of 4 subblocks of rank t of a single basic block of rank $t+1$, see Fig. 13. Hence the number of generated tuples is of the same order as the time complexity of our algorithm. The number of generated tuples is at most the number of basic blocks of T which is bounded by $O(n^{2.5})$.

We need also to initialize the construction by finding the smallest basic block of T containing the subsegment R, see Fig. 14.

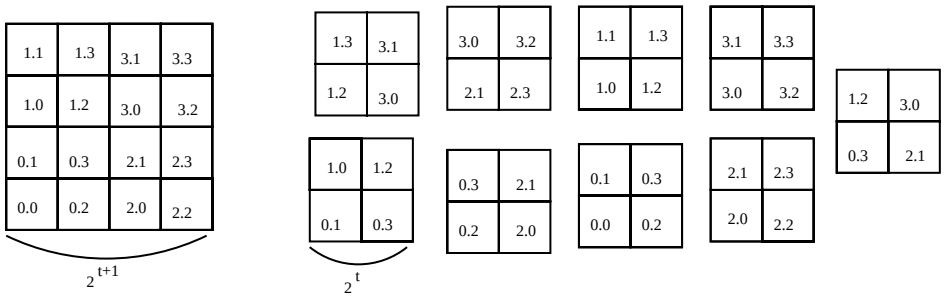

Fig. 13. The name of each basic block of rank t is derived from the name of a basic block of rank $t+1$. The t-basic blocks which are not quadrants of $t+1$-basic blocks are shown.

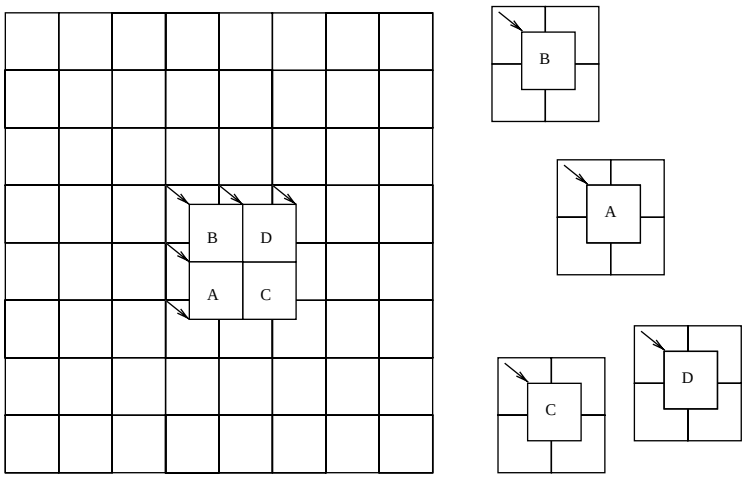

Fig. 14. Initialization of the computation of names for the work blocks of P.

The complexity is dominated by the number of regular blocks which are constructed for R, which, due to Theorem 3.4, is $O(n^{2.5})$. Hence the time complexity of the whole construction is $O(n^{2.5})$. □

6. The subsegment compression problem for weighted automata

For weighted automata, contrary to deterministic ones, the compressed size of the subsegment can grow only linearly. This surprising phenomenon is due to the fact that for weighted automata we can have many edges from the same state labeled with the same symbol, but having possibly different weights. This enables to do operations similar to matrix addition, such a trick is not possible in the deterministic case.

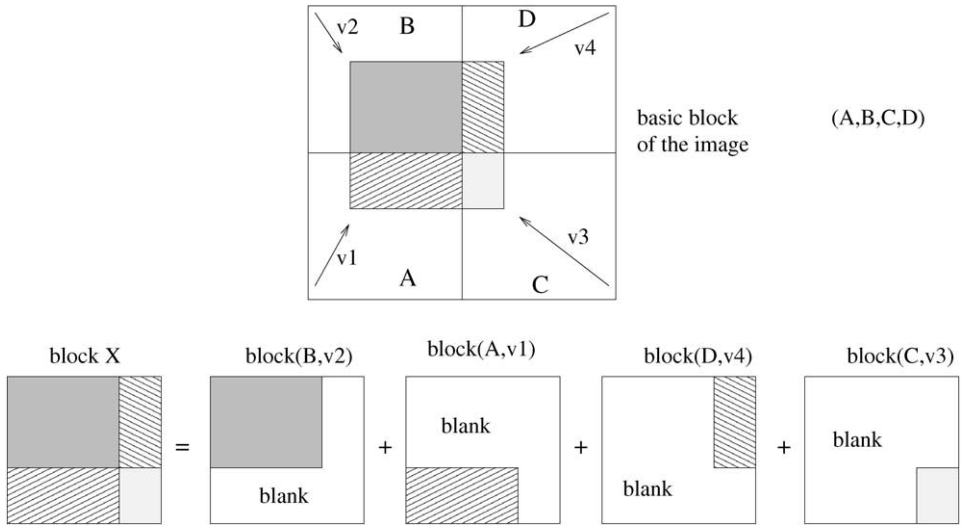

Fig. 15. Regular block X cut off from a basic block of T (as a matrix) can be treated as the sum: $X = block(A, v1) + block(B, v2) + block(C, v3) + block(D, v4)$.

A weighted finite automaton describing an image is specified by (see [12] for details):

- the set of states Q,
- the alphabet $\{0, 1, 2, 3\}$,
- weight of edges given by the function $W_a : Q \times Q \to (-\infty, \infty)$ for edges labeled by the symbol a, for $a \in \{0, 1, 2, 3\}$,
- a function $I : Q \to (-\infty, \infty)$ called the *initial distribution function* and
- a function $F : Q \to (-\infty, \infty)$ called the *final distribution function*.

