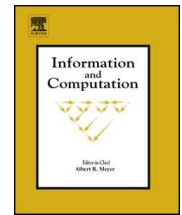


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Efficient representation and counting of antipower factors in words

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ARTICLE INFO

Article history:

Received 12 August 2019

Received in revised form 14 October 2020

Accepted 24 March 2021

Available online xxxx

Keywords:

Antipower

Gapped repeat

Run (maximal repetition)

ABSTRACT

A k -antipower (for $k \geq 2$) is a concatenation of k pairwise distinct words of the same length. The study of fragments of a word being antipowers was initiated by Fici et al. (ICALP 2016) and first algorithms for computing such fragments were presented by Badkobeh et al. (Inf. Process. Lett., 2018). We address two open problems posed by Badkobeh et al. We propose efficient algorithms for counting and reporting fragments of a word which are k -antipowers. They work in $\mathcal{O}(nk \log k)$ time and $\mathcal{O}(nk \log k + C)$ time, respectively, where C is the number of reported fragments. We also show that the number of different k -antipower factors of a word of length n can be computed in $\mathcal{O}(nk^4 \log k \log n)$ time. Our main algorithmic tools are runs and gapped repeats.

This is a full and extended version of a paper presented at LATA 2019.

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1. Introduction

One of the most common types of regular words are powers. If equality is replaced by inequality, other versions of powers are obtained. Antipowers are a new type of regularity of words, based on diversity rather than on equality, that was recently introduced by Fici et al. [1,2]. Algorithmic study of antipowers was initiated by Badkobeh et al. [3]. Very recently, a related concept of antiperiods was considered by Alamro et al. [4].

Let us assume that $x = y_0 \cdots y_{k-1}$, where $k \geq 2$ and y_i are words of the same length d . We then say that:

- x is a k -power if all y_i 's are the same;
- x is a k -antipower (or a (k, d) -antipower) if all y_i 's are pairwise distinct;
- x is a weak k -power (or a weak (k, d) -power) if it is not a k -antipower, that is, if $y_i = y_j$ for some $i \neq j$;
- x is a gapped (q, d) -square if $y_0 = y_{k-1}$ and $q = k - 2$.

In the first three cases, the length d is called the *base* of the power or antipower x .

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¹ Supported by the Polish National Science Center, grant no 214/13/B/ST6/00770.

² Supported by the "Algorithms for text processing with errors and uncertainties" project carried out within the HOMING program of the Foundation for Polish Science co-financed by the European Union under the European Regional Development Fund, grant no POIR.04.04.00-00-24BA/16.

<https://doi.org/10.1016/j.ic.2021.104779>

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If w is a word, then by $w[i..j]$ we denote a *fragment* of w composed of letters $w[i], \dots, w[j]$. The corresponding word $w[i] \dots w[j]$ is called a *factor* of w . The fragment $w[i..j]$ (which is an occurrence of the factor $w[i] \dots w[j]$) can be concisely represented by the indices i and j . Badkobeh et al. [3] considered fragments of a word that are antipowers and obtained the following result.

Fact 1.1 ([3]). *The maximum number of k -antipower fragments in a word of length n is $\Theta(n^2/k)$, and they can all be reported in $\mathcal{O}(n^2/k)$ time. In particular, all k -antipower fragments of a specified base d can be reported in $\mathcal{O}(n)$ time.*

Badkobeh et al. [3] asked for an output-sensitive algorithm that reports all k -antipower fragments in a given word. We present such an algorithm. En route to enumerating k -antipowers, we (complementarily) find weak k -powers. Also gapped (q, d) -squares play an important role in our algorithm.

2-antipowers can be called *antisquares*. An antisquare is simply an even-length word that is not a square. The number of fragments of a word of length n being squares can obviously be $\Theta(n^2)$, e.g., for the word a^n . However, the number of different square factors in a word of length n is $\mathcal{O}(n)$; see [5,6]. In comparison, the number of different antisquare factors of a word of length n can already be $\Theta(n^2)$. For example, this is true for a de Bruijn word. Still, we show that the number of different antisquare factors of a word can be computed in $\mathcal{O}(n)$ time and that the number of different k -antipower factors for relatively small values of k can also be computed efficiently.

For a given word w , an *antipower query* (i, j, k) asks to check if a fragment $w[i..j]$ is a k -antipower. Badkobeh et al. [3] proposed the following data structures for answering such queries.

Fact 1.2 ([3]). *Antipower queries can be answered (a) in $\mathcal{O}(k)$ time with a data structure of size $\mathcal{O}(n)$; (b) in $\mathcal{O}(1)$ time with a data structure of size $\mathcal{O}(n^2)$.*

In either case, answering n antipower queries using Fact 1.2 requires $\Omega(n^2)$ time in the worst case (including construction of the data structure). We show a trade-off between the data structure space (and construction time) and query time that allows answering any n antipower queries more efficiently.

Our results We assume an integer alphabet $\{1, \dots, n^{\mathcal{O}(1)}\}$. Our first result is an algorithm that computes the number C of k -antipower fragments of a word of length n in $\mathcal{O}(nk \log k)$ time and reports them in $\mathcal{O}(nk \log k + C)$ time.

Our second result is an algorithm that computes the number of different factors of a word of length n that are k -antipowers in $\mathcal{O}(nk^4 \log k \log n)$ time.

Our third result is a construction in $\mathcal{O}(n^2/r)$ time of a data structure of size $\mathcal{O}(n^2/r)$, for any $r \in \{1, \dots, n\}$, which answers antipower queries in $\mathcal{O}(r)$ time. Thus, any n antipower queries can be answered in $\mathcal{O}(n\sqrt{n})$ time and space.

This is a full and extended version of [7].

Structure of the paper Our algorithms are based on a relation between weak powers and two notions of periodicity of words: gapped repeats and runs. In Section 2, we recall important properties of these notions. Section 4 shows a simple algorithm that counts k -antipower fragments in a word of length n in $\mathcal{O}(nk^3)$ time. In Section 5, it is improved in three steps to an $\mathcal{O}(nk \log k)$ -time algorithm. One of the steps applies static range trees that are recalled in Section 3. Algorithms for reporting k -antipower fragments and answering antipower queries are presented in Section 6. The reporting algorithm makes a more sophisticated application of the static range tree that is also described in Section 3. Finally, an algorithm that counts the number of different k -antipower factors in a word of length n in $\mathcal{O}(nk^4 \log k \log n)$ time is shown in Section 7.

2. Preliminaries

The length of a word w is denoted by $|w|$ and the letters of w are numbered 0 through $|w| - 1$, with $w[i]$ representing the i th letter. Let $[i..j]$ denote the integer interval $\{i, i+1, \dots, j\}$ and $[i..j)$ denote $[i..j-1]$. By $w[i..j]$ we denote the *fragment* of w between the i th and the j th letter, inclusively. If $i > j$, the fragment is empty. Let us further denote $w[i..j) = w[i..j-1]$. The word $w[i] \dots w[j]$ that corresponds to the fragment $w[i..j]$ is called a *factor* of w . Thus the two main counting algorithms that we develop count different k -antipower fragments and different k -antipower factors of the input word, respectively.

By w^R we denote the reversed word w . We say that p is a *period* of the word w if $w[i] = w[i+p]$ holds for all $i \in [0..|w|-p)$.

An α -gapped repeat γ (for $\alpha \geq 1$) in a word w is a triple (i, j, p) such that $w[i..j]$ is of the form uvu for $p = |uv| \leq \alpha|u|$. The two occurrences of u are called *arms* of the α -gapped repeat and p , denoted $\text{per}(\gamma)$, is called *the period* of the α -gapped repeat. Note that an α -gapped repeat is also an α' -gapped repeat for every $\alpha' > \alpha$. An α -gapped repeat is called *maximal* if its arms can be extended simultaneously with the same character neither to the right nor to the left. In short, we call maximal α -gapped repeats α -MGRs and the set of α -MGRs in a word w is further denoted by $\text{MGR}_{\alpha}(w)$.

Kolpakov et al. [8] showed the first upper bound $\mathcal{O}(n\alpha^2)$ on the number of α -MGRs and proposed an $\mathcal{O}(n\alpha^2)$ -time algorithm computing α -MGRs. This result was later improved by Tanimura et al. [9] and Crochemore et al. [10] resulting

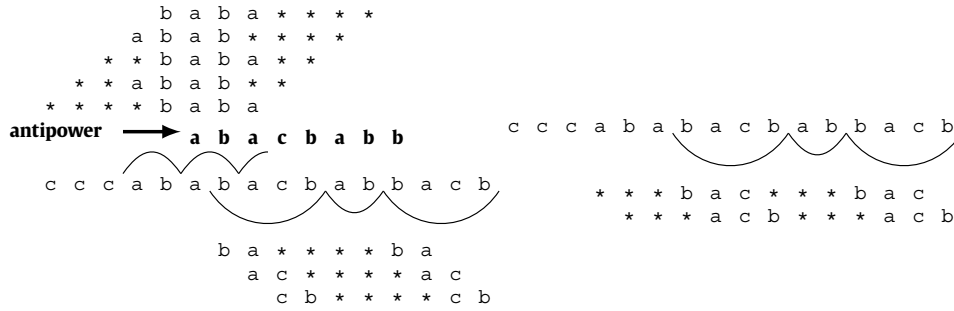


Fig. 1. To the left: all weak (4,2)-powers and the only (4,2)-antipower in a word of length 16. An asterisk denotes any character. The first five weak (4,2)-powers are generated by the run *ababa* with period 2, and the last three are generated by the 1.5-MGR *bacbabbacb*, whose period 6 is divisible by 2. To the right: all weak (4,3)-powers in the same word are generated by the same MGR because its period is a multiple of 3.

in an upper bound of $\mathcal{O}(n\alpha)$ and an $\mathcal{O}(n\alpha)$ -time algorithm working for constant-sized alphabets. Gawrychowski et al. [11] provided a constant of 18 in the upper bound and obtained an algorithm that works for integer alphabets. The constant was later improved by I and Köppl [12] to $3(\pi^2/6 + 5/2) \approx 12.435$. We use the following fact that summarizes the state of the art.

