Computing the Number of Cubic Runs in Standard Sturmian Words

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Abstract. The *standard Sturmian* words are extensively studied in combinatorics of words. They are enough complicated to have many interesting properties and at the same time they are highly compressible. In this paper we design an efficient algorithm computing the number of cubic runs in standard Sturmian words. Our algorithm runs in linear time with respect to the size of the compressed representation (recurrences) describing the word. The explicit size of the word can be exponential with respect to this representation. This is yet another example of a very fast computation on highly compressible texts.

Keywords: standard Sturmian words, repetitions, cubic runs, algorithms.

1 Introduction

Repetitions in strings are important in combinatorics on words and many practical applications, see for instance [6], [11], [19] and [20]. The structure of repetitions is almost completely understood for the class of Fibonacci words, see [15], [17], [24], however it is not well understood for general words.

Runs are repetitions in which the period repeats at least twice. Highly repetitive segments, in which the repetitions ratio is at least 3, called the *cubic runs*, were introduced and studied in [10]. In this paper we investigate the structure of cubic runs in class $\mathcal{S}$ of standard Sturmian words and give recurrence formulas for their number. We show also the algorithm, which computes this number in any standard word in linear time with respect to the size of its compressed representation – the directive sequence – hence in time logarithmic with respect to the length of the word.

Recall that a number i is a period of the word w if $w[j] = w[i + j]$ for all i with $i + j \leq |w|$. The minimal period of w will be denoted by $period(w)$. We say that a word w is periodic if $period(w) \leq \frac{|w|}{2}$. A word w is said to be *primitive* if w is not of the form z^k, where z is a finite word and $k \geq 2$ is a natural number.

A *maximal repetition* (a *run*, in short) in a word w is an interval $\alpha = [i..j]$ such that $w[i..j] = u^k v$ ($k \geq 2$) is a nonempty periodic subword of w, where u is of the minimal length and v is a proper prefix (possibly empty) of u, that can not be extended

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(neither $w[i-1..j]$ nor $w[i..j+1]$ is a run with the period $|u|$). *Cubic runs* are defined in the same way, but we require that the period repeat at least three times ($k \geq 3$).

A run α can be properly included as an interval in another run β, but in this case $period(\alpha) < period(\beta)$. The value of the run $\alpha = [i...j]$ is the factor $val(\alpha) = w[i...j]$. When it creates no ambiguity we identify sometimes run with its value and the period of the run $\alpha = [i...j]$ with the subword $w[i..period(w)]$ – called also the *generator* of the repetition. The meaning will be clear from the context. Observe that two different runs could correspond to the identical subwords, if we disregard their positions. Hence runs are also called the maximal *positioned* repetitions.

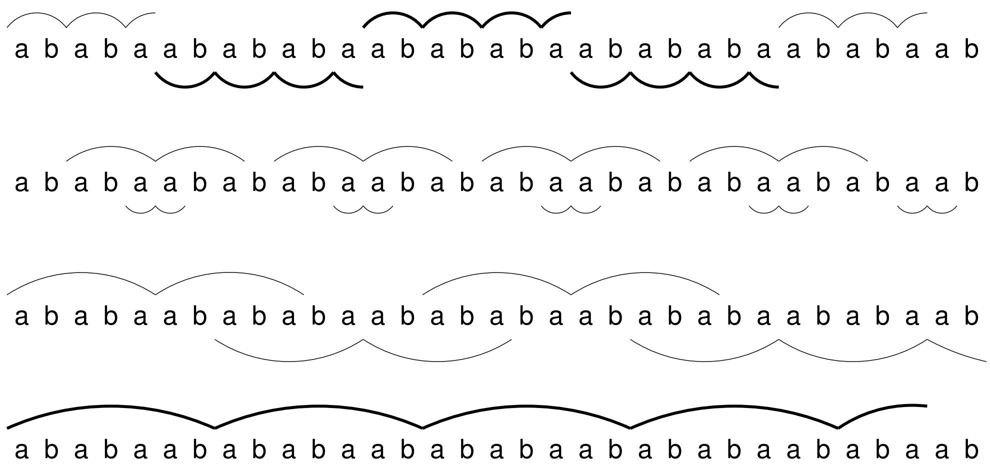

Figure 1. The structure of repetitions in the word Sw(1, 2, 1, 3, 1). There are 19 runs and 4 cubic runs (marked in **bold**).

