On Periodicity Lemma for Partial Words*

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Abstract

We investigate the function $L(h,p,q)$, called here the *threshold function*, related to periodicity of partial words (words with holes). The value $L(h,p,q)$ is defined as the minimum length threshold which guarantees that a natural extension of the periodicity lemma is valid for partial words with h holes and (strong) periods p,q. We show how to evaluate the threshold function in $\mathcal{O}(\log p + \log q)$ time, which is an improvement upon the best previously known $\mathcal{O}(p+q)$-time algorithm. In a series of papers, the formulae for the threshold function, in terms of p and q, were provided for each fixed $h \le 7$. We demystify the generic structure of such formulae, and for each value h we express the threshold function in terms of a piecewise-linear function with $\mathcal{O}(h)$ pieces.

1 Introduction

Consider a word X of length $|X| = n$, with its positions numbered 0 through $n-1$. We say that X has a period p if $X[i] = X[i+p]$ for all $0 \le i < n-p$. In this case, the prefix $P = X[0..p-1]$ is called a *string period* of X. Our work can be seen as a part of the quest to extend Fine and Wilf's Periodicity Lemma [11], which is a ubiquitous tool of combinatorics on words, into partial words.

Lemma 1.1 (Periodicity Lemma [11])**.** *If p,q are periods of a word X of length $|X| \ge p+q-\gcd(p,q)$, then $\gcd(p,q)$ is also a period of X.*

A partial word is a word over the alphabet $\Sigma \cup \{\diamondsuit\}$, where $\diamondsuit$ denotes a hole (a don't care symbol). In what follows, by n we denote the length of the partial word and by h the number of holes. For $a,b \in \Sigma \cup \{\diamondsuit\}$, the relation of matching $\approx$ is defined so that $a \approx b$ if $a = b$ or either of these symbols is a hole. A (solid) word P of length p is a *string period* of a partial word X if $X[i] \approx P[i \bmod p]$ for $0 \le i < n$. In this case, we say that the integer p is a *(strong) period* of X.

We aim to compute the optimal thresholds $L(h,p,q)$ which make the following generalization of the periodicity lemma valid:

Lemma 1.2 (Periodicity Lemma for Partial Words)**.** *If X is a partial word with h holes with periods p,q and $|X| \ge L(h,p,q)$, then $\gcd(p,q)$ is also a period of X.*

If $\gcd(p,q) \in \{p,q\}$, then Lemma 1.2 trivially holds for each partial word X. Otherwise, as proved by Fine and Wilf [11], the threshold in Lemma 1.1 is known to be optimal, so $L(0,p,q) = p+q-\gcd(p,q)$.

Example 1.3. $L(1,5,7) = 12$, because:

- each partial word of length at least 12 with one hole and periods 5, 7 has also period $1 = \gcd(5,7)$,
- the partial word ababaababa$\diamondsuit$ of length 11 has periods 5, 7 and does not have period 1.

As our main aim, we examine the values $L(h,p,q)$ as a function of p,q for a given h. Closed-form formulae for $L(h,\cdot,\cdot)$ with $h \le 7$ were given in [2, 5, 22]. In these cases, $L(h,p,q)$ can be expressed using a constant number of functions linear in p, q, and $\gcd(p,q)$. We discover a common pattern in such formulae which lets us derive a closed-form formula for $L(h,p,q)$ with arbitrary fixed h using a sequence of $\mathcal{O}(h)$ fractions. Our construction relies on the theory of continued fractions; we also apply this link to describe $L(h,p,q)$ in terms of standard Sturmian words.

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h	$L(h,5,7)$	example of length $L(h,5,7)-1$
0	11	ababaababa
1	12	ababaababa♢
2	16	ababaababa♢♢aba
3	19	aaaabaaaa♢a♢aa♢aaa
4	21	aba♢♢ababaababa♢♢aba
5	25	aaaabaaaa♢a♢aa♢aaa♢♢aaaa

n :	10	**11**	**12**	13	14	15	**16**	17	18	**19**	20	**21**	22	23	24	**25**
$H(n,5,7)$:	0	1	2	2	2	2	3	3	3	4	4	5	5	5	5	6

Table 1: The optimal non-unary partial words with periods 5,7 and $h = 0, \dots, 5$ holes (of length $L(h,5,7)-1$) and the values $H(n,5,7)$ for $n = 10, \dots, 25$.

As an intermediate step, we consider a dual *holes function* $H(n,p,q)$, which gives the minimum number of holes h for which there is a partial word of length n with h holes and periods p, q which do not satisfy Lemma 1.2.

Example 1.4. We have $H(11,5,7) = 1$ because:

- $H(11,5,7) \geq 1$: due to the classic periodicity lemma, every solid word of length 11 with periods 5 and 7 has period $1 = \gcd(5,7)$, and
- $H(11,5,7) \leq 1$: ababaababa♢ is non-unary, has one hole and periods 5, 7.

We have $H(12,5,7) \leq H(11,5,7) + 1 = 2$ since appending ♢ preserves periods. In fact $H(12,5,7) = H(15,5,7) = 2$. However, there is no non-unary partial word of length 16 with 2 holes and periods 5, 7, so $L(2,5,7) = 16$; see Table 1.

For a function $f(n,p,q)$ monotone in n, we define its *generalized inverse* as:

$$\widetilde{f}(h,p,q) \;=\; \min\{n : f(n,p,q) > h\}.$$

Observation 1.5. $L = \widetilde{H}$.

As observed above, Lemma 1.2 becomes trivial if $p \mid q$. The case of $p \mid 2q$ is known to be special as well, but it has been fully described in [22]. Furthermore, it was shown in [5, 21] that the case of $\gcd(p,q) > 1$ is easily reducible to that of $\gcd(p,q) = 1$. We recall these existing results in Section 4, while in the other sections we assume that $\gcd(p,q) = 1$ and $p, q > 2$.

Previous results The study of periods in partial words was initiated by Berstel and Boasson [2], who proved that $L(1,p,q) = p+q$. They also showed that the same bound holds for *weak* periods[1] p and q. Shur and Konovalova [22] developed exact formulae for $L(2,p,q)$ and $L(h,2,q)$, and an upper bound for $L(h,p,q)$. A formula for $L(h,p,q)$ with small values h was shown by Blanchet-Sadri et al. [3], whereas for large h, Shur and Gamzova [21] proved that the optimal counterexamples of length $L(h,p,q)-1$ belong to a very restricted class of *special arrangements*. The latter contribution leads to an $\mathcal{O}(p+q)$-time algorithm for computing $L(h,p,q)$. An alternative procedure with the same running time was shown by Blanchet-Sadri et al. [5], who also stated closed-form formulae for $L(h,p,q)$ with $h \leq 7$. Weak periods were further considered in [4, 6, 23].

Other known extensions of the periodicity lemma include a variant with three [8] and an arbitrary number of specified periods [13, 24], the so-called new periodicity lemma [1, 10], a periodicity lemma for repetitions with morphisms [17], extensions into abelian [9] and k-abelian [14] periodicity, into abelian periodicity for partial words [7], into bidimensional words [18], and other variations [12, 19].

Our results First, we show how to compute $L(h,p,q)$ using $\mathcal{O}(\log p + \log q)$ arithmetic operations, improving upon the state-of-the-art complexity $\mathcal{O}(p+q)$.

[1] An integer p is a weak period of X if $X[i] \approx X[i+p]$ for all $0 \leq i < n-p$.

Furthermore, for any fixed h in $\mathcal{O}(h \log h)$ time we can compute a compact description of the threshold function $L(h,p,q)$. For the base case of $p<q$, $\gcd(p,q)=1$, and $h<p+q-2$, the representation is piecewise linear in p and q. More precisely, the interval $[0,1]$ can be split into $\mathcal{O}(h)$ subintervals I so that $L(h,p,q)$ restricted to $\frac{p}{q} \in I$ is of the form $a \cdot p + b \cdot q + c$ for some integers a, b, c.

Overview of the paper We start by introducing two auxiliary functions H^s and H^d which correspond to two restricted families of partial words. Our first key step is to prove that the value $H(n,p,q)$ is always equal to $H^s(n,p,q)$ or $H^d(n,p,q)$ and to characterize the arguments n for which either case holds. The final function L is then obtained as a combination of the generalized inverses L^s and L^d of H^s and H^d, respectively. Developing the closed-form formula for L^d requires considerable effort; this is where continued fractions arise.

2 Functions H^s and L^s

For relatively prime integers p, q, $1<p<q$, and an integer $n \geq q$, let us define

$$H^s(n,p,q) = \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n-q+1}{p} \right\rfloor .$$

We shall prove that $H(n,p,q) \leq H^s(n,p,q)$ for a suitable range of lengths n.

Fine and Wilf [11] constructed a word of length $p+q-2$ with periods p and q and without period 1. For given p, q we choose such a word $S_{p,q}$ and, we define a partial word $W_{p,q}$ as follows, setting $k = \lfloor q/p \rfloor$ (see Fig. 1):

$$W_{p,q} = (S_{p,q}[0..p-3]\diamondsuit\diamondsuit)^k \cdot S_{p,q} \cdot (\diamondsuit\diamondsuit S_{p,q}[q..q+p-3])^k .$$

Example 2.1. For $p=5$ and $q=7$, we can take $S_{5,7} = \mathsf{ababaababa}$ and

$$W_{5,7} \;=\; \mathsf{aba}\diamondsuit\diamondsuit\,\mathsf{ababaababa}\,\diamondsuit\diamondsuit\mathsf{aba}.$$

This partial word has length 20 and 4 holes. Hence, $H(20,5,7) \leq 4 = H^s(20,5,7)$ and $L(4,5,7) \geq 21$. In fact, these bounds are tight; see Table 1.

Intuitively, the partial word $W_{p,q}$ is an extension of $S_{p,q}$ preserving the period p, in which a small number of symbols is changed to holes to guarantee the periodicity with respect to q.