Fact 2.1 ([11,12]). *Given a word w of length n and a parameter α , the set $\text{MGR}_{\alpha}(w)$ can be computed in $\mathcal{O}(n\alpha)$ time and satisfies $|\text{MGR}_{\alpha}(w)| \leq 3(\pi^2/6 + 5/2)\alpha n$.*

Other results related to gapped repeats were shown in [13,14].

A *run* (a maximal repetition) in a word w is a triple (i, j, p) such that $w[i..j]$ is a fragment with the smallest period p , $2p \leq j - i + 1$, that can be extended neither to the left nor to the right preserving the period p . Its *exponent* e is defined as $e = (j - i + 1)/p$. Kolpakov and Kucherov [15] showed that a word of length n has $\mathcal{O}(n)$ runs, that the sum of their exponents is $\mathcal{O}(n)$, and that they can be computed in $\mathcal{O}(n)$ time. Bannai et al. [16] recently refined these combinatorial results and developed a simpler the linear-time algorithm.

Fact 2.2 ([16]). *A word of length n has at most n runs, and the sum of their exponents does not exceed $3n$. All these runs can be computed in $\mathcal{O}(n)$ time.*

Further work [17,18] led to improved constants in the upper bound, although only for the case of binary strings.

A *generalized run* in a word w is a triple $\gamma = (i, j, p)$ such that $w[i..j]$ is a fragment with a period p , not necessarily the shortest one, $2p \leq j - i + 1$, that can be extended neither to the left nor to the right preserving the period p . By $\text{per}(\gamma)$ we denote p , called *the period* of the generalized run γ . The set of generalized runs in a word w is denoted by $\text{GRuns}(w)$.

A run (i, j, p) with exponent e corresponds to $\lfloor \frac{e}{2} \rfloor$ generalized runs: (i, j, p) , $(i, j, 2p)$, $(i, j, 3p)$, ..., $(i, j, \lfloor \frac{e}{2} \rfloor p)$. By Fact 2.2, we obtain the following.

Corollary 2.3. *For a word w of length n , $|\text{GRuns}(w)| \leq 1.5n$ and this set can be computed in $\mathcal{O}(n)$ time.*

Our algorithm uses a relation between weak powers, α -MGRs, and generalized runs; see Fig. 1 for an example presenting the interplay of these notions.

The *interval representation* of a finite set X of integers is

$$X = [i_1 .. j_1] \cup [i_2 .. j_2] \cup \dots \cup [i_t .. j_t],$$

where $i_1 \leq j_1$, $j_1 + 1 < i_2$, $i_2 \leq j_2$, ..., $j_{t-1} + 1 < i_t$, and $i_t \leq j_t$. We denote this representation by $\mathcal{R}(X)$. The value t , denoted $|\mathcal{R}(X)|$, is called *the size of the representation*. The following simple lemma allows implementing basic operations on interval representations.

Lemma 2.4. *Assume that $\mathcal{X}_1, \dots, \mathcal{X}_r$ are non-empty families of subintervals of $[0..n]$. The interval representations of $\bigcup \mathcal{X}_1, \bigcup \mathcal{X}_2, \dots, \bigcup \mathcal{X}_r$ can be computed in $\mathcal{O}(n + m)$ time, where m is the total size of the families \mathcal{X}_i . Similarly, the interval representation of $\bigcap_{i=1}^r (\bigcup \mathcal{X}_i)$ can be computed in $\mathcal{O}(n + m)$ time.*

Proof. We start by sorting the endpoints of the intervals and grouping them by the index i of the family \mathcal{X}_i . This can be done in $\mathcal{O}(n + m)$ time using bucket sort [19]. Next, for each i , to compute the interval representation of $\bigcup \mathcal{X}_i$, we scan the endpoints left to right maintaining the number of intervals containing the current point. We start an interval when this number becomes positive and end one when the number drops to 0. This processing takes $\mathcal{O}(m)$ time.

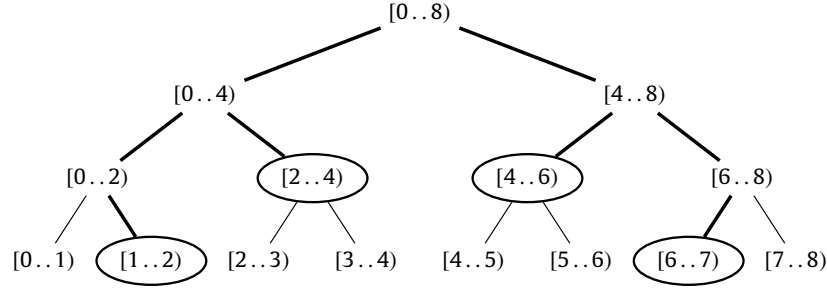


Fig. 2. A static range tree for $m = 8$ with the set of nodes that comprises $\text{Decomp}([1..7])$. The paths visited in the recursive decomposition algorithm are shown in bold.

In order to compute the representation of the intersection, we use the same type of a counter when simultaneously processing the interval representations of $\bigcup \mathcal{X}_1, \bigcup \mathcal{X}_2, \dots, \bigcup \mathcal{X}_r$, but start an interval only when the counter becomes equal to r and end one when the counter drops below r . \square

Let \mathcal{J} be a family of subintervals of $[0..m)$, initially empty. Let us consider the following operations on \mathcal{J} , where I is an interval: $\text{insert}(I): \mathcal{J} := \mathcal{J} \cup \{I\}$; $\text{delete}(I): \mathcal{J} := \mathcal{J} \setminus \{I\}$ for $I \in \mathcal{J}$; and count , which returns $|\bigcup \mathcal{J}|$. It is folklore knowledge that all these operations can be performed efficiently using a static range tree (sometimes called a segment tree; see, e.g., [20, Section 2.1]). In Section 3, we prove the following lemma for completeness.

Lemma 2.5. *There exists a data structure of size $\mathcal{O}(m)$ that, after $\mathcal{O}(m)$ -time initialization, supports insert and delete in $\mathcal{O}(\log m)$ time and count in $\mathcal{O}(1)$ time.*

Let us introduce another operation report that returns all elements of the set $A = [0..m) \setminus \bigcup \mathcal{J}$. We also show in Section 3 that a static range tree can be augmented to support this operation efficiently.

Lemma 2.6. *There exists a data structure of size $\mathcal{O}(m)$ that, after $\mathcal{O}(m)$ -time initialization, supports insert and delete in $\mathcal{O}(\log m)$ time and report in $\mathcal{O}(|A|)$ time.*

3. Applications of static range tree

Let m be a power of two. We define a *basic interval* as an interval of the form $[2^i a .. 2^i(a+1)) \subseteq [0..m-1)$ such that i and a are non-negative integers. For example, the basic intervals for $m = 8$ are $[0..1), \dots, [7..8), [0..2), [2..4), [4..6), [6..8), [0..4), [4..8), [0..8)$. In a *static range tree* (sometimes called a segment tree; see [20, Section 2.1]), each node is identified with a basic interval. The children of a node $J = [2^i a .. 2^i(a+1))$, for $i > 0$, are $\text{lchild}(J) = [2^{i-1} \cdot 2a .. 2^{i-1}(2a+1))$ and $\text{rchild}(J) = [2^{i-1}(2a+1) .. 2^{i-1}(2a+2))$. Thus, a static range tree is a full binary tree of size $\mathcal{O}(m)$ rooted at $\text{root} = [0..m)$.

Every interval $I \subseteq [0..m)$ can be decomposed into a disjoint union of at most $2 \log m$ basic intervals. The decomposition can be computed in $\mathcal{O}(\log m)$ time recursively starting from the root. Let J be a node considered in the algorithm. If $J \subseteq I$, the algorithm adds J to the decomposition. Otherwise, for each child J' of the node J , if $J' \cap I \neq \emptyset$, the algorithm makes a recursive call to the child. At each level of the tree, the algorithm makes at most two recursive calls. The resulting set of basic intervals is denoted by $\text{Decomp}(I)$; see Fig. 2.

Proofs of the Lemmas 2.5 and 2.6 follow. Instead of Lemma 2.5, we show an equivalent lemma with an operation count' which returns $|[0..m) \setminus \bigcup \mathcal{J}|$.

Lemma 3.1. *There exists a data structure of size $\mathcal{O}(m)$ that, after $\mathcal{O}(m)$ -time initialization, supports insert and delete in $\mathcal{O}(\log m)$ time and count' in $\mathcal{O}(1)$ time.*

Proof. Let m' be the smallest power of two satisfying $m' \geq m$. Observe that the data structure for m can be simulated by an instance constructed for m' : it suffices to insert an interval $[m..m')$ in the initialization phase to make sure that integers $i \geq m$ will not be counted when count' is invoked. Henceforth, we may assume without loss of generality that m is a power of two.

We maintain a static range tree; every node J stores two values (see Fig. 3):

- $\text{bi}(J) = |\{I \in \mathcal{J} : J \in \text{Decomp}(I)\}|$
- $\text{val}(J) = |J \setminus \bigcup \{J' : J' \subseteq J, J' \in \text{Decomp}(I), I \in \mathcal{J}\}|$.