Example 1.
Let $w = ababaabababaabababaabababaababaab$.
There are 5 runs with the period $|a|$:

$$w[5..6] = a^2, \quad w[12..13] = a^2, \quad w[19..20] = a^2,$$
$$w[26..27] = a^2, \quad w[31..32] = a^2,$$

5 runs with the period $|ab|$ (including 3 cubic runs):

$$w[1..5] = (ab)^2a, \quad w[6..12] = (ab)^3a, \quad w[13..19] = (ab)^3a,$$
$$w[20..26] = (ab)^3a, \quad w[27..31] = (ab)^2a,$$

4 runs with the period $|aba|$:

$$w[3..8] = (aba)^2, \quad w[10..15] = (aba)^2,$$
$$w[17..22] = (aba)^2, \quad w[24..29] = (aba)^2,$$

4 runs with the period $|ababa|$:

$$w[1..10] = (ababa)^2, \quad w[8..17] = (ababa)^2,$$
$$w[15..24] = (ababa)^2, \quad w[22..33] = (ababa)^2ab,$$

and 1 (cubic) run with the period $|ababaab|$:

$$w[1..31] \;=\; (ababaab)^4 aba.$$

All together we have 19 runs and 4 cubic runs , see Figure 1 for comparison.

Denote by $\rho(w)$ and $\rho^{(3)}(w)$ the number of runs and cubic runs in the word w and by $\rho(n)$ and $\rho^{(3)}(n)$ the maximal number of runs and cubic runs in the words of length n respectively. The most interesting and open conjecture about maximal repetitions is:

$$\rho(n) \;<\; n.$$

In 1999 Kolpakov and Kucherov (see [16]) showed that the number $\rho(w)$ of runs in a string w is $O(|w|)$, but the exact multiplicative constant coefficient is still unknown. The best known results related to the value of $\rho(n)$ are

$$0.944575712\, n \;\leq\; \rho(n) \leq\; 1.029\, n.$$

The upper bound is by [8], [9] and the lower bound is by [13], [14], [18], [27]. The best known results related to $\rho^{(3)}(n)$ are (due to [10]):

$$0.41\, n \;\leq\; \rho^{(3)}(n) \;\leq\; 0.5\, n.$$

For the class $\mathcal{S}$ of standard Sturmian words there are known exact formulas for the number of runs and squares and their asymptotic behavior, see [2] and [22] for details. In this case we have

$$\lim_{n\to\infty} \frac{\rho(n)}{n} \;=\; 0.8.$$

This paper is devoted to the investigation of the structure of cubic runs in standard Sturmian words. We present exact formulas for $\rho^{(3)}(w)$ and the algorithm computing $\rho^{(3)}(w)$ for any word $w \in \mathcal{S}$ in linear time with respect to the compressed representation of w (logarithmic time with respect to the length of w). Some useful applets related to problems considered in this paper can be found on the web site:

`http://www.mat.umk.pl/~martinp/stringology/applets/`

2 Standard Sturmian words

Standard Sturmian words (standard words in short) are one of the most investigated class of strings in combinatorics on words, see for instance [1], [4], [5], [7], [19], [25], [26], [28] and references therein. They have very compact representations in terms of sequences of integers, which has many algorithmic consequences.

The *directive sequence* is the integer sequence: $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$, where $\gamma_0 \geq 0$ and $\gamma_i > 0$ for $i = 1, 2, \ldots, n$. The standard word corresponding to γ, denoted by $\mathrm{Sw}(\gamma)$, is described by the recurrences of the form:

$$\begin{aligned}
&x_{-1} = b, && x_0 = a,\\
&x_1 = x_0^{\gamma_0} x_{-1}, && x_2 = x_1^{\gamma_1} x_0,\\
&\vdots && \vdots\\
&x_n = x_{n-1}^{\gamma_{n-1}} x_{n-2}, && x_{n+1} = x_n^{\gamma_n} x_{n-1},
\end{aligned} \tag{1}$$

where $\mathrm{Sw}(\gamma) = x_{n+1}$.