Lemma 2.2. *The partial word $W_{p,q}$ has periods p and q.*

Proof. Let $n = |W_{p,q}|$. It is easy to observe that p is a period of $W_{p,q}$. We now show that q is a period of $W_{p,q}$ as well. Let X and Y be the prefix and the suffix of $W_{p,q}$ of length $p\lfloor q/p \rfloor$ (so that $W_{p,q} = X \cdot S_{p,q} \cdot Y$). Note that $|X|, |Y| < q \leq |S_{p,q}|$.

Let us start by showing that $W_{p,q}[i] \approx W_{p,q}[i+q]$ for $0 \leq i < n-q$. First, suppose that $W_{p,q}[i]$ is contained in X. The claim is obvious if $i \bmod p \geq p-2$, because in this case we have $W_{p,q}[i] = \diamondsuit$. Otherwise

$$W_{p,q}[i] = S_{p,q}[i \bmod p] \overset{(1)}{=} S_{p,q}[i \bmod p + q] \overset{(2)}{=} S_{p,q}[i+q-\lfloor \tfrac{q}{p} \rfloor p] = W_{p,q}[i+q],$$

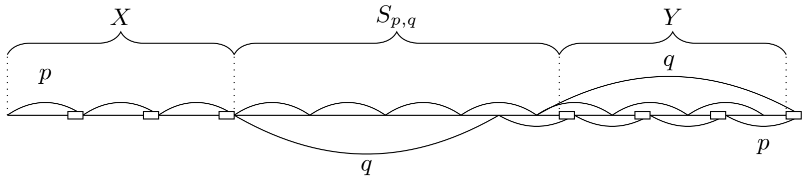

Figure 1: The structure of the partial word $W_{p,q}\diamondsuit\diamondsuit \;=\; X \cdot S_{p,q} \cdot Y \diamondsuit\diamondsuit$ for $\lfloor q/p \rfloor = 3$. Tiny rectangles correspond to two holes $\diamondsuit\diamondsuit$. We have $|X| = |Y| = p\lfloor q/p \rfloor = 3p$ and $|W_{p,q}| \;=\; p+q+2p\lfloor q/p \rfloor - 2 = q+7p-2$. There are $4 \cdot \lfloor q/p \rfloor = 12$ holes.

where (1) follows from the fact that $S_{p,q}$ has period q and $i \bmod p < p-2$, and (2) from the fact that $S_{p,q}$ has period p. By symmetry of our construction, we also have $W_{p,q}[i] \approx W_{p,q}[i+q]$ if $W_{p,q}[i+q]$ is contained in Y. In the remaining case, $W_{p,q}[i]$ and $W_{p,q}[i+q]$ are both contained in $S_{p,q}$, which yields $W_{p,q}[i+q] = W_{p,q}[i]$.

Next, we claim that $W_{p,q}[i] \approx W_{p,q}[i+kq]$ for every $k \geq 2$. Observe that $W_{p,q}[i+q], \ldots, W_{p,q}[i+(k-1)q]$ are contained in $S_{p,q}$ and thus they are equal solid symbols. Hence,

$$W_{p,q}[i] \approx W_{p,q}[i+q] = \cdots = W_{p,q}[i+(k-1)q] \approx W_{p,q}[i+kq].$$

The intermediate symbols are solid, so this implies $W_{p,q}[i] \approx W_{p,q}[i+kq]$, as claimed. Consequently, q is indeed a period of $W_{p,q}$. □

We use the word $S_{p,q}$ and the partial word $W_{p,q}\diamond\diamond$ to show that H^s is an upper bound for H for all intermediate lengths n ($|S_{p,q}| \leq n \leq |W_{p,q}\diamond\diamond|$).

Lemma 2.3. *Let $1 < p < q$ be relatively prime integers. For each length $p+q-2 \leq n \leq p+q+2p\lfloor q/p \rfloor$, we have $H(n,p,q) \leq H^s(n,p,q)$.*

Proof. We extend $S_{p,q}$ to $W_{p,q}\diamond\diamond$ symbol by symbol, first prepending the characters before $S_{p,q}$, and then appending the characters after $S_{p,q}$. By Lemma 2.2, the resulting partial word has periods p and q because it is contained in $W_{p,q}\diamond\diamond$. Moreover, it is not unary because it contains $S_{p,q}$.

A hole is added at the first two iterations among every p iterations. Hence, the total number of holes is as claimed:

$$\left\lceil \tfrac{n-|S_{p,q}|}{p} \right\rceil + \left\lceil \tfrac{n-|S_{p,q}|-1}{p} \right\rceil = \left\lfloor \tfrac{n-q+1}{p} \right\rfloor + \left\lfloor \tfrac{n-q}{p} \right\rfloor = H^s(n,p,q),$$

because $\lceil \frac{x}{p} \rceil = \lfloor \frac{x+p-1}{p} \rfloor$ for every integer x.

□

Finally, the function $L^s = \widetilde{H^s}$ is very simple and easily computable.

Lemma 2.4. *If $h \geq 0$ is an integer, then $L^s(h,p,q) = \lceil \frac{h+1}{2} \rceil p + q - (h+1) \bmod 2$.*

Proof. We have to determine the smallest n such that $\left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor = h+1$. There are two cases, depending on parity of h:

Case 1: $h = 2k$. In this case $\left\lfloor \frac{n-q}{p} \right\rfloor = k$ and $\left\lfloor \frac{n-q+1}{p} \right\rfloor = k+1$. Hence, $n-q+1 = p(k+1)$, i.e., $n = p(k+1)+q-1 = \lceil \frac{h+1}{2} \rceil p + q - (h+1) \bmod 2$.

Case 2: $h = 2k+1$. In this case $\left\lfloor \frac{n-q}{p} \right\rfloor = k+1$ and $\left\lfloor \frac{n-q+1}{p} \right\rfloor = k+1$. Hence, $n-q = p(k+1)$, i.e., $n = p(k+1)+q = \lceil \frac{h+1}{2} \rceil p + q - (h+1) \bmod 2$. □

3 Functions H^d and L^d

In this section, we study a family of partial words corresponding to the *special arrangements* introduced in [21]. For relatively prime integers $p, q > 1$, we say that a partial word S of length $n \geq \max(p,q)$ is (p,q)-*special* if it has a position l such that for each position i:

$$S[i] = \begin{cases} \mathsf{a} & \text{if } p \nmid (l-i) \text{ and } q \nmid (l-i), \\ \mathsf{b} & \text{if } p \mid (l-i) \text{ and } q \mid (l-i), \\ \diamond & \text{otherwise.} \end{cases}$$

Let $H^d(n,p,q)$ be the minimum number of holes in a (p,q)-special partial word of length n.

Fact 3.1. *For each $n \geq \max(p,q)$, we have $H(n,p,q) \leq H^d(n,p,q)$.*

Proof. Observe that every (p,q)-special partial word has periods p and q. However, due to $p,q > 1$, it does not have period $1 = \gcd(p,q)$. □

h	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\widetilde{G}(h,5,7)$	5	7	10	14	15	20	21	25	28	30	40	42	45	49	50	55	56	60	63	65	75
$L^d(h,5,7)$	10	12	15	19	21	25	28	30	34	35	45	47	50	54	56	60	63	65	69	70	80

Table 2: Functions $\widetilde{G}$ and L^d for $p = 5$, $q = 7$, and $h = 0, \ldots, 20$. By Lemma 3.3, we have, for example, $L^d(8,5,7) = \max\left(\widetilde{G}(0,5,7) + \widetilde{G}(8,5,7), \ldots, \widetilde{G}(4,5,7) + \widetilde{G}(4,5,7)\right) = \max(5+28,\ 7+25,\ 10+21,\ 14+20,\ 15+15) = 34$.

Example 3.2. The partial word aaaabaaaa◇a◇aa◇aaa is $(5,7)$-special (with $l = 4$), so $H(18,5,7) \le H^d(18,5,7) \le 3$ and $L(3,5,7) \ge 19$. In fact, these bounds are tight; see Table 1.

To derive a formula for $H^d(n,p,q)$, let us introduce an auxiliary function G, which counts integers $i \in \{1, \ldots, n\}$ that are multiples of p or of q but not both:

$$G(n,p,q) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 2 \left\lfloor \frac{n}{pq} \right\rfloor .$$

The function H^d can be characterized using G, while the generalized inverse $L^d = \widetilde{H^d}$ admits a dual characterization in terms of $\widetilde{G}$; see also Table 2.

Lemma 3.3. *Let $p, q > 1$ be relatively prime integers.*

(a) *If $n \ge \max(p,q)$, then $H^d(n,p,q) = \min_{l=0}^{n-1} \left(G(l,p,q) + G(n-l-1,p,q)\right)$.*

(b) *If $h \ge 0$, then $L^d(h,p,q) = \max_{k=0}^{h} \left(\widetilde{G}(k,p,q) + \widetilde{G}(h-k,p,q)\right)$.*

Proof. Let S be a (p,q)-special partial word of length n with h holes, k of which are located to the left of position l. Observe that $k = G(l,p,q)$ (so $l + 1 \le \widetilde{G}(k,p,q)$) and $h - k = G(n-l-1,p,q)$ (so $n - l \le \widetilde{G}(h-k,p,q)$). Hence, $h = G(l,p,q) + G(n-l-1,p,q)$ and $n+1 \le \widetilde{G}(k,p,q) + \widetilde{G}(h-k,p,q)$. The claimed equalities follow from the fact that these bounds can be attained for each l and k, respectively. □

4 Characterizations of H and L

Shur and Gamzova in [21] proved that $H(n,p,q) = H^d(n,p,q)$ for $n \ge 3q + p$. In this section, we give a complete characterization of H in terms of H^d and H^s, and we derive an analogous characterization of L in terms of L^d and L^s. Our proof is based on a graph-theoretic approach similar to that in [5].

Let us define the *(n,p,q)-graph* $\mathbf{G} = (V,E)$ as an undirected graph with vertices $V = \{0, \ldots, n-1\}$. The vertices i and j are connected if and only if $p \mid (j-i)$ or $q \mid (j-i)$. Observe that $H(n,p,q)$ is the minimum size of a *vertex separator* in $\mathbf{G}$, i.e., the minimum number of vertices to be removed from $\mathbf{G}$ so that the resulting graph is no longer connected; see Fig. 2.