The value $\text{val}(J)$ can also be defined recursively:

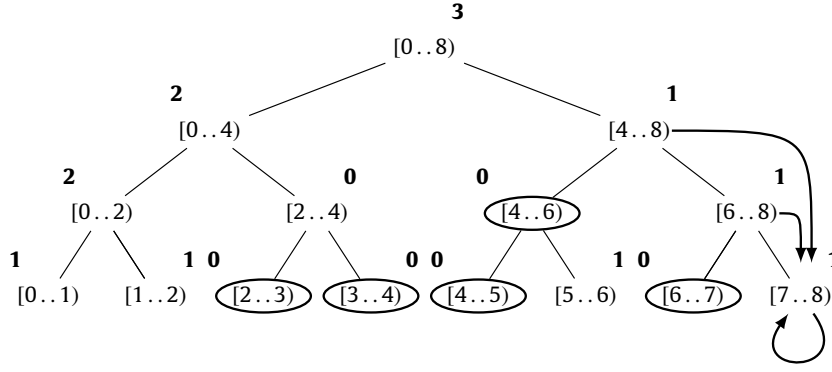


Fig. 3. A static range tree for $m = 8$ that stores the family $\mathcal{J} = \{[2..3], [3..5], [4..7], [6..7]\}$. The values $val(J)$ are shown in bold. The arrows present selected *jump* pointers (described in the proof of Lemma 2.6).

- If $bi(J) > 0$, then $val(J) = 0$.
- If $bi(J) = 0$ and J is a leaf, then $val(J) = 1$.
- Otherwise, $val(J) = val(lchild(J)) + val(rchild(J))$.

This allows computing $val(J)$ from $bi(J)$ and the values in the children of J .

The data structure can be initialized bottom-up in $\mathcal{O}(m)$ time. The respective operations on the data structure are now implemented as follows:

- **insert(I):** Compute $Decomp(I)$ recursively. For each $J \in Decomp(I)$, increment $bi(J)$. For each node J encountered in the recursive computation, recompute $val(J)$.
- **delete(I):** Similar to **insert**, but we decrement $bi(J)$ for each node $J \in Decomp(I)$.
- **count':** Return $val(root)$.

The complexities of the respective operations follow. \square

Lemma 2.6. *There exists a data structure of size $\mathcal{O}(m)$ that, after $\mathcal{O}(m)$ -time initialization, supports **insert** and **delete** in $\mathcal{O}(\log m)$ time and **report** in $\mathcal{O}(|A|)$ time.*

Proof. As in the proof of Lemma 3.1, we assume without loss of generality that m is a power of two. Again, the data structure applies a static range tree. We also reuse the values $bi(J)$ for nodes; we generalize the $val(J)$ values, though.

If J and J' are basic intervals and $J' \subseteq J$, then we define $val_J(J')$ as 0 if there exists a basic interval J'' on the path from J to J' (i.e., such that $J' \subseteq J'' \subseteq J$) for which $bi(J'') > 0$, and as $val(J')$ otherwise. These values satisfy the following properties.

Observation 3.2. *For every node J , (a) $val_J(J) = val(J)$ and (b) $val_{root}(J) = |J \setminus \bigcup \mathcal{J}|$.*

By point (b) of the observation, our goal in a **report** query is to report all leaves J such that $val_{root}(J) = 1$. The first idea how to do it would be to recursively visit all the nodes J' of the tree such that $val_{root}(J') > 0$. However, this approach would cost $\Omega(|A| \log m)$ time since, for every leaf, all the nodes on the path to the root would need to be visited.

In order to efficiently answer **report** queries, we introduce *jump* pointers, stored in each node J , such that $jump(J)$ is the lowest such node J' in the subtree of J such that $val_J(J') = val_J(J)$; see Fig. 3.

The pointer $jump(J)$ can be computed in $\mathcal{O}(1)$ time from the values in the children of J :

$$jump(J) = \begin{cases} J & \text{if } J \text{ is a leaf or } 0 < val(lchild(J)) < val(J), \\ jump(lchild(J)) & \text{if } val(rchild(J)) = 0, \\ jump(rchild(J)) & \text{otherwise.} \end{cases}$$

This formula allows recomputing the *jump* pointers on the paths visited during a call to **insert** or **delete** without altering the time complexity.

Let us consider a subtree that is composed of all the nodes J with positive values $val_{root}(J)$. Using *jump* pointers, we make a recursive traversal of the subtree that avoids visiting long paths of non-branching nodes of the subtree. It visits all the leaves and branching nodes of the subtree and, in addition, both children of each branching node. With this traversal, a **report** query is therefore answered in $\mathcal{O}(|A|)$ time. \square

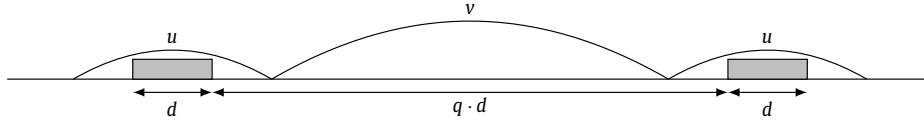


Fig. 4. An occurrence of a gapped (q, d) -square generated by a gapped repeat with period $(q + 1)d$. Gray rectangles represent equal words.

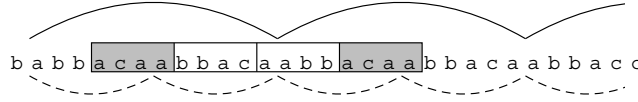


Fig. 5. An occurrence of a gapped $(2, 4)$ -square $acaaabbaacaaabbaacaa$ generated by a generalized run with period 12. Note that the generalized run has its origin in a (generalized) run with period 6 (depicted below) that *does not* generate this gapped square.

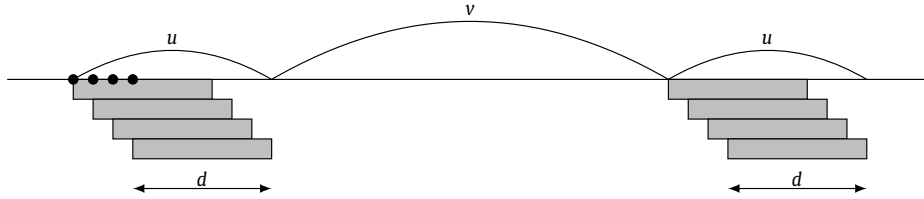


Fig. 6. An interval, represented as a sequence of four consecutive positions (black dots), of starting positions of gapped (q, d) -square fragments generated by a gapped repeat with period $(q + 1)d$.

4. Computing a compact representation of weak k -powers

Let us denote by $Squares(q, d)$ the set of starting positions of gapped (q, d) -square fragments in the input word w .

We say that an occurrence at position i of a gapped (q, d) -square is *generated* by a gapped repeat γ if $\text{per}(\gamma) = (q + 1)d$ and $w[i \dots i + (q + 2)d]$ is fully contained in this gapped repeat. In other words, if γ is of the form uvu , then $u = u_1u_2u_3$, $|u_2| = d$, $|u_3vu_1| = qd$, and γ starts at position $i - |u_1|$ of the input word; cf. Fig. 4.

Similarly, an occurrence at position i of a gapped (q, d) -square is *generated* by a generalized run γ if $\text{per}(\gamma) = (q + 1)d$ and $w[i \dots i + (q + 2)d]$ is fully contained in this generalized run. See Fig. 5 for a concrete example.

Lemma 4.1. Let $q \geq 0$ and $1 \leq d \leq n/(q + 2)$ be integers.

- (a) Every gapped (q, d) -square fragment is generated by a $(q + 1)$ -MGR with period $(q + 1)d$ or by a generalized run with period $(q + 1)d$.
- (b) Each gapped repeat with period $(q + 1)d$ and each generalized run γ with period $(q + 1)d$ generates a single interval of positions where gapped (q, d) -squares occur; see Fig. 6. Moreover, this interval can be computed in constant time.

Proof. (a) Let i be the starting position of an occurrence of a gapped (q, d) -square x of length $\ell := |x| = (q + 2)d$. Observe that x has period $p := (q + 1)d$. We denote by $w[i' \dots j']$ the longest factor with period p that contains x (i.e., such that $i' \leq i$ and $i + \ell - 1 \leq j'$), and denote $\gamma := (i', j', p)$.

If $|w[i' \dots j']| < 2p$, then γ is a gapped repeat with period p , and it is maximal by definition. Moreover, it is a $(q + 1)$ -MGR since its arms have length at least d .

If $|w[i' \dots j']| \geq 2p$, then γ is a generalized run that generates the gapped square x . In particular, this happens for $q = 0$.

(b) Let $\gamma = (i, j, p)$ be a gapped repeat or a generalized run with period $p = (q + 1)d$. Then γ generates gapped (q, d) -squares that start at positions in the interval $[i \dots j + 1 - (p + d)]$. \square

The interval defined in Lemma 4.1(b) will be further denoted by $Squares(q, d, \gamma)$.

Corollary 4.2. For all integers $q \geq 0$ and $d \geq 1$, we have $\mathcal{R}(Squares(q, d)) = \{Squares(q, d, \gamma) : \gamma \in \text{MGReps}_{q+1}(w) \cup \text{GRuns}(w) \text{ and } \text{per}(\gamma) = (q + 1)d\}$. Moreover, for every $q \geq 0$,

$$\sum_{d=1}^{\lfloor n/(q+2) \rfloor} |\mathcal{R}(Squares(q, d))| = \mathcal{O}(n(q + 1))$$

and these representations $\mathcal{R}(Squares(q, d))$ can be computed in $\mathcal{O}(n(q + 1))$ time.