The sequence of words $\{x_i\}_{i=0}^{n+1}$ is called the standard sequence. Every word occurring in a standard sequence is a standard word, and every standard word occurs in some standard sequence. We assume that the standard word given by the empty directive sequence is a and $\mathrm{Sw}(0) = b$. The class of all standard words is denoted by $\mathcal{S}$.

Example 2.
Consider the directive sequence $\gamma = (1, 2, 1, 3, 1)$. We have:

$$\begin{aligned}
x_{-1} &= b \\
x_0 &= a \\
x_1 &= (x_0)^1 \cdot x_{-1} = a \cdot b \\
x_2 &= (x_1)^2 \cdot x_0 \quad = ab \cdot ab \cdot a \\
x_3 &= (x_2)^1 \cdot x_1 \quad = ababa \cdot ab \\
x_4 &= (x_3)^3 \cdot x_2 \quad = ababaab \cdot ababaab \cdot ababaab \cdot ababa \\
x_5 &= (x_4)^1 \cdot x_3 \quad = ababaababaababaababaababa \cdot ababaab
\end{aligned}$$

and finally

$$\mathrm{Sw}(1, 2, 1, 3, 1) \;=\; ababaababaababaababaababaababaab.$$

For $\gamma_0 > 0$ we have standard words starting with the letter a and for $\gamma_0 = 0$ we have standard words starting with the letter b. In fact the word $\mathrm{Sw}(0, \gamma_1, \ldots, \gamma_n)$ can be obtained from $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$ by switching the letters a and b.

Observe that for even $n > 0$ the standard word x_n has the suffix ba, and for odd $n > 0$ it has the suffix ab. Moreover, every standard word consists either of repeated occurrences of the letter a separated by single occurrences of the letter b or repeated occurrences of the letter b separated by single occurrences of the letter a. Those letters are called the *repeating letter* and the *single letter*, respectively. If the repeating letter is a (letter b respectively), the word is called the Sturmian word of the type a (type b respectively), see the definition 6.1.4 in [23] for comparison.

Remark 3.
Without loss of generality we consider here the standard Sturmian words of the type a, therefore we assume that $\gamma_0 > 0$. The words of the type b can be considered similarly and all the results hold.

The number $N = |\mathrm{Sw}(\gamma)|$ is the (real) size of the word, while $(n + 1) = |\gamma|$ can be thought as its compressed size. Observe that, by the definition of standard words, N is exponential with respect to n. Each directive sequence corresponds to a *grammar-based compression*, which consists in describing a given word by a context-free grammar G generating this (single) word. The size of the grammar G is the total length of all productions of G. In our case the size of the grammar is proportional to the length of the directive sequence.

3 Morphic reduction of standard words

The recurrent definition of standard words leads to the simple characterization by the composition of morphisms. Let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be a directive sequence. We associate with γ a sequence of morphisms $\{h_i\}_{i=0}^{n}$, defined as

$$h_i : \begin{cases} a \longrightarrow a^{\gamma_i} b \\ b \longrightarrow a \end{cases} \qquad \text{for } 0 \le i \le n. \tag{2}$$

Lemma 4.
For $0 \le i \le n$ the morphism h_i transforms a standard word into another standard word, and we have:

$$\begin{aligned} \mathrm{Sw}(\gamma_n) &= h_n(a), \\ \mathrm{Sw}(\gamma_i, \gamma_{i+1}, \ldots, \gamma_n) &= h_i(\mathrm{Sw}(\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_n)). \end{aligned}$$

Proof.
We will prove the lemma by the induction on the length of the directive sequence.

Recall that the standard word given by the empty directive sequence is a. For $|\gamma| = 1$ we have, by definition of standard words and the morphism h_n,

$$\mathrm{Sw}(\gamma_n) \;=\; a^{\gamma_n} b \;=\; h_n(a).$$

Assume now that $|\gamma| = k \ge 2$ and for directive sequences shorter than k the thesis holds. We have then:

$$\begin{aligned} \mathrm{Sw}(\gamma_i, \ldots, \gamma_n) &= [\mathrm{Sw}(\gamma_i, \ldots, \gamma_{n-1})]^{\gamma_n} \cdot \mathrm{Sw}(\gamma_i, \ldots, \gamma_{n-2}) \\ &\overset{ind.}{=} \Big[h_i\Big(\mathrm{Sw}(\gamma_{i+1}, \ldots, \gamma_{n-1})\Big)\Big]^{\gamma_n} \cdot h_i\Big(\mathrm{Sw}(\gamma_{i+1}, \ldots, \gamma_{n-2})\Big) \\ &= h_i\Big([\mathrm{Sw}(\gamma_{i+1}, \ldots, \gamma_{n-1})]^{\gamma_n} \cdot \mathrm{Sw}(\gamma_{i+1}, \ldots, \gamma_{n-2})\Big) \\ &= h_i\Big(\mathrm{Sw}(\gamma_{i+1}, \ldots, \gamma_n)\Big), \end{aligned}$$

which concludes the proof.

As a direct conclusion from Lemma 4 we have that the standard word corresponding to the directive sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ is given as:

$$\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n) \;=\; h_0 \circ h_1 \circ \ldots \circ h_n(a). \tag{3}$$

The inverse morphism h_i^{-1} can be seen as a reduction of the word $\mathrm{Sw}(\gamma_i, \ldots, \gamma_n)$ to the word $\mathrm{Sw}(\gamma_{i+1}, \ldots, \gamma_n)$ and allows us to reduce the computation of cubic runs in $\mathrm{Sw}(\gamma_i, \ldots, \gamma_n)$ to the same computation in $\mathrm{Sw}(\gamma_{i+1}, \ldots, \gamma_n)$.

Recall that $|w|_a$ denotes the number of occurrences of the letters a in the word w. We define the function, which will be useful in the rest of this paper. For a directive sequence $\gamma = (\gamma_0, \ldots, \gamma_n)$ and an integer $0 \le k \le n+1$ we define

$$N_\gamma(k) = |S(\gamma_k, \gamma_{k+1}, \ldots, \gamma_n)|_a, \tag{4}$$

Moreover, for $k > n+1$, we define $N_\gamma(k) = 0$.

Remark 5.
As a direct conclusion from the above definition, the equation (1) and the equation (2) we have that for $0 \leq k \leq n$ the numbers $N_\gamma(k)$ satisfy:

$$N_\gamma(k) = \gamma_k \, N_\gamma(k+1) + N_\gamma(k+2), \tag{5}$$

Example 6.
Let $\gamma = (1,2,1,3,1)$ be a directive sequence. We have then

$$\begin{aligned}
\mathrm{Sw}(1,2,1,3,1) &= ababaababababaabababaababababaababaab & N_\gamma(0) &= 19 \\
\mathrm{Sw}(2,1,3,1) &= aabaaabaaabaaabaaba & N_\gamma(1) &= 14 \\
\mathrm{Sw}(1,3,1) &= abababaab & N_\gamma(2) &= 5 \\
\mathrm{Sw}(3,1) &= aaaba & N_\gamma(3) &= 4 \\
\mathrm{Sw}(1) &= ab & N_\gamma(4) &= 1 \\
\mathrm{Sw}(\varepsilon) &= a & N_\gamma(5) &= 1
\end{aligned}$$

4 Formulas for the number of cubic runs

In this section we present and prove formulas for the number of cubic runs in standard words. The following zero-one functions for testing the parity of a nonnegative integer i will be useful to simplify those formulas:

$$even(i) = \begin{cases} 1 & \text{for even } i \\ 0 & \text{for odd } i \end{cases} \quad \text{and} \quad odd(i) = \begin{cases} 1 & \text{for odd } i \\ 0 & \text{for even } i \end{cases}.$$

We begin with the characterization of possible periods of cubic runs in standard words. The following lemma is a consequence of the very special structure of subword graphs (especially their compacted versions) of standard words. See [3] and [25] for more information.

Lemma 7.
The period of each cubic run in the standard word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ *is of the form* x_i, *where* x_i*'s are as in equation (1).*

To prove the above lemma it is sufficient to show that no factor of the word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$, that does not satisfy the condition given there, could be the generator of a cubic runs. We can use similar argumentation as in proof of Theorem 1 in [12]. The details are omitted in this version.

The main idea of the computation of cubic runs in a standard word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ is the partition of them into three separate categories depending on the length of their periods. We say that a cubic run is:

short – with the periods x_0 and x_1,
medium – with the period x_2,
large – with the periods x_i, for $i > 2$.