We say that an edge (i,j) of the (n,p,q)-graph is a *p-edge* if $p \mid (j-i)$ and a *q-edge* if $q \mid (j-i)$. The set of all nodes giving the same remainder modulo p (modulo q) is called a *p-class* (*q-class*, respectively). Each p-class and each q-class forms a clique in the (n,p,q)-graph.

Fact 4.1 (see [5])**.** *Let $1 < p < q$ be relatively prime integers. If $n < pq$, then $H^d(n,p,q)$ is the minimal degree of a vertex in the (n,p,q)-graph.*

Proof. Observe that vertex number l has $G(l,p,q)$ neighbors $i < l$ and $G(n-l-1,p,q)$ neighbors $i > l$. Consequently, by Lemma 3.3, $H^d(n,p,q) = \min_{l=0}^{n-1}(G(l,p,q) + G(n-l-1,p,q)) = \min_{l=0}^{n-1} \deg_{\mathbf{G}}(l)$. □

Let $\mathbf{G} = (V,E)$ be the (n,p,q)-graph. For each $i \in \{0, \ldots, p-1\}$ let C_i be the p-class containing the vertex i; see Fig. 2. We slightly abuse the notation and use arbitrary integers for indexing the p-classes: $C_i = C_{i \bmod p}$ for $i \in \mathbb{Z}$. We denote by E_i the set of q-edges of the form $(j, j+q)$ for $j \in C_i$. Let us start with two auxiliary facts.

Fact 4.2. *Let $1 < p < q$ be relatively prime integers.*

(a) *For $j \in \{0, \ldots, p-1\}$, we have $|E_j| = \left\lceil \frac{n-j-q}{p} \right\rceil$.*

(b) *$H^s(n,p,q) = |E_{p-1}| + |E_{p-2}| = \min_{i \ne j} (|E_i| + |E_j|)$.*

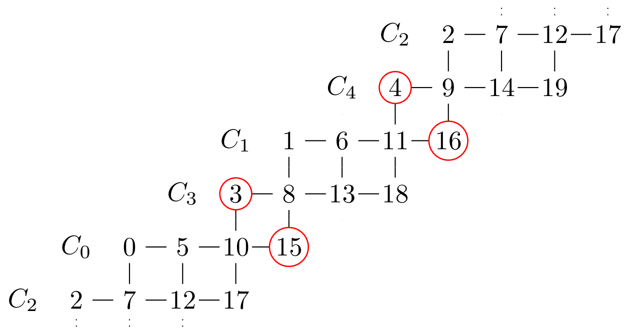

Figure 2: The structure of the $(20, 5, 7)$-graph. Each 5-clique C_i is actually a clique; same applies for the vertical 7-cliques. The 5-clique C_2 is repeated to show the cyclicity. The set $U = \{3, 4, 5, 7\}$ of encircled vertices is a minimum-size vertex separator. It corresponds to the partial word $W_{5,7}$ from Example 2.1: holes of $W_{5,7}$ are located at positions $i \in U$, the positions i with $W_{5,7}[i] = \mathtt{b}$ form a connected component $(C_1 \cup C_4) \setminus U$, while the positions i with $W_{5,7}[i] = \mathtt{a}$ form a connected component $(C_0 \cup C_2 \cup C_3) \setminus U$.

Proof. Let $i = kp + j$, where $0 \le j < p$. There is a q-edge $(i, i + q)$ if and only if

$$kp + j + q \le n - 1, \text{ so } k \le \left\lfloor \frac{n-1-j-q}{p} \right\rfloor.$$

The number of such values of k is $\left\lfloor \frac{n-1-j-q}{p} \right\rfloor + 1 = \left\lceil \frac{n-j-q}{p} \right\rceil$.

As for the second statement of the fact, we have:

$$|E_j| \ge |E_{p-1}| = \left\lceil \frac{n-p+1-q}{p} \right\rceil = \left\lfloor \frac{n-q}{p} \right\rfloor$$

for $0 \le j < p$ and, similarly, $|E_j| \ge |E_{p-2}| = \left\lfloor \frac{n-q+1}{p} \right\rfloor$ for $0 \le j < p - 1$. ☐

Fact 4.3. *Let U be a vertex separator in the (n, p, q)-graph $\mathbf{G} = (V, E)$ and let $\mathbf{G}' = \mathbf{G} \setminus U$. One can color the vertices of $\mathbf{G}$ in two colors so that every edge in $\mathbf{G}'$ and every p-class in $\mathbf{G}$ is monochromatic, but $\mathbf{G}'$ is not monochromatic.*

Proof. Recall that each p-class C_i is a clique in $\mathbf{G}$, so $C_i \setminus U$ is still a clique in $\mathbf{G}'$. We distinguish a connected component M of $\mathbf{G}'$ and color the vertices of C_i depending on whether $C_i \setminus U \subseteq M$. It is easy to verify that this coloring satisfies the claimed conditions. ☐

The following lemma provides lower bounds on $H(n, p, q)$.

Lemma 4.4. *Let $1 < p < q$ be relatively prime integers.*

(1) *If $n < 2q$, then $H(n, p, q) \ge H^s(n, p, q)$.*
(2) *If $p \ge 3$ and $n \ge 2q$, then $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q))$.*
(3) *If $p \ge 5$ and $n \ge 4q$, then $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q) + 1)$.*

Proof. Let U be a minimum-size vertex separator of hte (n, p, q)-graph $\mathbf{G} = (V, E)$; recall that $|U| = H(n, p, q)$. Let us fix a coloring of $\mathbf{G}$ using colors $\{A, B\}$ satisfying Fact 4.3; without loss of generality we assume that the number of p-classes with color A is at least the number of p-classes with color B. We have the following two cases.

Case a: Exactly one p-class has color B. Let C_j be the unique p-class with color B. By definition, the edges in $E_{j-q} \cup E_j$ are bichromatic. If $n < 2q$, then all the q-edges form a matching in $\mathbf{G}$. In particular, in order to disconnect $\mathbf{G}$, we need to remove at least one endpoint of each edge in $E_{j-q} \cup E_j$. Hence, $H(n, p, q) \ge |E_{j-q}| + |E_j| \ge H^s(n, p, q)$, where the second inequality follows from Fact 4.2(b). This concludes the proof of (1) in this case.

Now assume that $n \geq 2q$ and $p \geq 3$. We will show that $H(n,p,q) \geq H^d(n,p,q)$ holds in this case. Consider any q-class D and let k be its size; we have $k \geq \left\lfloor \frac{n}{q} \right\rfloor \geq 2$. In this q-class, every p-th element has color B. Let $\#_A(D)$ and $\#_B(D)$ denote the number of vertices in D colored with A and B, respectively. Then:

$$\#_B(D) \leq \left\lceil \frac{k}{p} \right\rceil \leq \left\lceil \frac{k}{3} \right\rceil = \left\lfloor \frac{k+2}{3} \right\rfloor \leq \left\lfloor \frac{2k}{3} \right\rfloor = k - \left\lceil \frac{k}{3} \right\rceil \leq k - \left\lceil \frac{k}{p} \right\rceil \leq \#_A(D).$$

The set U contains all B-colored vertices or all A-colored vertices of every q-class D, as otherwise there would be a non-monochromatic edge in $\mathbf{G}'$ connecting two vertices of $D \setminus U$, contradicting Fact 4.3. At least one vertex of $\mathbf{G}'$ is B-colored, so in at least one q-class, U must contain all A-colored vertices; assume that this is the q-class D_0. Consequently,

$$|U| = \sum_{D:q\text{-class}} |U \cap D| \geq \#_A(D_0) + \sum_{D:q\text{-class},\, D \neq D_0} \#_B(D) =$$

$$|D_0| + \sum_{D:q\text{-class}} \#_B(D) - 2\#_B(D_0) = |D_0| + |C_j| - 2|D_0 \cap C_j| \geq H^d(n,p,q).$$

The last inequality follows from Fact 4.1. This concludes (2) and (3) in this case.

Case b: There are at least two p-classes with each color. In particular, $p \geq 4$. We consider two subcases based on the colors c_i of classes C_i. In each case we will show that $H(n,p,q)$ is bounded from below by $H^s(n,p,q)$ or $H^s(n,p,q)+1$.

First, suppose that there is exactly one p-class C_i such that $c_i = A$ and $c_{i+q} = B$. Equivalently, there is exactly one p-class C_j such that $c_j = B$ and $c_{j+q} = A$. Since there are at least two p-classes with each color, $c_{i+2q} = B$ and $c_{j+2q} = A$, so $C_{i+q} \neq C_j$ and $C_{j+q} \neq C_i$. This means that $E_i \cup E_j$ forms a bichromatic matching in $\mathbf{G}$. Consequently,

$$H(n,p,q) \geq |E_i| + |E_j| \geq H^s(n,p,q).$$

This concludes the proof of (1) and (2) in the current subcase.

For the proof of (3), observe that $p \geq 5$ and the choice of A as the more frequent color yields that $c_{j+3q} = A$, so C_{j+2q} is distinct from C_i. Hence, we can extend the matching $E_i \cup E_j$ with an edge (x,y) where $x = (j-q) \bmod p \in C_{j-q}$ and $y = x + 3q \in C_{j+2q}$. This edge exists because $n \geq 4q > p + 3q > y$. It forms a matching with $E_i \cup E_j$ because no edge in $E_i \cup E_j$ is incident to C_{j+2q}, while the only edges incident to C_{j-q} could be the edges in E_i provided that $C_i = C_{j-2q}$. However, $x < q$, so x is not an endpoint of any edge in E_{j-2q}. This concludes the proof of (3) in the current subcase.

Let us proceed to the second subcase. Let C_i, C_j be two distinct p-classes such that $c_i = c_j = A$ and $c_{i+q} = c_{j+q} = B$. It is easy to see that $E_i \cup E_j$ forms a bichromatic matching in $\mathbf{G}$, so $H(n,p,q) \geq |E_i| + |E_j| \geq H^s(n,p,q)$. Thus, it remains to prove (3) in this subcase.