Proof. Lemma 4.1 shows that $Squares(q, d)$ can be expressed as the union of intervals $Squares(q, d, \gamma)$, where γ is a generalized run with period $(q + 1)d$ or a $(q + 1)$ -MGR with period $(q + 1)d$. Moreover, by maximality of generalized runs

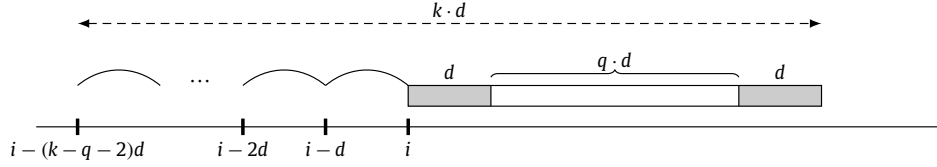


Fig. 7. The fact that $i \in \text{Squares}(q, d)$ witnesses inclusion $(\text{Chain}_k(q, d, i) \cap [0..n - kd]) \subseteq \text{WeakP}_k(d)$.

and $(q + 1)$ -MGRs, no two such intervals $\text{Squares}(q, d, \gamma)$ overlap (contain a common position) or touch (contain adjacent positions).

As for the second claim, the total number of interval chains in these representations is $\mathcal{O}(n(q + 1))$ because the number of $(q + 1)$ -MGRs and generalized runs γ is bounded by $\mathcal{O}(n(q + 1))$ due to Facts 2.1 and 2.2, respectively. Since each interval $\text{Squares}(q, d, \gamma)$ can be computed in constant time (by Lemma 4.1(b)), this also yields an $\mathcal{O}(n(q + 1))$ bound on the construction time. \square

Let us denote

$$\text{Chain}_k(q, d, i) = \{i, i - d, i - 2d, \dots, i - (k - q - 2)d\}.$$

This definition can be extended to intervals I . To this end, let us denote

$$I \ominus r = \{i - r : i \in I\}$$

and define

$$\text{Chain}_k(q, d, I) = I \cup (I \ominus d) \cup (I \ominus 2d) \cup \dots \cup (I \ominus (k - q - 2)d).$$

This set, further referred to as an *interval chain*, can be stored in $\mathcal{O}(1)$ space. A *chain representation* of a set of integers is its representation as a union of interval chains, limited to some base interval. The size of the chain representation is the number of chains.

We denote by $\text{WeakP}_k(d)$ the set of starting positions in w of weak (k, d) -power fragments. The following lemma shows how to compute small chain representations of the sets $\text{WeakP}_k(d)$ (with base intervals $[0..n - kd]$).

Lemma 4.3.

(a) $\text{WeakP}_k(d) = \bigcup_{q=0}^{k-2} \bigcup_{I \in \mathcal{R}(\text{Squares}(q, d))} \text{Chain}_k(q, d, I) \cap [0..n - kd]$.

(b) For $d = 1, \dots, \lfloor n/k \rfloor$, the sets $\text{WeakP}_k(d)$ have chain representations of total size $\mathcal{O}(nk^2)$; these representations can be computed in $\mathcal{O}(nk^2)$ time.

Proof. As for point (a), $x = y_0 \dots y_{k-1}$ for $|y_0| = \dots = |y_{k-1}| = d$ is a weak (k, d) -power if and only if $y_i \dots y_j$ is a gapped $(j - i - 1, d)$ -square for some $0 \leq i < j < k$. Conversely, a gapped (q, d) -square occurring at position i implies occurrences of weak (k, d) -powers at positions in the set $\text{Chain}_k(q, d, i)$, limited to the interval $[0..n - kd]$ due to the length constraint; see Fig. 7.

We obtain point (b) by applying the formula from point (a) to compute the chain representations of sets $\text{WeakP}_k(d)$ for all $d = 1, \dots, \lfloor n/k \rfloor$. The interval representations $\mathcal{R}(\text{Squares}(q, d))$ are computed using Corollary 4.2, which costs $\mathcal{O}(n(q + 1))$ time for each $q \in [0..k - 2]$ and $\mathcal{O}(nk^2)$ time in total. The total size of the interval representations and the chain representations is also $\mathcal{O}(nk^2)$. \square

Lemma 4.3 lets us count k -antipowers by computing the sizes of the complementary sets $\text{WeakP}_k(d)$. Thus, we obtain the following preliminary result.

Proposition 4.4. The number of k -antipower fragments in a word of length n can be computed in $\mathcal{O}(nk^3)$ time.

Proof. See Algorithm 1. We use Corollary 4.2 and Lemma 4.3 to express the sets $\text{WeakP}_k(d)$ for all $d = 1, \dots, \lfloor n/k \rfloor$ as unions of $\mathcal{O}(nk^2)$ interval chains. That is, the total size of the sets \mathcal{C}_d is $\mathcal{O}(nk^2)$. Each of the interval chains consists of at most k intervals. Hence, Lemma 2.4 can be applied to compute interval representations of the sets $\text{WeakP}_k(d)$ in $\mathcal{O}(nk^3)$ total time. Finally, the size of the complement of the set $\text{WeakP}_k(d)$ (in $[0..n - kd]$) is the number of (k, d) -antipowers. \square

Next, we improve the time complexity of this algorithm to $\mathcal{O}(nk \log k)$.

Algorithm 1: SimpleCount(w, n, k).

```

1   $(C_d)_{d=1}^{\lfloor n/k \rfloor} := (\emptyset, \dots, \emptyset)$ 
2  for  $q := 0$  to  $k - 2$  do
3      foreach  $(q + 1)$ -MGR or generalized run  $\gamma$  in  $w$  do
4           $p := \text{per}(\gamma)$ 
5           $d := \frac{p}{q+1}$ 
6          if  $d \in \mathbb{Z}$  then
7               $I := \text{Squares}(q, d, \gamma)$ 
8               $C_d := C_d \cup \{\text{Chain}_k(q, d, I)\}$ 
9   $\text{antipowers} := 0$ 
10 for  $d := 1$  to  $\lfloor n/k \rfloor$  do
11      $\text{WeakP}_k(d) := (\bigcup C_d) \cap [0..n - kd]$ 
12      $\text{antipowers} := \text{antipowers} + (n - kd + 1) - |\text{WeakP}_k(d)|$ 
13 return  $\text{antipowers}$ 

```

5. Counting k -antipower fragments in $\mathcal{O}(nk \log k)$ time

We improve the algorithm SimpleCount threefold. First, we show that the chain representation of weak k -powers actually consists of only $\mathcal{O}(nk)$ chains. Then, instead of converting the chain representations to the interval representations, we introduce a geometric interpretation that reduces the problem to computing the area of the union of $\mathcal{O}(nk)$ axis-aligned rectangles. This area could be computed directly in $\mathcal{O}(nk \log n)$ time, but we improve this complexity to $\mathcal{O}(nk \log k)$ by exploiting properties of the dimensions of the rectangles.

5.1. First improvement of SimpleCount

First, we improve the $\mathcal{O}(nk^2)$ bounds of Lemma 4.3(b). By inspecting the structure of MGRs, we show that the formula from Lemma 4.3(a) generates only $\mathcal{O}(nk)$ interval chains in total for $d \geq 2k - 2$. Trivially, the sets $\text{WeakP}_k(d)$ for $d < 2k - 2$ can be represented using $\mathcal{O}(n)$ interval chains each. A careful implementation lets us compute all such chain representations in $\mathcal{O}(nk)$ time.

We say that an α -MGR for integer α with period p is *nice* if $\alpha \mid p$ and $p \geq 2\alpha^2$. Let $\text{NMGR}_{\alpha}(w)$ denote the set of nice α -MGRs in the word w . The following lemma provides a combinatorial foundation of the improvement.

Lemma 5.1. *For a word w of length n and an integer $\alpha > 1$, $|\text{NMGR}_{\alpha}(w)| \leq 9(\pi^2/6 + 5/2)n$.*

Proof. Let us consider a partition of the word w into blocks of α letters (the final $n \bmod \alpha$ letters are not assigned to any block). Let uvu be a nice α -MGR in w . We know that $2\alpha^2 \leq |uv| \leq \alpha|u|$, so $|u| \geq 2\alpha$. Now, let us fit the considered α -MGR into the structure of blocks. Since $\alpha \mid |uv|$, the indices in w of the occurrences of the left and the right arm are equal modulo α . We shrink both arms to u' such that u' is the maximal inclusion-wise interval of blocks which is encompassed by each arm u . Then, let us expand v to v' so that it fills the space between the two occurrences of u' .

We have $u = xu'y$ and $v' = yvx$ for some words x, y , so $|u'v'| = |u'| + |x| + |y| + |v| = |uv|$. Moreover, $|u'| \geq \frac{1}{3}|u|$ since u encompasses at least one full block of w . Consequently, $|u'v'| \leq 3\alpha|u'|$.

Let t be a word whose letters correspond to whole blocks in w and u'', v'' be factors of t that correspond to u' and v' , respectively. We have $|u''| = |u'|/\alpha$ and $|v''| = |v'|/\alpha$, so $u''v''u''$ is a 3α -gapped repeat in t . It is also a 3α -MGR because it can be expanded by one block neither to the left nor to the right, as it would contradict the maximality of the original nice α -MGR. This concludes that every nice α -MGR in w has a corresponding 3α -MGR in t . Also, every 3α -MGR in t corresponds to at most one nice α -MGR in w , as it can be translated into blocks of w and expanded in a single way to a 3α -MGR (that can happen to be a nice α -MGR).

We conclude that the number of nice α -MGRs in w is at most the number of 3α -MGRs in t . As $|t| \leq n/\alpha$, this is at most $9(\pi^2/6 + 5/2)n$ due to Fact 2.1. \square

Lemma 5.2. *For $d = 1, \dots, \lfloor n/k \rfloor$, the sets $\text{WeakP}_k(d)$ have chain representations of total size $\mathcal{O}(nk)$ which can be computed in $\mathcal{O}(nk)$ time. Moreover, $\sum_{d=2k-2}^{\lfloor n/k \rfloor} \sum_{q=0}^{k-2} |\mathcal{R}(\text{Squares}(q, d))| = \mathcal{O}(nk)$.*

Proof. The chain representations of sets $\text{WeakP}_k(d)$ are computed for $d < 2k - 2$ and for $d \geq 2k - 2$ separately.