Denote by $\rho_S^{(3)}(w)$, $\rho_M^{(3)}(w)$ and $\rho_L^{(3)}(w)$ the number of short, medium and large cubic runs in the word w, respectively. We will consider each type separately.

Example 8.
Recall the word $w = \mathrm{Sw}(1, 2, 1, 3, 1)$ from Example 1. In this case we have:
- 3 short runs (period ab),
- no medium run,
- 1 large run (the period $ababaab$),

see Figure 1 for comparison.

4.1 Short runs

We start with the computation of the *short* cubic runs. These are the cubic runs with the periods of the form a or $a^k b$. Their number depends on the values of γ_0 and γ_1.

Lemma 9.
The number $\rho^{(3)}_{S_1}$ of cubic runs with the period a in the word $w = \mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n)$ equals:

$$\rho^{(3)}_{S_1}(w) \;=\; \begin{cases} 0 & \text{for } \gamma_0 = 1 \\ N_\gamma(2) - odd(n) & \text{for } \gamma_0 = 2 \\ N_\gamma(1) & \text{for } \gamma_0 > 2 \end{cases} . \tag{6}$$

Proof.
First assume that $\gamma_0 > 2$. Every cubic run with the period a in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ equals a^{γ_0} or a^{γ_0+1} and is followed by the single letter b. Due to Lemma 4 every such cubic run in $\mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n)$ corresponds to the letter a in $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$. Hence in this case we have $N_\gamma(1)$ cubic runs with the period a.

Assume now that $\gamma_0 = 2$. In this case the word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ consists of the blocks of the two types: aab and $aaab$. Only the blocks of the second type include the cubic run with the period a. Due to Lemma 4 every such cubic run in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ corresponds to the letter b followed by the letter a in $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$. Hence the number of such cubic runs equals the number of blocks ba in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$.

Recall that for an even length of the directive sequence $|(\gamma_1, \ldots, \gamma_n)|$ (n is even) the word $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$ ends with ba and in this case the number of cubic runs with the period a in $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$ equals the number of the letters b in $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$, namely $N_\gamma(2)$. For an odd length of the directive sequence $|(\gamma_1, \ldots, \gamma_n)|$ (n is odd) the word $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$ ends with ab and the last letter b does not correspond to a cubic run in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$. In this case, the number of runs with the period a in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ is one less than the number of the letters b in $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$, namely $N_\gamma(2) - 1$.

Finally assume that $\gamma_0 = 1$. In this case the word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ consists of the blocks of the two types: ab and aab. None of them includes a cubic run with the period a. This completes the proof.

Lemma 10.
The number $\rho^{(3)}_{S_2}$ of cubic runs with the period $a^k b$ in the word $w = \mathrm{Sw}(\gamma_0, \gamma_1, \ldots, \gamma_n)$ equals:

$$\rho^{(3)}_{S_2}(w) \;=\; \begin{cases} 0 & \text{for } \gamma_1 = 1 \\ N_\gamma(3) - even(n) & \text{for } \gamma_1 = 2 \\ N_\gamma(2) & \text{for } \gamma_1 > 2 \end{cases} . \tag{7}$$

Proof.
Notice that, due to the equation (2) and Lemma 4, cubic runs with the periods $a^{\gamma_0}b$ and $a^{\gamma_0+1}b$ in $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$ correspond to the runs with the period a in $\mathrm{Sw}(\gamma_1, \dots, \gamma_n)$. Similar reasoning as above establishes the desired formula.

4.2 Medium runs

Recall that the cubic run is called the *medium* if it has the period of the form x_2. Observe that the medium cubic runs appear in the standard words generated by the directive sequences of the length at least 3. We have to consider two cases: the directive sequences of the length 3 and the longer directive sequences. The values of γ_0 and γ_1 does not affect the number of the medium cubic runs, hence to simplify the calculations we can assume in further proofs that $\gamma_0 = \gamma_1 = 1$.

We start with counting the medium runs in the standard words generated by the directive sequences of the length greater than 3.