If $p \geq 5$, then there is a third p-class C_k with color A. Moreover, we may choose k so that $c_{k-q} = B$ and $c_k = A$. We extend $E_i \cup E_j$ with an edge (x,y) where $x = (k-q) \bmod p \in C_{k-q}$ and $y = x+q \in C_k$. This edge exists because $n > q + p > y$. It forms a matching with $E_i \cup E_j$ because no edge in $E_i \cup E_j$ is incident to C_k, while the only edges incident to C_{k-q} might be the edges in E_i or E_j provided that $i = k - 2q$ or $j = k - 2q$. However, $x < q$, so x is not an endpoint of any edge in E_{k-2q}. This concludes the proof of (3) in the current subcase, and the proof of the entire lemma. ◻

Lemma 4.5. *If $n \geq q$, then*

$$\left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor - 2\left\lceil \frac{n}{pq} \right\rceil \leq H^d(n,p,q) \leq \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor - 1.$$

Proof. Recall that $H^d(n,p,q) = \min_{l=0}^{n-1} \left(G(l,p,q) + G(n-l-1,p,q)\right)$ due to Lemma 3.3. The first part of the claim holds because for $0 \leq l < n$ we have:

$$G(l,p,q) + G(n-l-1,p,q) = \left\lfloor \frac{l}{p} \right\rfloor + \left\lfloor \frac{l}{q} \right\rfloor - 2\left\lfloor \frac{l}{pq} \right\rfloor + \left\lfloor \frac{n-l-1}{p} \right\rfloor + \left\lfloor \frac{n-l-1}{q} \right\rfloor - 2\left\lfloor \frac{n-l-1}{pq} \right\rfloor =$$

$$\left\lceil \frac{l-p+1}{p} \right\rceil + \left\lceil \frac{l-q+1}{q} \right\rceil - 2\left\lfloor \frac{l}{pq} \right\rfloor + \left\lceil \frac{n-l-p}{p} \right\rceil + \left\lfloor \frac{n-l-q}{q} \right\rfloor - 2\left\lfloor \frac{n-l-1}{pq} \right\rfloor \geq$$

$$\left\lceil \frac{n-2p+1}{p} \right\rceil + \left\lceil \frac{n-2q+1}{q} \right\rceil - 2\left\lfloor \frac{n-1}{pq} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor - 1 + \left\lfloor \frac{n}{q} \right\rfloor - 1 - 2\left\lceil \frac{n}{pq} \right\rceil + 2 = \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor - 2\left\lceil \frac{n}{pq} \right\rceil.$$

As for the second part, due to $n \geq q$ we have:

$$H^d(n,p,q) \geq G(q-1,p,q) + G(n-q,p,q) = \left\lfloor \tfrac{q-1}{p} \right\rfloor + \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n-q}{q} \right\rfloor - 2\left\lfloor \tfrac{n-q}{pq} \right\rfloor \leq \left\lfloor \tfrac{q-1}{p} \right\rfloor + \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n}{q} \right\rfloor - 1,$$

which completes he proof. ☐

Theorem 4.6. *Let p and q be relatively prime integers such that $2 < p < q$. For each integer $n \geq p+q-2$, we have*

$$H(n,p,q) = \begin{cases} H^s(n,p,q) & \text{if } n \leq q + p\lceil \frac{q}{p} \rceil - 1 \text{ or } 3q \leq n \leq q+3p-1, \\ H^d(n,p,q) & \text{otherwise.} \end{cases}$$

Moreover, for each integer $h \geq 0$:

$$L(h,p,q) = \begin{cases} L^s(h,p,q) & \text{if } \frac{q}{p} > \lceil \frac{h}{2} \rceil \text{ or } (h=4 \text{ and } \frac{q}{p} < \frac{3}{2}) \\ L^d(h,p,q) & \text{otherwise.} \end{cases}$$

Proof. First, we prove the claim concerning H by analyzing several cases.

Case 0. $pq \leq n$**.**

By Fact 3.1, we have $H(n,p,q) \leq H^d(n,p,q)$. Moreover, using Lemma 4.5, we obtain:

$$H^s(n,p,q) = \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n-q+1}{p} \right\rfloor \geq \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n-pq}{p} \right\rfloor + \left\lfloor \tfrac{q-1}{p} \right\rfloor + \left\lfloor \tfrac{q(p-2)+2}{p} \right\rfloor \geq$$
$$\geq \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n}{q} \right\rfloor - p + \left\lfloor \tfrac{q-1}{p} \right\rfloor + \left\lfloor \tfrac{(p+1)(p-2)+2}{p} \right\rfloor = \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n}{q} \right\rfloor + \left\lfloor \tfrac{q-1}{p} \right\rfloor + \left\lfloor \tfrac{p^2-p-p^2}{p} \right\rfloor =$$
$$= \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n}{q} \right\rfloor + \left\lfloor \tfrac{q-1}{p} \right\rfloor - 1 \geq H^d(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n,p,q) \geq \min(H^d(n,p,q), H^s(n,p,q)) = H^d(n,p,q)$, which completes the proof.

Henceforth we assume that $n < pq$.

Case 1. $p+q-1 \leq n < 2q$**.**

We get $H(n,p,q) = H^s(n,p,q)$ directly from Lemma 4.4(1) and Lemma 2.3.

Case 2. $2q \leq n \leq q + \left\lceil \frac{q}{p} \right\rceil p - 1$**.**

Note that $n \leq q + \left\lceil \frac{q}{p} \right\rceil p - 1 = p + q + \left\lfloor \frac{q}{p} \right\rfloor p - 1 < p + q + 2p \left\lfloor \frac{q}{p} \right\rfloor$, so $H(n,p,q) \leq H^s(n,p,q)$ due to Lemma 2.3. Moreover, using Lemma 4.5, we obtain:

$$H^s(n,p,q) = \left\lfloor \tfrac{n-q}{p} \right\rfloor + \left\lfloor \tfrac{n-q+1}{p} \right\rfloor \leq \left\lfloor \tfrac{\lceil \frac{q}{p} \rceil p - 1}{p} \right\rfloor + \left\lfloor \tfrac{n-q+1}{p} \right\rfloor = \left\lceil \tfrac{q}{p} \right\rceil - 1 + \left\lfloor \tfrac{n-q+1}{p} \right\rfloor = \left\lfloor \tfrac{q-1}{p} \right\rfloor + \left\lfloor \tfrac{n-q+1}{p} \right\rfloor \leq$$
$$\left\lfloor \tfrac{n}{p} \right\rfloor \leq \left\lfloor \tfrac{n}{p} \right\rfloor + \left\lfloor \tfrac{n}{q} \right\rfloor - 2 \leq H^d(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n,p,q) \geq \min(H^d(n,p,q), H^s(n,p,q)) = H^s(n,p,q)$, which completes the proof.

Case 3. $q + \left\lceil \frac{q}{p} \right\rceil p - 1 \leq n < 3q$**.**

By Fact 3.1, we have $H(n,p,q) \leq H^d(n,p,q)$. Moreover, using Lemma 4.5, we obtain:

$$H^s(n,p,q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{\left\lceil \frac{q}{p} \right\rceil p}{p} \right\rfloor = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lceil \frac{q}{p} \right\rceil =$$
$$\left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + 1 + \left\lfloor \frac{n}{q} \right\rfloor - 2 \geq H^d(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n,p,q) \geq \min(H^d(n,p,q), H^s(n,p,q)) = H^d(n,p,q)$, which completes the proof.

Case 4. $3q \leq n \leq 3p + q - 1$**.**

Note that $n \leq 3p + q - 1 < p + q + 2p \leq p + q + 2p \left\lfloor \frac{q}{p} \right\rfloor$, so $H(n,p,q) \leq H^s(n,p,q)$ due to Lemma 2.3. Moreover, using Lemma 4.5, we obtain:

$$H^s(n,p,q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq \left\lfloor \frac{3p-1}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor = 2 + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq 1 + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq$$
$$\leq 1 + \left\lfloor \frac{n}{p} \right\rfloor = 3 + \left\lfloor \frac{n}{p} \right\rfloor - 2 \leq \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor - 2 \leq H^d(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n,p,q) \geq \min(H^d(n,p,q), H^s(n,p,q)) = H^s(n,p,q)$, which completes the proof.

Case 5. $\max(3q, 3p + q - 1) \leq n < 4q$ **and** $p < q < 2p$**.**

By Fact 3.1, we have $H(n,p,q) \leq H^d(n,p,q)$. Moreover, using Lemma 4.5, we obtain:

$$H^s(n,p,q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{3p}{p} \right\rfloor = \left\lfloor \frac{n-q}{p} \right\rfloor + 3 = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 1 \geq H^d(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n,p,q) \geq \min(H^d(n,p,q), H^s(n,p,q)) = H^d(n,p,q)$, which completes the proof.

Case 6. $3q \leq n$ **and** $q > 2p$**.**

By Fact 3.1, we have $H(n,p,q) \leq H^d(n,p,q)$. Moreover, using Lemma 4.5, we obtain:

$$H^s(n,p,q) \geq 2 \left\lfloor \frac{n-q}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-3q}{p} \right\rfloor + 2 \left\lfloor \frac{q}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-3q}{q} \right\rfloor + 2 \left\lfloor \frac{q}{p} \right\rfloor \geq$$
$$\geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 3 + \left\lfloor \frac{q-1}{p} \right\rfloor + 2 = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 1 \geq H^d(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n,p,q) \geq \min(H^d(n,p,q), H^s(n,p,q)) = H^d(n,p,q)$, which completes the proof.

Case 7. $4q \leq n$ **and** $p \geq 5$ **(and** $q < 2p$**).**

By Fact 3.1, we have $H(n,p,q) \leq H^d(n,p,q)$. Moreover, using Lemma 4.5, we obtain:

$$H^s(n,p,q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n+1-q}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-2q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-2q}{q} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor =$$
$$= \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 2 + \left\lfloor \frac{q-1}{p} \right\rfloor \geq H^d(n,p,q) - 1.$$

Finally, Lemma 4.4(3) yields that $H(n,p,q) \geq \min(H^d(n,p,q), H^s(n,p,q) + 1) = H^d(n,p,q)$, which completes the proof.