From Fact 1.1, we know that all (k, d) -antipowers for given k and d can be found in $\mathcal{O}(n)$ time. This lets us compute the set $\text{WeakP}_k(d)$ (and its trivial chain representation) in $\mathcal{O}(n)$ time. Across $d \in [1..2k - 2]$, this gives $\mathcal{O}(nk)$ chains and $\mathcal{O}(nk)$ time.

Henceforth we consider the case that $d \geq 2k - 2$. Let us note that if a gapped (q, d) -square with $d \geq 2(q + 1)$ is generated by a $(q + 1)$ -MGR, then this $(q + 1)$ -MGR is nice. Indeed, by Lemma 4.1(a) this $(q + 1)$ -MGR has period $p = (q + 1)d \geq 2(q + 1)(k - 1) \geq 2(q + 1)^2$. This observation lets us express the formula of Corollary 4.2 for $\mathcal{R}(\text{Squares}(q, d))$ using $\text{NMGR}_{q+1}(w)$ instead of $\text{MGR}_{q+1}(w)$ provided that $d \geq 2(q + 1)$.

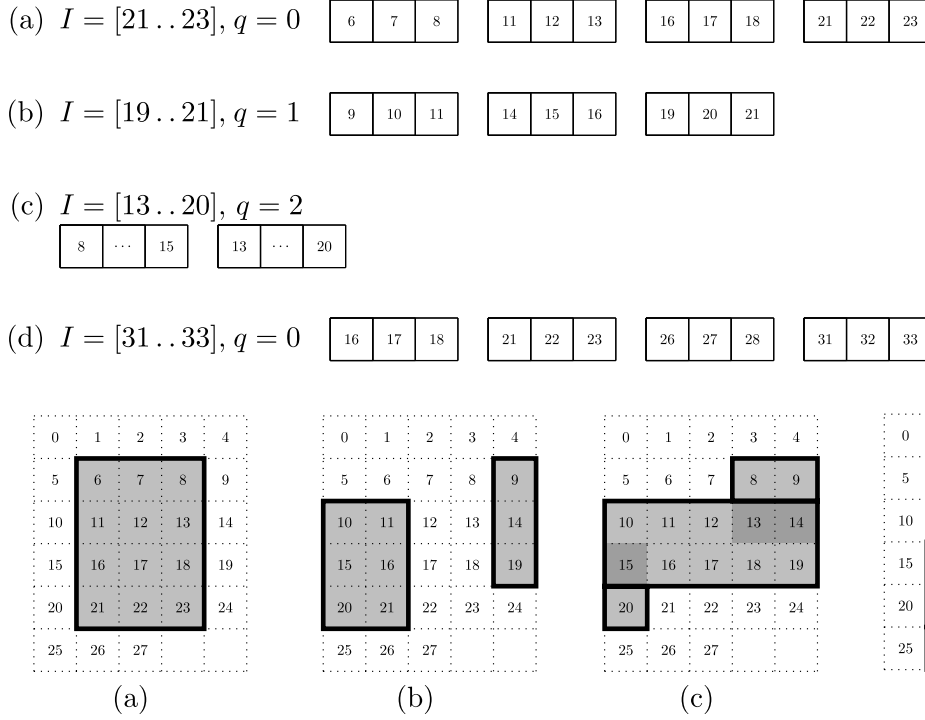


Fig. 8. Examples of decompositions of various interval chains $\text{Chain}_k(q, d, I)$ into orthogonal rectangles in the grid \mathcal{G}_d for $d = 5$, $k = 5$, $n = 52$.

By Fact 2.2 and Lemma 5.1, for every q we have only

$$|\text{NMGR}_{q+1}(w) \cup \text{GR}_{q+1}(w)| = \mathcal{O}(n)$$

MGRs and generalized runs to consider. Hence, the total size of chain representations of sets $\text{WeakP}_k(d)$ for $d \geq 2k - 2$ is $\mathcal{O}(nk)$ as well. The same applies to the total size of interval representations of sets $\text{Squares}(q, d)$ for $d \geq 2k - 2$.

The last piece of the puzzle is the following claim.

Claim 5.3. *The sets $\text{NMGR}_{\alpha}(w)$ for $\alpha \in [1 \dots k - 1]$ can be built in $\mathcal{O}(nk)$ time.*

Proof. The union of those sets is a subset of $\text{MGR}_{k-1}(w)$. Therefore, we can consider each $(k - 1)$ -MGR uvu with period $p = |uv|$ and report all $\alpha \in [\alpha_L \dots \alpha_R]$ such that $\alpha \mid p$, where

$$\alpha_L = \left\lceil \frac{p}{|u|} \right\rceil, \quad \alpha_R = \min \left(k - 1, \left\lfloor \sqrt{\frac{p}{2}} \right\rfloor \right).$$

We will use an auxiliary table next_p such that³

$$\text{next}_p[\alpha] = \min \{ \alpha' \in [\alpha + 1 \dots k] : \alpha' \mid p \}.$$

This table has size $\mathcal{O}(nk)$. For every $p \in [1 \dots n]$, all values $\text{next}_p[\alpha]$ for $\alpha \in [1 \dots k]$ can be computed, right to left, in $\mathcal{O}(k)$ time. Then, all values α for which uvu is a nice α -MGR can be computed by iterating $\alpha := \text{next}_p[\alpha]$ until a value greater than α_R is reached, starting from $\alpha = \alpha_L - 1$. Thus, the total time of constructing the sets $\text{NMGR}_{\alpha}(w)$ is $\mathcal{O}(|\text{MGR}_{k-1}(w)| + \sum_{\alpha=1}^{k-1} |\text{NMGR}_{\alpha}(w)|) = \mathcal{O}(nk)$. \square

This concludes the proof. \square

5.2. Second improvement of SimpleCount

We reduce the problem to computing unions of sets of orthogonal rectangles with bounded integer coordinates.

For a given value of d , let us fit the integers from $[0 \dots n - kd]$ into the cells of a grid of width d so that the first row consists of numbers 0 through $d - 1$, the second of numbers d to $2d - 1$, etc. Let us call this grid \mathcal{G}_d . The main idea behind the lemma presented below is shown in Fig. 8.

³ We assume that $\min \emptyset = \infty$.

Lemma 5.4. *The set $\text{Chain}_k(q, d, I)$ is a union of $\mathcal{O}(1)$ orthogonal rectangles in \mathcal{G}_d , each of height at most k or width exactly d . The coordinates of the rectangles can be computed in $\mathcal{O}(1)$ time.*

Proof. Translating the set $\text{Chain}_k(q, d, I)$ onto our grid representation, it becomes a union of horizontal strips, each corresponding to an interval $I \ominus ad$, for $a \in [0 \dots k - q - 2]$, that possibly wrap around into the subsequent rows. Those strips have their beginnings in the same column, occupying consecutive positions. Depending on the column index of the beginning of a strip and its length, we have three cases:

- The strip does not wrap around at all (Fig. 8(a)). Then, the union of all strips is simply a single rectangle. Its height is exactly $k - q - 1$.
- The strip's length is smaller than the length of the row, but it wraps around at some point (Fig. 8(b)). Then, there exists a column which does not intersect with any strip. The strips' parts that have wrapped around (that is, to the left of the column) form a rectangle and similarly the strips' parts that have not wrapped around form a rectangle as well. Both of these rectangles have height equal to $k - q - 1$.
- The strip's length is greater than or equal to the length of the row. In this case, excluding the first and the last row, the union of the strips is actually a rectangle fully encompassing all columns (Fig. 8(c)). Therefore the union of all strips can be represented as a union of three rectangles: the first row, the last row and what is in between. Both the first and the last row have height equal to 1 and the rectangle in between has width equal to d .

In some cases, such decomposition into orthogonal rectangles may include cells that are not on the grid, that is, negative numbers or numbers greater than $n - kd$; see Fig. 8(d). Then, the rectangles need to be trimmed to fit inside the grid. Moreover, in the case of numbers greater than $n - kd$, we may need to consider the part of the union that is located in the last row of the grid as a separate rectangle, since it may have a smaller width than the rectangle it originates from (as shown on Fig. 8(d)). \square

Thus, by Lemma 5.2, our problem reduces to computing the area of unions of rectangles in subsequent grids \mathcal{G}_d . In total, the number of rectangles is $\mathcal{O}(nk)$.

5.3. Third improvement of SimpleCount

Assume that r axis-aligned rectangles in the plane are given. The area of their union can be computed in $\mathcal{O}(r \log r)$ time using a classic sweep line algorithm of Bentley [21] (obtaining a more efficient solution to this problem, even for integer coordinates bounded by $\mathcal{O}(n)$, seems hard [22]). This approach would yield an $\mathcal{O}(nk \log n)$ -time algorithm for counting k -antipowers. We refine this approach in the case that the rectangles have bounded height or maximum width and their coordinates are bounded.

Lemma 5.5. *Assume that r axis-aligned rectangles in $[0 \dots m_h] \times [0 \dots m_w]$ with integer coordinates are given and that each rectangle has height at most k or width exactly m_w . The area of their union can be computed in $\mathcal{O}(r \log k + m_h + m_w)$ time and $\mathcal{O}(r + m_h + m_w)$ space.*

Proof. We assume first that all rectangles have height at most k .

Let us partition the plane into horizontal strips of height k . Thus, each of the rectangles is divided into at most two pieces. The algorithm performs a line sweep in each of the strips.

Let the sweep line move from left to right. The events in the sweep correspond to the left and right sides of rectangles. The events can be sorted left-to-right, across all strips simultaneously, in $\mathcal{O}(r + m_h)$ time using bucket sort [19].