Lemma 11.
Let $w = \mathrm{Sw}(\gamma_0, \dots, \gamma_n)$ be a standard word and $n \geq 3$. The number of medium cubic runs in w equals:

$$\rho^{(3)}_M(w) \;=\; \begin{cases} N_\gamma(4) - 1 & \text{for } \gamma_2 = 1 \\ N_\gamma(3) & \text{for } \gamma_2 \geq 2 \end{cases}. \tag{8}$$

Proof.
We start with assumption that $\gamma_2 > 2$. In this case every factor of the form $x_3 = x_2^{\gamma_2}x_1$ includes one cubic runs with the period x_2. Hence the number of such cubic runs equals the number factors x_3 in $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$, namely $N_\gamma(3)$ (due to Lemma 4).

Assume now that $\gamma_2 = 2$. The word $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$ can be represented as a sequence of concatenated words x_3 and x_2 and has the form:

$$x_3^{\alpha_1}x_2x_3^{\alpha_2}x_2\dots x_3^{\alpha_s}x_2x_3 \qquad \text{or} \qquad x_3^{\beta_1}x_2x_3^{\beta_2}x_2\dots x_3^{\beta_s}x_2.$$

Observe that $x_3 = x_2x_2x_1$ and every occurrence of x_3 in $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$ either follows some occurrence of x_2 or is followed by some occurrence of x_2. In the first case we have $x_2 \cdot x_3 = x_2 \cdot x_2x_2x_1$ and there is a cubic run with period x_2. In the second case we have $x_3 \cdot x_2 = x_2x_2x_1 \cdot x_2$, and there is also a cubic run with period x_2, since x_1 is a prefix of x_2. Therefore the number of medium cubic runs in this case equals the number of the factors x_3 in $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$, namely $N_\gamma(3)$.

Finally assume that $\gamma_2 = 1$. The word $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$ can be represented as a sequence of concatenated words x_3 and x_4 and has the form:

$$x_4^{\alpha_1}x_3x_4^{\alpha_2}x_3\dots x_4^{\alpha_s}x_3x_4 \qquad \text{or} \qquad x_4^{\beta_1}x_3x_4^{\beta_2}x_3\dots x_4^{\beta_s}x_3.$$

We have $x_3 = x_2x_1$ and $x_4 = x_2x_1 \cdot \ldots \cdot x_2x_1 \cdot x_2$. Therefore only the last one occurrence of x_4 in $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$ does not correspond to a cubic run with the period x_2 and we have $N_\gamma(4) - 1$ such cubic runs in this case. This completes the proof.

Lemma 12.
The number of medium cubic runs in the standard word $w = \mathrm{Sw}(\gamma_0, \gamma_1, \gamma_2)$ *equals:*

$$\rho_M^{(3)}(w) \;=\; \begin{cases} 1 & \text{for } \gamma_2 > 2 \\ 0 & \text{for } \gamma_2 \le 2 \end{cases}. \tag{9}$$

Proof.
We have $\mathrm{Sw}(\gamma_0, \gamma_1, \gamma_2) \;=\; x_2^{\gamma_2} x_1$. Hence there is only one medium run (with period x_2) if $\gamma_2 > 2$ and no medium run otherwise.

4.3 Large runs

Recall that a cubic run is called *large* if it has the period of the length greater than $|x_2|$, where x_2 is as in the equation (1). We reduce the problem of counting the large cubic runs to the one for counting the medium cubic runs, using the morphic representation of the standard words introduced in the previous section.

Let h be a morphism and let $v = a_1 a_2 \dots a_k$ be the word of the length k. The morphism h defines the partition of the word $w = h(v)$ into segments $h(a_1)$, $h(a_2)$,..., $h(a_t)$. These segments are called the *h-blocks*. We say that a factor x of the word w is *synchronized* with the morphism h in w if and only if each occurrence of x in w starts at the beginning of some h-block and ends at the end of some h-block. Observe that every factor in w that is synchronized with h corresponds to some factor in v, hence the morphism h preserves the structure of the factors that are synchronized with it.