The only remaining case, that $4q \leq n$ and $p < 5$, is a subcase of Case 0. This completes the proof of the formula for $H(n,p,q)$.

The characterization of $L(h,p,q)$ is relatively easy to derive from that of $H(n,p,q)$. Recall that L, L^s, and L^d are generalized inverses of H, H^s, and H^d, respectively. Note that Cases 1. and 2. yield $H(n,p,q) = H^s(n,p,q)$ for $n \le q + p\lceil \frac{q}{p} \rceil - 1$ while Case 3. additionally implies

$$H^d(q + p\lceil \tfrac{q}{p} \rceil - 1, p, q) = H^s(q + p\lceil \tfrac{q}{p} \rceil - 1, p, q) = 2\lceil \tfrac{q}{p} \rceil - 1.$$

Consequently, $L(h,p,q) = L^s(h,p,q)$ if $h < 2\lceil \frac{q}{p} \rceil - 1$, i.e., if $\frac{q}{p} > \lceil \frac{h}{2} \rceil$. Moreover, if $\frac{3}{2} < \frac{q}{p}$, then $3q > q + 3p - 1$, and therefore $H(n,p,q) = H^d(n,p,q)$ for $n \ge q + p\lceil \frac{q}{p} \rceil - 1$ due to Cases 0. and 5.–7. Hence, if $h \ge 2\lceil \frac{q}{p} \rceil - 1$, i.e., $\frac{3}{2} < \frac{q}{p} < \lceil \frac{h}{2} \rceil$, then $L(h,p,q) = L^d(h,p,q)$.

Now, it suffices to consider the case of $\frac{q}{p} < \frac{3}{2} < \lceil \frac{h}{2} \rceil$. Then, by Cases 5., 7., and 0., $H(n,p,q) = H^d(n,p,q)$ for $n \ge q+3p-1$. Case 4. additionally yields $H^d(q+3p-1,p,q) = H^s(q+3p-1,p,q) = 5$, so $L(h,p,q) = L^d(h,p,q)$ if $h \ge 5$. Moreover, by Case 4., $H(n,p,q) = H^d(n,p,q)$ for $q + p - 1 \le n < 3q$, so $L(3,p,q) = L^d(3,p,q)$ due to $H^s(3q,p,q) \ge 4$. Finally, we note that Case 3. yields $H(n,p,q) = H^s(n,p,q)$ for $3q \le n \le 3p + q - 1$, so $L(4,p,q) = L^s(4,p,q)$ due to $H(3q-1,p,q) \le H^s(3q-1,p,q) \le 4$. ☐

The remaining cases have already been well understood:

Fact 4.7 ([21, 5])**.** *If $p, q > 1$ are integers such that $\gcd(p,q) \notin \{p,q\}$, then*

$$L(h,p,q) = \gcd(p,q) \cdot L\left(h, \tfrac{p}{\gcd(p,q)}, \tfrac{q}{\gcd(p,q)}\right).$$

Fact 4.8 ([22])**.** *If q, h are integers such that $q > 2$, $2 \nmid q$, and $h \ge 0$, then*

$$L(h,2,q) = (2p+1)\left\lfloor \tfrac{h}{p} \right\rfloor + h \bmod p.$$

The results above lead to our first algorithm for computing $L(h,p,q)$.

Corollary 4.9. *Given integers $p, q > 1$ such that $\gcd(p,q) \notin \{p,q\}$ and an integer $h \ge 0$, the value $L(h,p,q)$ can be computed in $\mathcal{O}(h + \log p + \log q)$ time.*

Proof. First, we apply Fact 4.7 to reduce the computation to $L(h,p',q')$ such that $\gcd(p',q') = 1$ and, without loss of generality, $1 < p' < q'$. This takes $\mathcal{O}(\log p + \log q)$ time. If $p' = 2$, we use Fact 4.8, while for $p' > 2$ we rely on the characterization of Theorem 4.6, using Lemmas 2.4 and 3.3 for computing L^s and L^d, respectively. The values $\widetilde{G}(h',p',q')$ form a sorted sequence of multiples of p' and q', but not of $p'q'$. Hence, it takes $\mathcal{O}(h)$ time to generate them for $0 \le h' \le h$. The overall running time is $\mathcal{O}(h + \log p + \log q)$. ☐

5 Faster Algorithm for Evaluating L

A more efficient algorithm for evaluating L relies on the theory of continued fractions; we refer to [15] and [20] for a self-contained yet compact introduction. A finite continued fraction is a sequence $[\gamma_0; \gamma_1, \ldots, \gamma_m]$, where $\gamma_0, m \in \mathbb{Z}_{\ge 0}$ and $\gamma_i \in \mathbb{Z}_{\ge 1}$ for $1 \le i \le m$. We associate it with the following rational number:

$$[\gamma_0; \gamma_1, \ldots, \gamma_m] = \gamma_0 + \cfrac{1}{\gamma_1 + \cfrac{1}{\ddots + \cfrac{1}{\gamma_m}}}.$$

Depending on the parity of m, we distinguish odd and even continued fractions. Often, an improper continued fraction $[;] = \frac{1}{0}$ is also introduced and assumed to be odd. Each positive rational number has exactly two representations as a continued fraction, one as an even continued fraction, and one as an odd continued fraction. For example, $\frac{5}{7} = [0;1,2,2] = [0;1,2,1,1]$.

Consider a continued fraction $[\gamma_0; \gamma_1, \ldots, \gamma_m]$. Its *convergents* are continued fractions of the form $[\gamma_0; \gamma_1, \ldots, \gamma_{m'}]$ for $0 \le m' < m$, and $[;] = \frac{1}{0}$. The *semiconvergents* also include continued fractions of the form $[\gamma_0; \gamma_1, \ldots, \gamma_{m'-1}, \gamma'_{m'}]$, where $0 \le m' \le m$ and $0 < \gamma'_{m'} < \gamma_{m'}$. The two continued fractions representing a positive rational number have the same semiconvergents.

Example 5.1. The semiconvergents of $[0;1,2,2] = \frac{5}{7} = [0;1,2,1,1]$ are $[;] = \frac{1}{0}$, $[0;] = \frac{0}{1}$, $[0;1] = \frac{1}{1}$, $[0;1,1] = \frac{1}{2}$, $[0;1,2] = \frac{2}{3}$, and $[0;1,2,1] = \frac{3}{4}$.

Semiconvergents of $\frac{p}{q}$ can be generated using the (slow) *continued fraction algorithm*, which produces a sequence of *Farey pairs* $(\frac{a}{b}, \frac{c}{d})$ such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$.

Algorithm 1: Farey process for a rational number $\frac{p}{q} > 0$

```
(a/b, c/d) := (0/1, 1/0);
while true do
    Report a Farey pair (a/b, c/d);
    if (a+c)/(b+d) < p/q then a/b := (a+c)/(b+d);
    else if (a+c)/(b+d) = p/q then break;
    else c/d := (a+c)/(b+d);
```

Example 5.2. For $\frac{p}{q} = \frac{5}{7}$, the Farey pairs are $(\frac{0}{1}, \frac{1}{0}) \rightsquigarrow (\frac{0}{1}, \frac{1}{1}) \rightsquigarrow (\frac{1}{2}, \frac{1}{1}) \rightsquigarrow (\frac{2}{3}, \frac{1}{1}) \rightsquigarrow (\frac{2}{3}, \frac{3}{4})$. The process terminates at $\frac{2+3}{3+4} = \frac{5}{7}$.

Consider the set $\mathcal{F} = \{\frac{a}{b} : a, b \in \mathbb{Z}_{\geq 0}, \gcd(a, b) = 1\}$ of reduced fractions (including $\frac{1}{0}$). We denote $\mathcal{F}_k = \{\frac{a}{b} \in \mathcal{F} : a + b \leq k\}$ and, for each $x \in \mathbb{R}_+$:

$$\mathrm{Left}_k(x) = \max\{a \in \mathcal{F}_k \ : \ a \leq x\} \quad \text{and} \quad \mathrm{Right}_k(x) = \min\{a \in \mathcal{F}_k \ : \ a \geq x\}.$$

We say that $\frac{a}{b} < x$ is a *best left approximation* of x if $\frac{a}{b} = \mathrm{Left}_k(x)$ for some $k \in \mathbb{Z}_{\geq 0}$. Similarly, $\frac{c}{d} > x$ is a *best right approximation* of x if $\frac{c}{d} = \mathrm{Right}_k(x)$.

Example 5.3. We have $\mathcal{F}_7 = (\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \frac{1}{0})$. Here, $\mathrm{Left}_7(\frac{5}{7}) = \frac{2}{3}$ and $\mathrm{Right}_7(\frac{5}{7}) = \frac{3}{4}$ are best approximations of $\frac{5}{7}$.

We heavily rely on the following extensive characterization of semiconvergents:

Fact 5.4 ([15], [25, Theorem 3.3], [20, Theorem 2])**.** *Let $\frac{p}{q} \in \mathcal{F} \setminus \{\frac{1}{0}, \frac{0}{1}\}$. The following conditions are equivalent for reduced fractions $\frac{a}{b} < \frac{p}{q}$:*

(a) *the Farey process for $\frac{p}{q}$ generates a pair $(\frac{a}{b}, \frac{c}{d})$ for some $\frac{c}{d} \in \mathcal{F}$,*

(b) *$\frac{a}{b}$ is an even semiconvergent of $\frac{p}{q}$,*

(c) *$\frac{a}{b}$ is a best left approximation of $\frac{p}{q}$,*

(d) *$b = \left\lfloor \frac{aq}{p} \right\rfloor + 1$ and $aq \bmod p > iq \bmod p$ for $0 \leq i < a$.*

By symmetry, the following conditions are equivalent for reduced fractions $\frac{c}{d} > \frac{p}{q}$:

(a) *the Farey process for $\frac{p}{q}$ generates a pair $(\frac{a}{b}, \frac{c}{d})$ for some $\frac{a}{b} \in \mathcal{F}$,*

(b) *$\frac{c}{d}$ is an odd semiconvergent of $\frac{p}{q}$,*

(c) *$\frac{c}{d}$ is a best right approximation of $\frac{p}{q}$,*

(d) *$c = \left\lfloor \frac{dp}{q} \right\rfloor + 1$ and $dp \bmod q > ip \bmod q$ for $0 \leq i < d$.*

Example 5.5. For $\frac{p}{q} = \frac{5}{7}$, the prefix maxima of $(iq \bmod p)_{i=0}^{p-1} = (0, 2, 4, 1, 3)$ are attained for $i = 0, 1, 2$ (numerators of $\frac{0}{1}, \frac{1}{2}, \frac{2}{3}$) while the prefix maxima of $(ip \bmod q)_{i=0}^{q-1} = (0, 5, 3, 1, 6, 4, 2)$ are attained for $i = 0, 1, 4$ (denominators $\frac{1}{0}, \frac{1}{1}, \frac{3}{4}$).