For each strip, the sweep line stores a data structure that allows insertion and deletion of intervals with integer coordinates in $[0 \dots k]$ and querying for the total length of the union of the intervals that are currently stored. This corresponds to the operations of the data structure from Lemma 2.5 for $m = k$ (with elements corresponding to unit intervals), which supports insertions and deletions in $\mathcal{O}(\log k)$ time and queries in $\mathcal{O}(1)$ time after $\mathcal{O}(k)$ -time preprocessing per strip. The total preprocessing time is $\mathcal{O}(m_h)$ and, since the total number of events in all strips is at most $2r$, the sweep works in $\mathcal{O}(r \log k)$ time.

Finally, let us consider the width- m_w rectangles. Each of them induces an interval on the second component. First, in $\mathcal{O}(r + m_h)$ time, the union S of these intervals, represented as a union of pairwise disjoint maximal intervals, is computed by bucket sorting the endpoints of the intervals. Then, each maximal interval in S is partitioned by the strips and the resulting subintervals are inserted into the data structures of the respective strips before the sweep. In total, at most $2r + m_h/k$ additional intervals are inserted, so the time complexity is still $\mathcal{O}((r + m_h/k) \log k + m_h + m_w) = \mathcal{O}(r \log k + m_h + m_w)$. \square

We arrive at the main result of this section.

Theorem 5.6. *The number of k -antipower fragments in a word of length n can be computed in $\mathcal{O}(nk \log k)$ time and $\mathcal{O}(nk)$ space.*

Proof. We use Lemma 5.2 to express the sets $WeakP_k(d)$ for $d = 1, \dots, \lfloor n/k \rfloor$ as unions of $\mathcal{O}(nk)$ interval chains. This takes $\mathcal{O}(nk)$ time. Using Lemma 5.4, each chain is represented on the corresponding grid \mathcal{G}_d as the union of a constant number of rectangles, with each rectangle of height at most k or width exactly d . Denoting by r_d the number of rectangles in \mathcal{G}_d , we can bound the total number of rectangles by $\sum_d r_d = \mathcal{O}(nk)$.

As the next step, in each grid we assign consecutive numbers to the (sorted) first coordinates of rectangle vertices and repeat the same procedure for the second coordinates. This can be done in $\mathcal{O}(nk)$ time, for all the grids simultaneously, using bucket sort [19]. This gives new coordinates to rectangle vertices; the new coordinates also store the original values. After this transformation, rectangles with height at most k retain this property and rectangles with width d retain maximal width. Moreover, the coordinates of the rectangles are now within $[0..2r_d)$, so the application of Lemma 5.5 costs $\mathcal{O}(r_d \log k)$ time and $\mathcal{O}(r_d)$ space. One can readily verify that the underlying procedure can be adapted to compute the total area in the original coordinate values. In total, computing $|WeakP_k(d)|$ for all d costs $\mathcal{O}(nk \log k)$ time and $\mathcal{O}(nk)$ space.

In the end, the number of (k, d) -antipower fragments is calculated as $n - kd + 1 - |WeakP_k(d)|$. \square

6. Reporting antipowers and answering antipower queries

The same technique can be used to report all k -antipower fragments. In the grid representation, they correspond to grid cells of \mathcal{G}_d that are not covered by any rectangle. Hence, in Lemma 5.5, instead of computing the area of the rectangles with the aid of Lemma 2.5, we need to report all grid cells excluded from rectangles using Lemma 2.6. The computation takes $\mathcal{O}(r \log k + d + C_d)$ time where C_d is the number of reported cells. By plugging this routine into the algorithm of Theorem 5.6, we obtain the following result.

Theorem 6.1. *All fragments of a word of length n being k -antipowers can be reported in $\mathcal{O}(nk \log k + C)$ time and $\mathcal{O}(nk)$ space, where C is the size of the output.*

Finally, we present our data structure for answering antipower queries that introduces a smooth trade-off between the two data structures of Badkobeh et al. [3] (see Fact 1.2). Let us recall that an antipower query (i, j, k) asks to check if a fragment $w[i..j]$ of the word w is a k -antipower.

Theorem 6.2. *Assume that a word of length n is given. For every $r \in [1..n]$, there is a data structure of size $\mathcal{O}(n^2/r)$ that can be constructed in $\mathcal{O}(n^2/r)$ time and answers antipower queries in $\mathcal{O}(r)$ time.*

Proof. Let w be a word of length n and let $r \in [1..n]$. If an antipower query (i, j, k) satisfies $k \leq r$, we answer it in $\mathcal{O}(k)$ time using Fact 1.2(a). This is always $\mathcal{O}(r)$ time, and the data structure requires $\mathcal{O}(n)$ space.

Otherwise, if $w[i..j]$ is a k -antipower, then its base is at most n/r . Our data structure will let us answer antipower queries for every such base in $\mathcal{O}(1)$ time.

Let us consider a positive integer $b \leq n/r$. We group the length- b fragments of w by the remainder modulo b of their starting position. For a remainder $g \in [0..b-1]$ and index $i \in [0..\lfloor \frac{n-g}{b} \rfloor)$, we store, as $A_g^b[i]$, the smallest index $j > i$ such that $w[jb + g..(j+1)b + g] = w[ib + g..(i+1)b + g]$ ($j = \infty$ if it does not exist). We also store a data structure for range minimum queries over A_g^b for each group; it uses linear space, takes linear time to construct, and answers queries in constant time (see [23]). The tables take $\mathcal{O}(n)$ space for a fixed b , which gives $\mathcal{O}(n^2/r)$ in total. They can also be constructed in $\mathcal{O}(n^2/r)$ total time, as shown in the following claim.

Claim 6.3. *The tables A_g^b for all $b \in [1..m]$ and $g \in [0..b-1]$ can be constructed in $\mathcal{O}(nm)$ time.*

Proof. Let us assign to each fragment of w of length at most m an identifier in $[0..n)$ such that factors corresponding to two equal-length fragments are equal if and only if their identifiers are equal. For length-1 fragments, this requires sorting the alphabet symbols, which can be done in $\mathcal{O}(n)$ time for an integer alphabet. For factors of length $\ell > 1$, we construct pairs that consist of the identifiers of the length- $(\ell-1)$ prefix and length-1 suffix and bucket sort the pairs. This gives $\mathcal{O}(nm)$ time in total.

To construct the tables A_g^b for a given b , we use an auxiliary array D that is indexed by identifiers in $[0..n)$. Initially, all its elements are set to ∞ . For a given g , the indices i are considered in descending order. For each i , we take as x the identifier of the factor $w[ib + g..(i+1)b + g]$, set $A_g^b[i]$ to $D[x]$ and then $D[x]$ to i . Afterwards, in the same loop, all such values $D[x]$ are reset to ∞ . For given b and g , both loops take $\mathcal{O}(n/b)$ time. \square

Given an antipower query (i, j, k) such that $(j-i+1)/k = b$, we set

$$g = i \bmod b, \quad i' = \left\lfloor \frac{i}{b} \right\rfloor, \quad j' = \left\lfloor \frac{j+1}{b} \right\rfloor - 2,$$

and ask a range minimum query on $A_g^b[i'], \dots, A_g^b[j']$. Then, $w[i..j]$ is a k -antipower if and only if the query returns a value that is at least $j' + 2$. \square

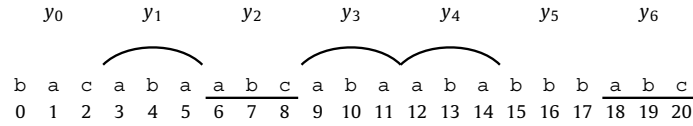


Fig. 9. This weak $(7, 3)$ -power is actually a weak $(7, 1, 3, 3)$ -power. We have $0 \in \text{WeakP}_{7,1,3}(3)$, since $3 \in \text{Squares}(1, 3)$, $3 \notin \text{Squares}(0, 3)$, and $0 \notin \text{Squares}(q, 3)$ for $q \in [0 \dots 5]$.

7. Counting different k -antipower factors

7.1. Warmup: counting different antisquare factors

Let us first show how to count different antisquare factors, that is, different 2-antipowers in a word w of length n . Recall that the *suffix tree* of a word w is a compact trie representing all the suffixes of the word $w\$$, where $\$$ is a special end-marker. The root, the branching nodes, and the leaves are explicit in the suffix tree, whereas the remaining nodes are stored implicitly. Explicit and implicit nodes of the suffix tree are simply called its nodes. Each implicit node is represented as its position within a compacted edge. The string-depth of a node v is the length of the path from v to the root in the uncompact version of the trie. The *locus* of a factor of w is the node it corresponds to. The suffix tree of a word of length n can be constructed in $\mathcal{O}(n)$ time [24].

Proposition 7.1. *The number of different antisquare factors in a word of length n can be computed in $\mathcal{O}(n)$ time.*

Proof. The algorithm counts different factors of even length and subtracts the number of different square factors. The latter can be computed in $\mathcal{O}(n)$ time [25,26]. The former can be computed by counting (explicit and implicit) nodes of the suffix tree of w at even string-depths. For every edge of the suffix tree, this number can be easily retrieved in constant time. \square

We will use the same idea, i.e., subtract the number of weak k -powers from the number of all factors of length divisible by k , to count the number of different k -antipower factors. The algorithm requires at some point an auxiliary data structure that answers the following queries related to the suffix tree.

A *weighted ancestor query* in the suffix tree, given a leaf v and a non-negative integer d , returns the ancestor of v located at depth d (being an explicit or implicit node). A weighted ancestor query can be used to compute, for a factor u of w given by its occurrence, the locus of u in the suffix tree.