Example 13.
Let $w = \mathrm{Sw}(1, 2, 1, 3, 1)$ and $v = \mathrm{Sw}(2, 1, 3, 1)$ be standard words and h_0 be the morphism defined as:

$$h_0 : \begin{cases} a \longrightarrow ab \\ b \longrightarrow a \end{cases}.$$

Recall that

$$\begin{aligned} \mathrm{Sw}(1, 2, 1, 3, 1) &= h_0(\mathrm{Sw}(2, 1, 3, 1)), \\ \mathrm{Sw}(1, 2, 1, 3, 1) &= ababaababababaababababaabababaababaab, \\ \mathrm{Sw}(2, 1, 3, 1) &= aabaaabaaabaaabaaba. \end{aligned}$$

The factors $w[6..8] = aba$ and $w[13..17] = abaab$ are not synchronized with the morphism h_0, because both of them ends in the middle some h_0-block. The factor $w[22..28]$ starts at the beginning of some h_0-block and ends at the end of some h_0-block, hence is synchronized with the morphism h_0. Moreover it corresponds with the factor $v[13..16] = aaba$, see Figure 2 for comparison.

Lemma 14.
The periods of the large *cubic runs in the word* $\mathrm{Sw}(\gamma_0, \dots, \gamma_n)$ *are synchronized with the morphism* h_0.

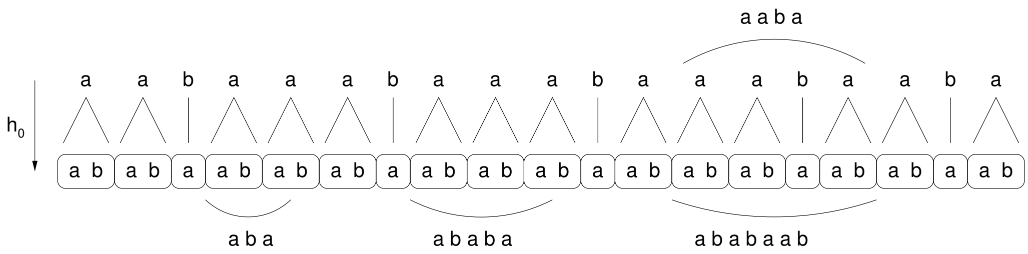

Figure 2. The factors *aba* and *ababa* do not synchronize with the morphism h_0 in the word $\mathrm{Sw}(1,2,1,3,1)$, while the factor *ababaab* (in fact the period of the large cubic run) is synchronized with h_0 and corresponds to the factor *aaba* in the word $\mathrm{Sw}(2,1,3,1)$.

Proof.
Let h_0 be the morphism defined as

$$h_0 : \begin{cases} a \longrightarrow a^{\gamma_0}b \\ b \longrightarrow a \end{cases} .$$

Due to Lemma 4 we have

$$\mathrm{Sw}(\gamma_0,\ldots,\gamma_n) \;=\; h_0(\mathrm{Sw}(\gamma_1,\ldots,\gamma_n)).$$

Moreover, h_0 determines the partition of $\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)$ into h_0-blocks of the form $a^{\gamma_0}b$ and a, see Figure 2 for the partition of $\mathrm{Sw}(1,2,1,3,1)$.

Recall that the period of each large cubic run in $\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)$ is of the form x_i, where $i \geq 3$. By the definition of standard words the factor x_i starts with $a^{\gamma_0}b$, hence at the beginning of some h_0-block.

For odd $i \geq 3$ the subword x_i ends with $x_1 = a^{\gamma_0}b$, hence at the end of some h_0-block, and is obviously synchronized with h_0.

For even $i \geq 3$ the factor x_i ends with

$$x_3 \cdot x_2 \;=\; x_2^{\gamma_2}x_1 \cdot x_1^{\gamma_1}x_0 \;=\; x_2^{\gamma_2} \cdot (a^{\gamma_0}b)^{\gamma_1+1}a.$$

First assume that x_i is followed by the block $a^{\gamma_0}b$. The single letter a at the end of x_i is then the whole h_0-block and x_i is synchronized with the morphism h_0.

Assume now that x_i ends with $(a^{\gamma_0}b)^{\gamma_1+1}a$ and is followed by $(a^{\gamma_0-1}b)$, namely it ends in the middle of some h_0-block. In this case we have the occurrence of the factor $(a^{\gamma_0}b)^{\gamma_1+2}$ in $\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)$, which is reduced by the morphism h_0^{-1} to the factor $a^{\gamma_1+2}b$ in $\mathrm{Sw}(\gamma_1,\ldots,\gamma_n)$. By definition the standard word $\mathrm{Sw}(\gamma_1,\ldots,\gamma_n)$ can include only the blocks of the two types: the short block – $a^{\gamma_1}b$ and the long block – $a^{\gamma_1+1}b$, hence we have the contradiction and the proof is complete.