Due to Fact 5.4, the best approximations can be efficiently computed using the *fast* continued fraction algorithm; see [20].

Corollary 5.6. *Given $\frac{p}{q} \in \mathcal{F}$ and a positive integer k, $1 \leq k < p + q$, the values $\mathrm{Left}_k(\frac{p}{q})$ and $\mathrm{Right}_k(\frac{p}{q})$ can be computed in $\mathcal{O}(\log k)$ time.*

Next, we characterize the function L^d.

Lemma 5.7. *Let $p, q > 2$ be relatively prime integers and let $h < p + q - 3$. If $\frac{a}{b} = \mathrm{Left}_{h+3}(\frac{p}{q})$ and $\frac{c}{d} = \mathrm{Right}_{h+3}(\frac{p}{q})$, then, assuming $G(-1, p, q) = 0$:*

$$L^d(h, p, q) = \begin{cases} \widetilde{G}(a + b - 2, p, q) + \widetilde{G}(c + d - 2, p, q) & \text{if } a + b + c + d = h + 4, \\ \widetilde{G}(h + 2, p, q) & \text{otherwise.} \end{cases}$$

Proof. Let us start with a special case of $\frac{a}{b}=\frac{0}{1}$. Then $\frac{c}{d}=\frac{1}{h+2}$, so $q>(h+2)p$ and $\widetilde{G}(k,p,q)=(k+1)p$ for $k\le h+1$. Consequently, by Lemma 3.3,

$$L^d(h,p,q)=\min_{k=0}^{h}\left(\widetilde{G}(k,p,q)+\widetilde{G}(h-k,p,q)\right)=(h+2)p.$$

Due to $a+b+c+d=0+1+1+h+2=h+4$, this is equal to the claimed value of $\widetilde{G}(-1,p,q)+\widetilde{G}(h+1,p,q)=0+(h+2)p$. Symmetrically, the lemma holds if $\frac{c}{d}=\frac{1}{0}$. Thus, below we assume $\frac{1}{h+2}<\frac{p}{q}<\frac{h+2}{1}$.

By Fact 5.4, $\frac{a+c}{b+d}$ is a best (left or right) approximation of $\frac{p}{q}$, so $\max(a+b,c+d)\le h+3<a+b+c+d$. Moreover,

$$G(aq,p,q)=\left\lfloor\tfrac{aq}{p}\right\rfloor+\left\lfloor\tfrac{aq}{q}\right\rfloor=b-1+a\quad\text{and}\quad G(dp,p,q)=\left\lfloor\tfrac{dp}{p}\right\rfloor+\left\lfloor\tfrac{dp}{q}\right\rfloor=d+c-1,$$

so $\widetilde{G}(a+b-2,p,q)+\widetilde{G}(c+d-2,p,q)=aq+dp$

First, suppose that $a+b+c+d<h+4$. Assume without loss of generality that $\widetilde{G}(h+2,p,q)=\alpha p$ is a multiple of p. Note that $d<\alpha<b+d$ due to

$$G((b+d)p,p,q)=b+d+\left\lfloor\tfrac{(b+d)p}{q}\right\rfloor\ge a+b+c+d-1\ge h+4.$$

Consequently, Fact 5.4 yields $\alpha p \bmod q < dp \bmod q$. Hence

$$G((\alpha-d)p,p,q)=(\alpha-d)+\left\lfloor\tfrac{\alpha p-dp}{q}\right\rfloor=(\alpha-d)+\left\lfloor\tfrac{\alpha p}{q}\right\rfloor+\left\lfloor\tfrac{-dp}{q}\right\rfloor=h+3-c-d,$$

and therefore

$$L^d(h,p,q)\ge\widetilde{G}(h+2-c-d,p,q)+\widetilde{G}(c+d-2,p,q)=(\alpha-d)p+dp=\widetilde{G}(h+2,p,q).$$

On the other hand, Lemma 4.5 yields $H^d(\alpha p,p,q)\ge G(\alpha p,p,q)-2=h+1$, so $L^d(h,p,q)\le\widetilde{G}(h+2,p,q)$.

Finally, suppose that $a+b+c+d=h+4$. Lemma 3.3 immediately yields $L^d(h,p,q)\ge\widetilde{G}(a+b-2,p,q)+\widetilde{G}(c+d-2,p,q)=aq+dp$. For the proof of the inverse inequality, let us take k such that $L^d(h,p,q)=\widetilde{G}(k,p,q)+\widetilde{G}(h-k,p,q)$, and define $x=\widetilde{G}(k,p,q)$ and $y=\widetilde{G}(h-k,p,q)$. Consequently,

$$\begin{aligned}a+b+c+d=h+4=\left\lfloor\tfrac{x-1}{p}\right\rfloor+\left\lfloor\tfrac{x-1}{q}\right\rfloor+\left\lfloor\tfrac{y-1}{p}\right\rfloor+\left\lfloor\tfrac{y-1}{q}\right\rfloor+4=\left\lceil\tfrac{x}{p}\right\rceil+\left\lceil\tfrac{y}{p}\right\rceil+\left\lceil\tfrac{x}{q}\right\rceil+\left\lceil\tfrac{y}{q}\right\rceil\ge\\ \left\lceil\tfrac{x+y}{p}\right\rceil+\left\lceil\tfrac{x+y}{q}\right\rceil\ge\left\lceil\tfrac{aq+dp}{p}\right\rceil+\left\lceil\tfrac{aq+dp}{q}\right\rceil=d+\left\lceil\tfrac{aq}{p}\right\rceil+a+\left\lceil\tfrac{dp}{q}\right\rceil=d+b+a+c.\end{aligned}$$

Each intermediate inequality must therefore be an equality, so we conclude that

$$\left\lceil\tfrac{x}{p}\right\rceil+\left\lceil\tfrac{y}{p}\right\rceil=\left\lceil\tfrac{x+y}{p}\right\rceil=\left\lceil\tfrac{aq+dp}{p}\right\rceil=b+d\quad\text{and}\quad\left\lceil\tfrac{x}{q}\right\rceil+\left\lceil\tfrac{y}{q}\right\rceil=\left\lceil\tfrac{x+y}{q}\right\rceil=\left\lceil\tfrac{aq+dp}{q}\right\rceil=a+c.$$

If $p\mid x$ and $p\mid y$, then $\frac{x+y}{p}=b+d$, so $\left\lceil\frac{(b+d)p}{q}\right\rceil=a+c$. Hence $\frac{a+c}{b+d}\ge\frac{p}{q}$, and consequently $\frac{a+c}{b+d}$ is either a right semiconvergent of $\frac{p}{q}$ or is equal to $\frac{p}{q}$. In both cases, Fact 5.4 implies $(-(b+d)p) \bmod q<\min((-x)\bmod q,(-y)\bmod q)$. This lets us derive a contradiction:

$$\left\lceil\tfrac{x}{q}\right\rceil+\left\lceil\tfrac{y}{q}\right\rceil=\tfrac{x+y+(-x)\bmod q+(-y)\bmod q}{q}>\tfrac{(b+d)p+2((-(b+d)p)\bmod q)}{q}\ge\left\lceil\tfrac{(b+d)p}{q}\right\rceil.$$

Symmetrically, $q\mid x$ and $q\mid y$ yields an analogous contradiction.

Thus, without loss of generality we may assume $p\mid x$ and $q\mid y$. However, the conditions $x+y\ge aq+dp$ and $\left\lceil\frac{x+y}{p}\right\rceil=\left\lceil\frac{aq+dp}{p}\right\rceil$ yield $(-y)\bmod p=(-(x+y))\bmod p\le(-(aq+dp))\bmod p=(-dp)\bmod p$. By Fact 5.4, this implies $y=dp$. Symmetrically, $x=aq$. Thus, $L^d(h,p,q)=aq+dp$, as claimed. ☐

Lemma 5.7 applies to $h<p+q-3$; the following fact lets us deal with $h\ge p+q-3$. It appeared in [5], but we provide an alternative proof for completeness.

Fact 5.8 ([5, Theorem 4])**.** *Let p, q be relatively prime positive integers. For each $h \ge 0$, we have*

$$L^d(h,p,q) = L^d(h \bmod (p+q-2), p, q) + \left\lfloor \tfrac{h}{p+q-2} \right\rfloor \cdot pq.$$

Moreover, $L^d(p+q-3,p,q) = pq$.

Proof. First, note that $\widetilde{G}(k,p,q) + \widetilde{G}(p+q-3-k,p,q) = pq$ holds for $0 \le k \le p+q-3$. Hence, $L^d(p+q-3,p,q) = pq$ holds as claimed due to Lemma 3.3.