Fact 7.2 ([27, Section 7]). *A batch of m weighted ancestor queries (for any rooted tree of n nodes with positive polynomially-bounded integer weights of edges) can be answered in $\mathcal{O}(n + m)$ time.*

7.2. Representing the set of weak powers

We say that $x = y_0 \cdots y_{k-1}$, where $|y_0| = \cdots = |y_{k-1}| = d$, is a *weak (k, i, j, d) -power* if $i < j$, $y_i = y_j$, and this is the “leftmost” pair of equal factors among y_0, \dots, y_{k-1} , i.e., for any $i' < j'$ such that $y_{i'} = y_{j'}$, either $i' > i$, or $i' = i$ and $j' > j$. This definition satisfies the following uniqueness property.

Observation 7.3. *A weak (k, d) -power is a weak (k, i, j, d) -power for exactly one pair of indices $0 \leq i < j < k$.*

We denote by $\text{WeakP}_{k,i,j}(d)$ the set of starting positions of weak (k, i, j, d) -powers in w ; see Fig. 9. The following lemma shows that this set can be computed efficiently. Let us recall that $\text{Squares}(q, d)$ denotes the set of starting positions of gapped (q, d) -square fragments.

Lemma 7.4. *For a given k , the sets $\text{WeakP}_{k,i,j}(d)$ for all $d = 1, \dots, \lfloor n/k \rfloor$ and $0 \leq i < j < k$ have interval representations of total size $\mathcal{O}(nk^4 \log k)$ which can be computed in $\mathcal{O}(nk^4 \log k)$ time.*

Proof. Let us note that $a \in \text{WeakP}_{k,i,j}(d)$ if and only if all the following conditions are satisfied:

1. $a + i \cdot d \in \text{Squares}(j - i - 1, d)$
2. $a + i \cdot d \notin \text{Squares}(q, d)$ for $q < j - i - 1$
3. for every $c \in [0 \dots i)$ and $q \leq k - c - 2$, we have $a + c \cdot d \notin \text{Squares}(q, d)$.

Intuitively, if $y_0 \cdots y_{k-1}$, with all factors of length d , is a weak (k, i, j, d) -power, then the first condition corresponds to $y_i = y_j$, the second condition to $y_i \neq y_{j'}$ for $i < j' < j$, and the third condition to $y_{i'} \neq y_{j'}$ for $i' < i$ and $i' < j' < k$.

Hence, $WeakP_{k,i,j}(d) = (A_{i,j}(d) \setminus (B_{i,j}(d) \cup C_{i,j}(d))) \cap [0..n - kd]$, where

$$A_{i,j}(d) = Squares(j - i - 1, d) \ominus (i \cdot d),$$

$$B_{i,j}(d) = \bigcup_{q=0}^{j-i-2} (Squares(q, d) \ominus (i \cdot d)),$$

$$C_{i,j}(d) = \bigcup_{c=0}^{i-1} \bigcup_{q=0}^{k-c-2} (Squares(q, d) \ominus (c \cdot d)).$$

By Corollary 4.2, the interval representations of all sets $Squares(q, d)$ for $0 \leq q \leq k - 2$ and $1 \leq d \leq n/k$ can be computed in $\mathcal{O}(nk^2)$ time. By Lemma 5.2, the total size of interval representations of sets $Squares(q, d)$ over all $d \geq 2k - 2$ is $\mathcal{O}(nk)$. We further have:

Claim 7.5. $\sum_{q=0}^{k-2} \sum_{d=1}^{2k-3} |\mathcal{R}(Squares(q, d))| = \mathcal{O}(nk \log k)$.

Proof. The interval representation of the set $Squares(q, d)$ has size $\mathcal{O}(n/d)$. Indeed, if $a < b < a + d$ and $a, b \in Squares(q, d)$, then $c \in Squares(q, d)$ for any $a < c < b$, so the endpoints of any two consecutive intervals in the representation are at least d positions apart. Hence, the total size of interval representations of the sets in question is $\mathcal{O}(k \sum_{d=1}^{2k-3} n/d) = \mathcal{O}(nk \log k)$. \square

In conclusion, $\sum_{q=0}^{k-2} \sum_{d=1}^{\lfloor n/k \rfloor} |\mathcal{R}(Squares(q, d))| = \mathcal{O}(nk \log k)$.

If $X_1, \dots, X_p \subseteq \mathbb{Z}$, then $|\mathcal{R}(\bigcup_{i=1}^p X_i)| \leq \sum_{i=1}^p |\mathcal{R}(X_i)|$. Consequently, for any i, j , and d ,

$$\begin{aligned} |\mathcal{R}(A_{i,j}(d))| + |\mathcal{R}(B_{i,j}(d))| + |\mathcal{R}(C_{i,j}(d))| &\leq \\ |\mathcal{R}(Squares(j - i - 1, d))| + \sum_{q=0}^{k-2} |\mathcal{R}(Squares(q, d))| + (k - 2) \sum_{q=0}^{k-2} |\mathcal{R}(Squares(q, d))| &\leq \\ \leq k \sum_{q=0}^{k-2} |\mathcal{R}(Squares(q, d))|. \end{aligned}$$

Hence, over all i, j, d , the size of these interval representations does not exceed

$$k^2(k + 2) \sum_{d=1}^{\lfloor n/k \rfloor} \sum_{q=0}^{k-2} |\mathcal{R}(Squares(q, d))| = \mathcal{O}(nk^4 \log k).$$

Finally, Lemma 2.4 can be used to compute the sets $A_{i,j}(d) \setminus (B_{i,j}(d) \cup C_{i,j}(d))$ in $\mathcal{O}(nk^4 \log k)$ total time (note that set subtraction can be computed as intersection with set complement). \square

We say that a weak (k, i, j, d) -power $y_0 \dots y_{k-1}$ is *generated* by an MGR or a generalized run γ if the $(j - i - 1, d)$ -square $y_i \dots y_j$ is generated by γ . We denote by $WeakP_{k,i,j}(d, \gamma)$ the set of starting positions of weak (k, i, j, d) -powers generated by γ . By Lemma 4.1(b), this set is an interval. It can be readily verified that the intervals generated in the above lemma can be labeled by the MGR or generalized run γ that generated them. This labelling is unique due to the following simple observation.

Observation 7.6. For any different MGRs or generalized runs γ_1, γ_2 , the intervals $WeakP_{k,i,j}(d, \gamma_1)$ and $WeakP_{k,i,j}(d, \gamma_2)$ neither overlap nor touch (contain adjacent positions).

Proof. It suffices to note that for any $q \leq k - 2$ and d , the sets $Squares(q, d, \gamma_1)$ and $Squares(q, d, \gamma_2)$ neither overlap nor touch (as already noted in the proof of Corollary 4.2). \square

Let us first show how to count different weak (k, i, j, d) -powers for $i > 0$. The case of $i = 0$ will be handled in Section 7.5.

Definition 7.7. Let $i > 0$. We say that a function g that assigns to every weak (k, i, j, d) -power factor x of w a position $g(x) \in [0..kd]$ is a *synchronizer* if for a given MGR or generalized run γ , for every $a \in WeakP_{k,i,j}(d, \gamma)$ the value $a + g(w[a] \dots w[a + kd - 1])$ is the same.

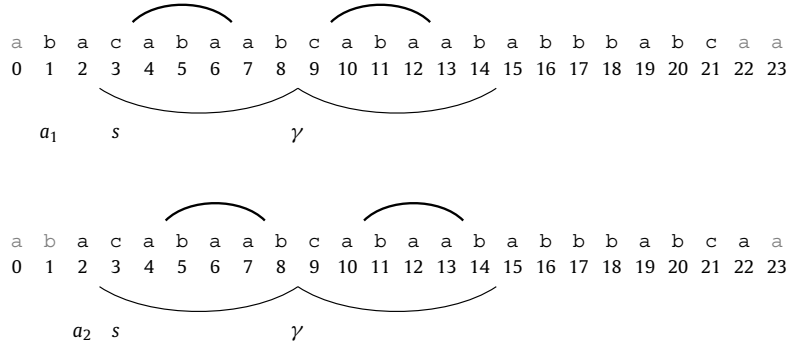


Fig. 10. Here, γ is a run starting at position $s = \text{start}(\gamma) = 2$ and we have $a_1 = 1, a_2 = 2 \in \text{WeakP}_{7,1,3}(3, \gamma)$. Moreover, $\text{synch}(w[a_1] \cdots w[a_1 + 20]) = s - a_1 = 2$ and $\text{synch}(w[a_2] \cdots w[a_2 + 20]) = s - a_2 = 1$.

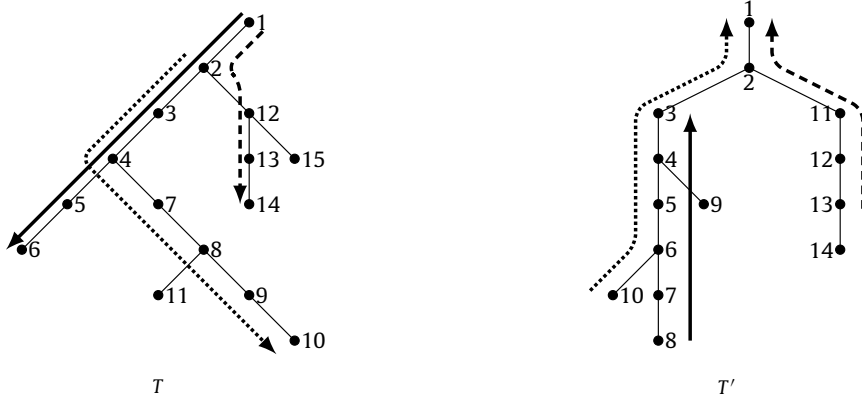


Fig. 11. Illustration of PATH PAIRS PROBLEM and Example 7.10. For simplicity, the trees in this example do not contain implicit nodes.