The weight of a word $w = a_1 a_2 \ldots a_k$ is interpreted to give a color $W(w)$ for the pixel $entry(w)$. It is defined as $W(w) = IW_{a_1} W_{a_2} \ldots W_{a_k} F$.

Our result for weighted automata is as follows.

Theorem 6.1. *Let A be a weighted automaton defining T and having n states and m edges, and let R be a subsegment of T. Then there is a weighted automaton A' which defines R and has $O(n)$ states and $O(m)$ edges. The Subsegment Compression Problem for weighted automata for a square subsegment R can be computed in linear time.*

Proof. For a regular block Y of T and a vector v, denote by $block(Y, v)$ the square which is identical to Y on the overlap of Y with the square of the same shape shifted from the corner of Y inside Y by the vector v, all other entries are blank (contain zeros), see Fig. 15.

We identify states of A with regular blocks. For each rank t a regular block of rank t of a subsegment is located in a basic block of T, where vectors $v1, v2, v3, v4$ are as in Fig. 15.

We create the automaton A' for R. Its states are identified with $block(Y, v)$, where Y is a regular block of T and v is one of the vectors $v1, \ldots, v4$ which depend on the rank. The correctness is based on the fact that the subblock X can be treated as a matrix and it is the sum of 4 matrices, as shown in the figure. For each rank there are 4 different vectors, so the number of states of A' is $O(n)$. The construction goes top down similarly as in Theorem 5.1 using the ideas from the proof of Theorem 4.2 in [10], where *atomic dependencies* are decomposed into smaller ones. In our case atomic dependencies correspond to subblocks of the type $block(Y, v)$. □

7. Conclusions

We have shown tight bounds for the compression sizes of subimages of images compressed in terms of deterministic and weighted automata, moreover we have shown how to construct efficiently compressed representation of subimages. A first simple application is an improvement (and simplification) upon a similar result in [10], and the second one gives the first polynomial time algorithm for the compressed checking problem for weighted automata.

Fact 7.1. *There is an algorithm for compressed pattern matching for deterministic automata working in time* $O(n^2(m + \sqrt{n}))$ *where n is the compressed representation of T and m is the total size of an uncompressed pattern P.*

Proof. In the construction of the representation for a subsegment we compute basic blocks. Clearly, P occurs in T if it occurs in a basic block of rank $t + 1$, where t is the rank of P. We can construct all basic blocks of rank $t + 1$ in time $O(n^{2.5})$ using top-down algorithm which constructs a set of basic blocks of rank i on the basis of the set of basic blocks of rank $i + 1$. By Lemma 3.2 there are $O(n^2)$ basic blocks of each rank in particular of rank $t + 1$. For each of them we check (by known linear time algorithms for uncompressed two-dimensional matching) if P occurs in one of those. It costs $O(n^2 m)$ time. □

Fact 7.2. *There is a polynomial time algorithm for the fully compressed checking problem for weighted automata.*

Proof. It follows directly from the proof of Theorem 8.2 in [6] that equivalence of two weighted automata can be checked in polynomial time. We can construct the compressed representation of the subsegment R of T which is of the same shape as the pattern P and starts at the same location. Then we check the equality of the images R, P in polynomial time using the polynomial time test for the equivalence of two weighted automata. □

References

[1] P. Berman, M. Karpinski, L. Larmore, W. Plandowski, W. Rytter, On the complexity of pattern matching of highly compressed two-dimensional texts, in: Proceedings of the CPM'97, Lecture Notes in Computer Science, Vol. 1264, 1997, pp. 40–51.
[2] K. Culik, J. Karhumäki, Finite automata computing real functions, SIAM J. Comput. 23 (4) (1994) 789–814.
[3] K. Culik, J. Kari, Image compression using weighted finite automata, Comput. Graph. 17 (1993) 305–313.
[4] K. Culik, J. Kari, in: Y. Fisher (Ed.), Fractal Image Compression: Theory and Applications, Springer, Berlin, 1995, pp. 243–258.
[5] D. Derencourt, J. Karhumäki, M. Latteux, A. Terlutte, On continuous functions computed by finite automata, RAIRO Theor. Inform. Appl. 28 (1994) 387–404.
[6] S. Eilenberg, Automata, Languages and Machines, Vol. A, Academic Press, New York, 1974.
[7] M. Farach, M. Thorup, String matching in Lempel-Ziv compressed strings, in: Proc. Twenty-Seventh Annual ACM Symp. on the Theory of Computing, ACM Press, 1995, pp. 703–712.
[8] L. Gąsieniec, M. Karpiński, W. Plandowski, W. Rytter, Efficient algorithms for Lempel-Ziv encoding (extended abstract), in: Proceedings of the SWAT'96, Lecture Notes in Computer Science, Vol. 1097, 1996, pp. 392–403.
[9] J. Karhumäki, W. Plandowski, W. Rytter, Pattern-matching problems for two-dimensional images described by finite automata, Nordic J. Comput. 7 (2000) 1–13.
[10] J. Karhumäki, W. Plandowski, W. Rytter, The compression of subsegments of images described by finite automata, in: Proceedings of the CPM'99, Lecture Notes in Computer Science, Vol. 1645, pp. 186–195.
[11] J. Kari, P. Franti, Arithmetic coding of weighted finite automata, RAIRO Theor. Inform. Appl. 28 (1994) 343–360.
[12] M. Karpinski, W. Rytter, A. Shinohara, Pattern-matching for strings with short description, in: Proceedings of the CPM'95, Lecture Notes in Computer Science, Vol. 937, 1995, pp. 205–214.