For the first part of the statement, it suffices to prove that $H^d(n+pq,p,q) = H^d(n,p,q)+p+q-2$ for each $n \ge q$. The function G satisfies an analogous equality, so Lemma 3.3 immediately yields $H^d(n+pq,p,q) \le p+q+2+H^d(n,p,q)$. The other inequality also follows from Lemma 3.3 unless each optimum value l for $n+pq$ satisfies $n \le l < pq$. However, for such l (and $q < n < pq$), we have

$$G(l,p,q)+G(n+pq-l-1,p,q) = \left\lfloor \tfrac{l}{p} \right\rfloor + \left\lfloor \tfrac{l}{q} \right\rfloor + \left\lfloor \tfrac{n+pq-l-1}{p} \right\rfloor + \left\lfloor \tfrac{n+pq-l-1}{p} \right\rfloor \ge \left\lfloor \tfrac{n+pq}{p} \right\rfloor - 1 + \left\lfloor \tfrac{n+pq}{q} \right\rfloor - 1 = G(n+pq,p,q)+G(0,p,q),$$

a contradiction. This concludes the proof. $\square$

Theorem 5.9. *Given integers $p, q \ge 1$ such that $\gcd(p,q) \notin \{p,q\}$ and an integer $h \ge 0$, the value $L(h,p,q)$ can be computed in $\mathcal{O}(\log p + \log q)$ time.*

Proof. We proceed as in the proof of Corollary 4.9, except that we apply Fact 5.8 and Lemma 5.7 to compute $L^d(h,p,q)$. Fact 5.8 reduces the problem to determining $L^d(h',p,q)$, where $h' = h \bmod (p+q-2)$. We use Corollary 5.6 to compute $\mathrm{Left}_{h'+3}(\frac{p}{q})$ and $\mathrm{Right}_{h'+3}(\frac{p}{q})$ in $\mathcal{O}(\log h')$ time. The values $\widetilde{G}(r,p,q)$ can be determined in $\mathcal{O}(\log r)$ time using binary search (restricted to multiples of p or q). The overall running time for $L^d(h,p,q)$ is $\mathcal{O}(\log h') = \mathcal{O}(\log p + \log q)$, so for $L(h,p,q)$ it is also $\mathcal{O}(\log p + \log q)$. $\square$

6 Closed-Form Formula for $L(h,\cdot,\cdot)$

In this section we show how to compute a compact representation of the function $L(h,\cdot,\cdot)$ in $\mathcal{O}(h \log h)$ time. We start with such representations for $\widetilde{G}$ and L^d.

Assume that $h < p+q-3$. For $0 < i \le h+4$, let us define fractions

$$l_i = \tfrac{i-1}{h+4-i}, \quad m_i = \tfrac{i}{h+4-i},$$

called the h-special points and the h-middle points, respectively. Now, The function $\widetilde{G}$ can be expressed as follows (see Fig. 3):

Lemma 6.1. *If $\gcd(p,q) = 1$ and $h < p+q-3$, then*

$$\widetilde{G}(h+2,p,q) = \begin{cases} (h+4-i)\cdot p & \text{if } l_i \le \frac{p}{q} \le m_i, \\ i \cdot q & \text{if } m_i \le \frac{p}{q} \le l_{i+1}. \end{cases}$$

Proof. Note that $\widetilde{G}(h+2,p,q) = n$ is equivalent to $G(n-1,p,q) \le h+2 < G(n,p,q)$. Additionally, observe that $\widetilde{G}(h+2,p,q)$ is a multiple of p or q. We have two cases.

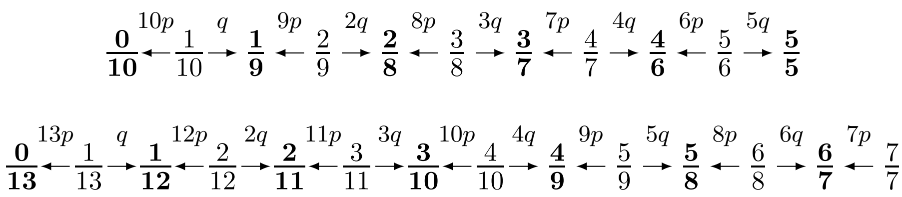

Figure 3: Graphical representations of the closed-form formulae for $\widetilde{G}(9,p,q)$ (above) and $\widetilde{G}(12,p,q)$ (below) for $p < q$: partitions of $[0,1]$ into intervals w.r.t. p/q and linear functions of p and q for each interval. The respective special points are shown in bold.

Case 1: The condition $\widetilde{G}(h+2,p,q) = j \cdot q$ for $j \in \mathbb{Z}_{>0}$ is equivalent to:

$$\left\lfloor \tfrac{jq}{p} \right\rfloor + j \ge h+3 \qquad \text{and} \qquad \left\lfloor \tfrac{jq-1}{p} \right\rfloor + j - 1 \le h+2,$$

i.e.,

$$\left\lfloor \tfrac{jq}{p} \right\rfloor \ge h+3-j \qquad \text{and} \qquad \left\lceil \tfrac{jq}{p} \right\rceil = \left\lfloor \tfrac{jq-1}{p} \right\rfloor + 1 \le h+4-j.$$

In other words, we have $h+3-j \le \frac{jq}{p} \le h+4-j$, i.e.,

$$m_j = \tfrac{j}{h+4-j} \le \tfrac{p}{q} \le \tfrac{j}{h+3-j} = l_{j+1}.$$

Case 2: The condition $\widetilde{G}(h+2,p,q) = j \cdot p$ for $j \in \mathbb{Z}_{>0}$ is equivalent to:

$$\left\lfloor \tfrac{jp}{q} \right\rfloor + j \ge h+3 \qquad \text{and} \qquad \left\lfloor \tfrac{jp-1}{q} \right\rfloor + j - 1 \le h+2,$$

i.e.,

$$\left\lfloor \tfrac{jp}{q} \right\rfloor \ge h+3-j \qquad \text{and} \qquad \left\lceil \tfrac{jq}{p} \right\rceil = \left\lfloor \tfrac{jp-1}{q} \right\rfloor + 1 \le h+4-j.$$

In other words, we have $h+3-j \le \frac{jp}{q} \le h+4-j$, i.e.,

$$l_{h+4-j} = \tfrac{h+3-j}{j} \le \tfrac{p}{q} \le \tfrac{h+4-j}{j} = m_{h+4-j}.$$

The family of intervals $[m_i, l_{i+1}]$ and $[l_i, m_i]$ has the property that any two distinct intervals in this family have disjoint interiors. Hence, the values of $\widetilde{G}(h,p,q)$ are as claimed. ☐

Combined with Lemma 5.7, Lemma 6.1 yields a closed-form formula for L^d. Note that for each i, we have $l_i \le \mathrm{Left}_{h+3}(m_i) \le m_i \le \mathrm{Right}_{h+3}(m_i) \le l_{i+1}$, but none of the inequalities is strict in general. In particular, $\mathrm{Left}_{h+3}(m_i) = m_i = \mathrm{Right}_{h+3}(m_i)$ if $\gcd(i, h+4-i) > 1$.

Corollary 6.2. *Let p,q be relatively prime positive integers and let $h \le p+q-3$ be a non-negative integer. Suppose that $l_i \le \frac{p}{q} \le l_{i+1}$ and define reduced fractions $\frac{a_i}{b_i} = \mathrm{Left}_{h+3}(m_i)$ and $\frac{c_i}{d_i} = \mathrm{Right}_{h+3}(m_i)$. Then:*

$$L^d(h,p,q) = \begin{cases} (h+4-i)\cdot p & \text{if } l_i \le \frac{p}{q} \le \frac{a_i}{b_i}, \\ a_i q + d_i p & \text{if } \frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}, \\ i \cdot q & \text{if } \frac{c_i}{d_i} \le \frac{p}{q} \le l_{i+1}. \end{cases}$$

Proof. First, observe that for $h = p+q-3$, we have $\frac{p}{q} = l_{p+1}$ and $m_p < \frac{p}{q} < m_{p+1}$, so $\frac{c_p}{d_p} \le \frac{p}{q} \le \frac{a_{p+1}}{b_{p+1}}$. As claimed, $L^d(h,p,q) = (h+4-(p+1))\cdot p = p \cdot q$.

Below, we assume $h < p+q-3$. Let $\frac{a}{b} = \mathrm{Left}_{h+3}(\frac{p}{q})$ and $\frac{c}{d} = \mathrm{Right}_{h+3}(\frac{p}{q})$. We shall prove that $a+b+c+d = h+4$ if and only if $\frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}$ for some i.

First, suppose that $a+b+c+d = h+4$. This means that $\frac{a+c}{b+d} \in \mathcal{F}_{h+4} \setminus \mathcal{F}_{h+3}$, so $\frac{a+c}{b+d} = m_i$ for some i, and therefore $\frac{a}{b} = \frac{a_i}{b_i}$ and $\frac{c}{d} = \frac{c_i}{d_i}$. Consequently, $\frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}$. In the other direction, $\frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}$ implies $\frac{a}{b} = \frac{a_i}{b_i}$ and $\frac{c}{d} = \frac{c_i}{d_i}$, so $\frac{a}{b} < m_i < \frac{c}{d}$. By Fact 5.4, this yields $a+b+c+d \le h+4$. Moreover, $\frac{a+c}{b+d} \notin \mathcal{F}_{h+3}$, so $a+b+c+d = h+4$.

Since $G(a_i q, p, q) = a_i + b_i - 1$ and $G(d_i p, p, q) = c_i + d_i - 1$ by Fact 5.4, we have $a_i q + d_i p = \widetilde{G}(a_i + b_i - 2, p, q) + \widetilde{G}(c_i + d_i - 2, p, q)$. Now, Lemmas 5.7 and 6.1 yield the final formula. ☐

Theorem 6.3. *Let $2 < p < q$ be relatively prime and let $4 < h < p+q-2$. Suppose that $l_i \le \frac{p}{q} \le l_{i+1}$ and define reduced fractions $\frac{a_i}{b_i} = \mathrm{Left}_{h+3}(m_i)$ and $\frac{c_i}{d_i} = \mathrm{Right}_{h+3}(m_i)$. Then:*

$$L(h,p,q) = \begin{cases} \left\lceil \frac{h+1}{2} \right\rceil p + q - (h+1) \bmod 2 & \text{if } 0 < \frac{p}{q} < 1/\left\lceil \frac{h}{2} \right\rceil \text{ else} \\ (h+4-i)\cdot p & \text{if } l_i \le \frac{p}{q} \le \frac{a_i}{b_i}, \\ a_i q + d_i p & \text{if } \frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}, \\ i \cdot q & \text{if } \frac{c_i}{d_i} \le \frac{p}{q} \le l_{i+1}. \end{cases}$$

This compact representation of $L(h,p,q)$ (see Fig. 4 for an example) for a given h has size $\mathcal{O}(h)$ and can be computed in time $\mathcal{O}(h \log h)$.