The following lemma, which is a direct conclusion from Lemma 14, allows us to reduce the problem of counting the large cubic runs in the standard word $\mathrm{Sw}(\gamma_0,\ldots,\gamma_n)$ to those in $\mathrm{Sw}(\gamma_1,\ldots,\gamma_n)$.

Lemma 15.
Let $w = \mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ *and* $v = \mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$ *be standard words. The number of* large *cubic runs in* w *is given by the recurrence*

$$\rho_L^{(3)}(w) \;=\; \rho_L^{(3)}(v) \;+\; \rho_M^{(3)}(v).$$

Proof.
Lemma 14 implies that the morphism defined in the equation (2) preserves the structure of the long cubic runs in standard words. Recall that the word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ is reduced by h_0^{-1} to the word $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$. Therefore, every large cubic run α in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ corresponds to some cubic run β in $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$.

Due to Lemma 7 the period of the cubic run α is of the form x_i, where $i \geq 3$. The corresponding cubic run β is either large (for $i > 3$) or medium (for $i = 3$). Hence to compute all large cubuc runs in $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ it is sufficient to compute all large and medium cubic runs in $\mathrm{Sw}(\gamma_1, \ldots, \gamma_n)$.

Proposition 16.
The number of cubic runs in a standard word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ *can be computed by combining the formulas (6), (7), the formula (8) repeated* $n-2$ *times, and finally the formula (9).*

Example 17.
Consider a directive sequence $\gamma = (1, 2, 1, 3, 1)$. We compute the number of cubic runs in the word $\mathrm{Sw}(1, 2, 1, 3, 1)$ using the formulas mentioned above. In this case we have:

cubic runs with the period a:	0
cubic runs with the period $a^k b$:	$\lvert aaaba\rvert_a - 1 = 3$
medium cubic runs:	$\lvert ab\rvert_a - 1 = 0$
large cubic runs:	$\rho_M^{(3)}(2, 1, 3, 1) + \rho_M^{(3)}(1, 3, 1) = \lvert ab\rvert_a + 0 = 1$

Altogether there is 4 cubic runs in $\mathrm{Sw}(1, 2, 1, 3, 1)$, see Example 1 for comparison.

4.4 Algorithm for computation of cubic runs

The formulas investigated above allow us to develop an efficient algorithm that computes the number of cubic runs in any standard Sturmian word.

Theorem 18.
We can count the number of cubic runs in any standard word $\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)$ *in linear time with respect to the length of the directive sequence* $|\gamma|$ *(logarithmic time with respect to the length of the whole word* $|\mathrm{Sw}(\gamma_0, \ldots, \gamma_n)|$*).*

Proof.
The formulas (6), (7), (8) and (9) for the number of cubic runs in a standard words $\mathrm{Sw}(\gamma)$ depend directly on the components of the directive sequence γ and the numbers $N_\gamma(k)$. We compute the numbers $N_\gamma(n)$, $N_\gamma(n-1)$, ..., $N_\gamma(1)$ consecutively iterating the equation (1). In each step i of the computation we remember the number of cubic runs related to the value of the γ_i. The algorithm performs n iteration, hence it has the time complexity $\mathrm{O}(|\gamma|)$.

Algorithm 1: *Counting-Cubic-Runs*(γ)

```
(x, y, cr) ⟵ (1, 0, 0);
if γ_n > 2 then  cr ⟵ cr + 1;
for i := n to 3 do
    (x, y) ⟵ (γ_i · x + y, x);
    if γ_{i−1} ≥ 2 then  cr ⟵ cr + x;
    else  cr ⟵ cr + y − 1;
if γ_1 = 2 then
    cr ⟵ cr + x;
    if n is even then cr ⟵ cr − 1;
(x, y) ⟵ (γ_2 · x + y, x);
if γ_1 > 2 then  cr ⟵ cr + x;
if γ_0 = 2 then
    cr ⟵ cr + x;
    if n is odd then cr ⟵ cr − 1;
if γ_0 > 2 then  cr ⟵ cr + γ_1 · x + y;
return cr;
```

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