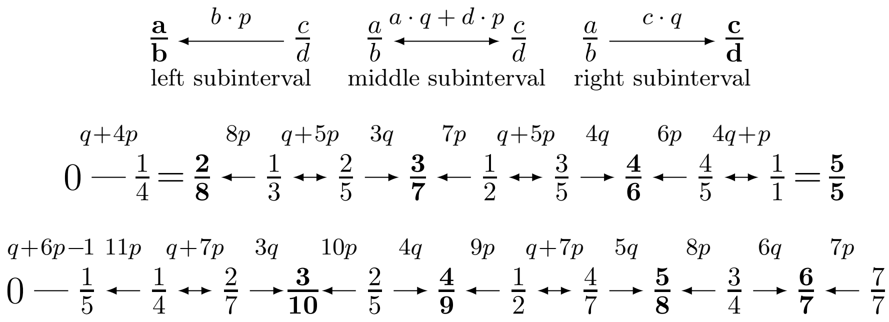

Figure 4: Graphical representations of the closed-form formulae for $L(7,p,q)$ (middle) and $L(10,p,q)$ (below). Compared to $\widetilde{G}(9,p,q)$ and $\widetilde{G}(12,p,q)$, respectively, an initial subinterval and several middle subintervals are added. A general pattern for the left, middle, and right subintervals, is presented above. However, the left subinterval $(\frac{1}{5},\frac{1}{4})$ within $L(10,p,q)$ is an exception because is has been trimmed by the initial interval.

Proof. The formula follows from the formulae for L^s (Lemma 2.4) and L^d (Corollary 6.2) combined using Theorem 4.6. To compute the table for L efficiently, we determine $\frac{a_i}{b_i} = \mathrm{Left}_{h+3}(m_i)$ and $\frac{c_i}{d_i} = \mathrm{Right}_{h+3}(m_i)$ using Corollary 5.6. ◻

7 Relation to Standard Sturmian Words

For a finite *directive sequence* $\gamma = (\gamma_1,\dots,\gamma_m)$ of positive integers, a Sturmian word $\mathrm{St}(\gamma)$ is recursively defined as X_m, where $X_{-1} = \mathsf{q}$, $X_0 = \mathsf{p}$, and $X_i = X_{i-1}^{\gamma_i} X_{i-2}$ for $1 \le i \le m$; see [16, Chapter 2]. We classify directive sequences γ (and the Sturmian words $\mathrm{St}(\gamma)$) into *even* and *odd* based on the *parity* of m.

Observation 7.1. *Odd Sturmian words of length at least 2 end with* pq*, while even Sturmian words of length at least 2 end with* qp*.*

For a directive sequence $\gamma = (\gamma_1,\dots,\gamma_m)$, we define $\mathrm{fr}(\gamma) = [0;\gamma_1,\dots,\gamma_m]$.

Fact 7.2 ([16, Proposition 2.2.24])**.** *If* $\mathrm{fr}(\gamma) = \frac{p}{q}$*, then* $\mathrm{St}(\gamma)$ *contains* p *characters* q *and* q *characters* p.

Example 7.3. We have $\frac{5}{7} = [0;1,2,2] = [0;1,2,1,1]$, so the Sturmian words with 5 q's and 7 p's are: $\mathrm{St}(1,2,2) =$ pqpqppqpqppq and $\mathrm{St}(1,2,1,1) =$ pqpqppqpqpqp.

For relatively prime integers $1 < p < q$, we define $\mathrm{St}_{p,q}$ as a Sturmian word with $\mathrm{fr}(\gamma) = \frac{p}{q}$. Note that we always have two possibilities for $\mathrm{St}_{p,q}$ (one odd and one even), but they differ in the last two positions only. In fact, the first $p+q-2$ characters of $\mathrm{St}_{p,q}$ are closely related to the values $\widetilde{G}(i,p,q)$.

Fact 7.4 ([16, Proposition 2.2.15])**.** *Let* $1 < p < q$ *be relatively prime integers. If* $i \le p+q-3$*, then*

$$\mathrm{St}_{p,q}[i] = \begin{cases} \mathsf{p} & \text{if } p \mid \widetilde{G}(i,p,q), \\ \mathsf{q} & \text{if } q \mid \widetilde{G}(i,p,q). \end{cases}$$

As a result, the values $\widetilde{G}(i,p,q)$ can be derived from $\mathrm{St}_{p,q}$; see Table 3.

Fact 7.5 ([16, Exercise 2.2.9])**.** $\mathrm{St}(\gamma'_0,\dots,\gamma'_{m'})$ *is a prefix of* $\mathrm{St}(\gamma)$ *if and only if* $[0;\gamma'_0,\dots,\gamma'_{m'}]$ *is a semiconvergent of* $\mathrm{fr}(\gamma)$.

Example 7.6. The semiconvergents of $[0;1,2,2] = \frac{5}{7} = [0;1,2,1,1]$ are $[0;1,2,1] = \frac{3}{4}$, $[0;1,2] = \frac{2}{3}$, $[0;1,1] = \frac{1}{2}$, $[0;1] = 1$, $[0;] = \frac{0}{1}$ (and $\frac{1}{0}$). They correspond to the following Sturmian prefixes of $\mathrm{St}(1,2,2) =$ pqpqppqpqppq: $\mathrm{St}(1,2,1) =$ pqpqpppq, $\mathrm{St}(1,2) =$ pqpqp, $\mathrm{St}(1,1) =$ pqp, $\mathrm{St}(1) =$ pq, and $\mathrm{St}() =$ p.

	0	1	2	3	4	5	6	7	8	9	10	11
$\mathrm{St}_{p,q}[i]$	p	q	p	q	p	p	q	p	q	p	p/q	q/p
$\widetilde{G}(i,p,q)$	p	q	$2p$	$2q$	$3p$	$4p$	$3q$	$5p$	$4q$	$6p$		
$\widetilde{G}(i,p,q)$	5	7	10	14	15	20	21	25	28	30		

Table 3: The Sturmian words $\mathrm{St}_{p,q}$ for $p = 5$ and $q = 7$ and the corresponding values of $\widetilde{G}(i,p,q)$ for $i < p + q - 2$.

Corollary 7.7. *Consider a proper prefix P of Sturmian word $\mathrm{St}(\gamma)$. Moreover, let $\frac{a}{b} = \mathrm{Left}_{|P|}(\mathrm{fr}(\gamma))$ and $\frac{c}{d} = \mathrm{Right}_{|P|}(\mathrm{fr}(\gamma))$. The longest even Sturmian prefix of P has length $a + b$, whereas the longest odd Sturmian prefix of P has length $c + d$.*

Proof. By Fact 7.5, the longest even Sturmian prefix of P is the longest Sturmian word $\mathrm{St}(\gamma')$ such that $\frac{a'}{b'} := \mathrm{fr}(\gamma')$ is an even semiconvergent of $\mathrm{fr}(\gamma)$. Its length $a' + b' \le |P|$ is largest possible, so by Fact 5.4 $\frac{a'}{b'}$ is the best left approximation of $\mathrm{fr}(\gamma)$ with $a' + b' \le |P|$. This is precisely how $\frac{a}{b} = \mathrm{Left}_{|P|}(\mathrm{fr}(\gamma))$ is defined.

The proof for odd Sturmian prefixes is symmetric. □

The following theorem can be seen as a restatement of Lemma 5.7 in terms of Sturmian words.

Theorem 7.8. *Let $\mathrm{St}_{p,q}$ be a standard Sturmian word corresponding to $\frac{p}{q}$ and let $0 \le h < p + q - 3$. If $\mathrm{St}_{p,q}[0..h+3]$ is a Sturmian word, then $L^d(h,p,q) = \widetilde{G}(l-2,p,q) + \widetilde{G}(r-2,p,q)$, where l, r are the lengths of the longest proper Sturmian prefixes of $\mathrm{St}_{p,q}[0..h+3]$ of different parities, and $\widetilde{G}(-1,p,q) = 0$. Otherwise, $L^d(h,p,q) = \widetilde{G}(h+2,p,q)$.*

Proof. To apply Lemma 5.7, we set $\frac{a}{b} = \mathrm{Left}_{h+3}(\frac{p}{q}$ and $\frac{c}{d} = \mathrm{Right}_{h+3}(\frac{p}{q})$. Observe that the mediant $\frac{a+c}{b+d}$ is a better approximation of $\frac{p}{q}$ than $\frac{a}{b}$ or $\frac{c}{d}$, and thus it is a semiconvergent of $\frac{p}{q}$. Thus, we always have $a+b+c+d \ge h+4$ and, by Fact 7.5, equality holds if and only if $\mathrm{St}_{p,q}$ has a Sturmian prefix of length $h+4$. In other words, the case distinction here coincides with the one in Lemma 5.7. If $a+b+c+d > h+4$, then we have $L^d(h,p,q) = \widetilde{G}(h+2,p,q)$. Otherwise, $L^d(h,p,q) = \widetilde{G}(a+b-2,p,q) + \widetilde{G}(c+d-2,p,q)$. However, due to Corollary 7.7, $\mathrm{St}_{p,q}[0..a+b-1]$ is an even Sturmian word corresponding to (a,b), $\mathrm{St}_{p,q}[0..c+d-1]$ is an odd Sturmian word corresponding to (c,d), and these are the longest Sturmian prefixes of $\mathrm{St}_{p,q}[0..h+2]$ of each parity. □

Example 7.9. Consider a word $\mathrm{St}_{5,7}$ as in Table 3. The lengths of its proper even Sturmian prefixes are $2, 7$, whereas the lengths of its proper odd Sturmian prefixes are $1, 3, 5$. Hence, $L^d(7,5,7) = \widetilde{G}(9,5,7) = 30$, since $\mathrm{St}_{5,7}[0..10]$ is not a Sturmian word. Moreover, $L^d(8,5,7) = \widetilde{G}(5,5,7) + \widetilde{G}(3,5,7) = 20 + 14 = 34$, since $\mathrm{St}_{5,7}[0..11] = \mathrm{St}_{5,7}$ is a Sturmian word.

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