Note that a synchronizer function is defined on factors of w , not on fragments; i.e., it has the same value for every occurrence of the same weak (k, i, j, d) -power factor.

We will now show how to efficiently construct a synchronizer in the case of $i > 0$. For a fragment $\alpha = w[a..b]$ of w , let us denote $\text{start}(\alpha) = a$ and $\text{end}(\alpha) = b$.

Lemma 7.8. A function synch that assigns to every weak (k, i, j, d) -power x , for $i > 0$, such that $x = w[a] \cdots w[a + kd - 1]$ and $a \in \text{WeakP}_{k,i,j}(d, \gamma)$, the position $\text{start}(\gamma) - a$, is a synchronizer; see Fig. 10.

Proof. Clearly, for any positions $a_1, a_2 \in \text{WeakP}_{k,i,j}(d, \gamma)$ we have

$$a_1 + \text{synch}(w[a_1] \cdots w[a_1 + kd - 1]) = a_2 + \text{synch}(w[a_2] \cdots w[a_2 + kd - 1]) = \text{start}(\gamma).$$

Now let us show that synch is indeed a function on the set of weak k -power factors, i.e., that its value does not depend on the particular occurrence of a weak k -power and that it satisfies $\text{synch}(x) \in [0..|x|)$. Let $w[a] \cdots w[a + kd - 1] = y_0 \cdots y_{k-1} = x$ be an occurrence of a weak (k, i, j, d) -power for length- d words y_0, \dots, y_{k-1} and let γ be the MGR or generalized run that generates it. We have $y_i = y_j$ and γ has period $p = (j - i)d$. Let $r = \max\{b < i \cdot d : x[b] \neq x[b + p]\}$. We have $r > (i - 1)d$, since otherwise we would have $y_{i-1} = y_{j-1}$ and x would not be a weak (k, i, j, d) -power. Then position $r + 1$ corresponds to the starting position of γ , i.e., $\text{synch}(x) = \text{start}(\gamma) - a = r + 1$. Hence, indeed this value does not depend on the position a and $\text{synch}(x) \in [0..|x|)$. \square

7.3. Reduction to PATH PAIRS PROBLEM

We say that T is a *compact tree* if it is a rooted tree with positive integer weights on edges. If an edge weight is $e > 1$, this edge contains $e - 1$ implicit nodes. We make an assumption that the depth of a compact tree with N explicit nodes does not exceed N . A *path* in a compact tree is an upwards or downwards path that connects two explicit nodes. Let us introduce the following convenient auxiliary problem.

Problem 7.9. PATH PAIRS PROBLEM

Input: Two compact trees T and T' containing up to N explicit nodes each and a set P of M pairs (π, π') of equal-length paths where π is a path going downwards in T and π' is a path going upwards in T' .

Output: $|\bigcup_{(\pi, \pi') \in P} \text{Induced}(\pi, \pi')|$, where by $\text{Induced}(\pi, \pi')$ we denote the set of pairs of (explicit or implicit) nodes (u, u') such that u is the i th node on π and u' is the i th node on π' , for some i .

Example 7.10. Let us consider the instance of PATH PAIRS PROBLEM from Fig. 11. We have $P = \{(\pi_1, \pi'_1), (\pi_2, \pi'_2), (\pi_3, \pi'_3)\}$, where

- $\pi_1 = 1 \rightarrow 6, \pi'_1 = 8 \rightarrow 3$ (solid lines),
- $\pi_2 = 2 \rightarrow 10, \pi'_2 = 10 \rightarrow 1$ (dotted lines),
- $\pi_3 = 1 \rightarrow 14, \pi'_3 = 13 \rightarrow 1$ (dashed lines).

Then

$$\text{Induced}(\pi_1, \pi'_1) = \{(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3)\},$$

$$\text{Induced}(\pi_2, \pi'_2) = \{(2, 10), (3, 6), (4, 5), (7, 4), (8, 3), (9, 2), (10, 1)\},$$

$$\text{Induced}(\pi_3, \pi'_3) = \{(1, 13), (2, 12), (12, 11), (13, 2), (14, 1)\}.$$

In total $|\bigcup_{i=1}^3 \text{Induced}(\pi_i, \pi'_i)| = 16$ and $\text{Induced}(\pi_1, \pi'_1) \cap \text{Induced}(\pi_2, \pi'_2) = \{(3, 6), (4, 5)\}$.

Synchronizers let us reduce the problem in scope to the auxiliary problem.

Lemma 7.11. Computing the number of different weak (k, i, j, d) -powers for given k and all $0 < i < j < k, d \leq \frac{n}{k}$ in a word of length n reduces in $\mathcal{O}(nk^4 \log k)$ time to an instance of the PATH PAIRS PROBLEM with $M, N = \mathcal{O}(nk^4 \log k)$.

Proof. Let us consider the suffix tree T of w and the suffix tree T' of w^R .

For an interval $[a..b]$ in the interval representation of $\text{WeakP}_{k,i,j}(d)$, let us denote $q = a + \text{synch}(w[a] \dots w[a+kd-1])$. By Observation 7.6, $q = c + \text{synch}(w[c] \dots w[c+kd-1])$ for all $c \in [a..b]$. We create a downwards path π in T that connects the loci of $w[q..a+kd]$ and $w[q..b+kd]$ and an upwards path π' in T' that connects the loci of $(w[a..q])^R$ and $(w[b..q])^R$. We use weighted ancestor queries (Fact 7.2) to find the endpoints of the paths in the suffix trees, which can be explicit or implicit nodes.

Finally, the endpoints of the paths are made explicit in both trees. This can be achieved by grouping the endpoints by the compact edges they belong to and sorting them, within each edge, in the order of non-decreasing string-depth, which can be done in linear time via radix sort.

The resulting instance of the PATH PAIRS PROBLEM is equivalent to counting the number of different weak powers thanks to the fact that synchronizers are defined for factors.

By Lemma 7.4, the number of intervals in the interval representation of $\text{WeakP}_{k,i,j}(d)$ over all $0 < i < j$ and d is $\mathcal{O}(nk^4 \log k)$. Each of them produces one pair of paths. In the end, we obtain $\mathcal{O}(nk^4 \log k)$ paths in two compact trees containing $\mathcal{O}(nk^4 \log k)$ explicit nodes each. The conclusion follows. \square

7.4. Solution to PATH PAIRS PROBLEM

Let us recall the notion of a *heavy-path decomposition* of a rooted tree T that was introduced in [28]. Here, we only consider explicit nodes of T . A path in T is a sequence of nodes. For each non-leaf node u of T , the heavy edge (u, v) is a downwards edge for which the subtree rooted at v has the maximal number of leaves (in case of several such subtrees, we fix one of them). The remaining edges are called light. A heavy path is a maximal path containing only heavy edges. A known property of the heavy-path decomposition is that the path from any leaf u in T towards the root visits at most $\log N$ heavy paths, where N is the number of nodes of T .

In the solution to the PATH PAIRS PROBLEM, we compute the heavy path decompositions of both trees T and T' . For each pair of paths (π, π') in P , we decompose each path π, π' into maximal fragments belonging to different heavy paths. Note that the decomposition of the upwards path π' can be computed in $\mathcal{O}(\log n)$ time assuming that each tree node stores the topmost node in its heavy path and the decomposition of the downwards path π can be computed in $\mathcal{O}(\log n)$ time by traversing π in the reverse direction. Then we further decompose the paths π, π' into maximal subpaths $\pi = \pi_1, \dots, \pi_\ell, \pi' = \pi'_1, \dots, \pi'_\ell$ so that the lengths of π_i and π'_i are the same and each π_i and each π'_i is a fragment of one heavy path in T and in T' , respectively; we have $\ell \leq 2 \log N$. For an illustration, see Fig. 12. Finally, we create new pairs of paths (π_i, π'_i) , label each of them by the pair of heavy paths they belong to, and group them by their labels. This can be done using radix

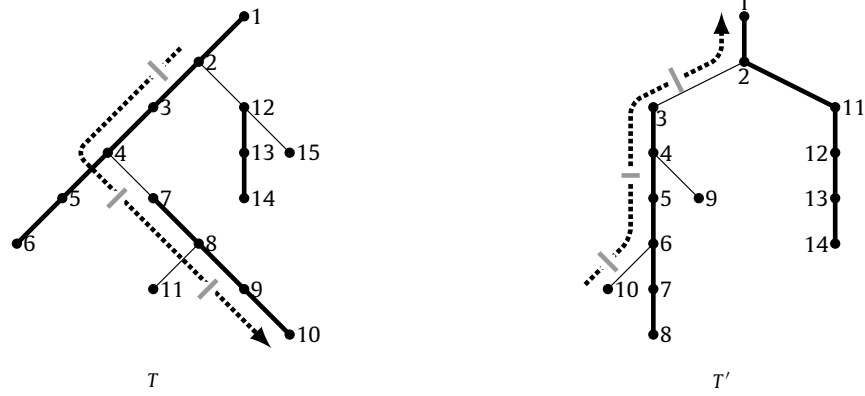


Fig. 12. Partitioning of the second pair of paths from Example 7.10 into $2 \rightarrow 2, 3 \rightarrow 4, 7 \rightarrow 8, 9 \rightarrow 10$ and $10 \rightarrow 10, 6 \rightarrow 5, 4 \rightarrow 3, 2 \rightarrow 1$ along the heavy paths (drawn as thick edges).

sort in $\mathcal{O}(N + M \log N)$ time, since the number of new path pairs is $\mathcal{O}(M \log N)$ and the number of heavy paths in each tree is $\mathcal{O}(N)$.

In the end, we obtain $\mathcal{O}(M \log N)$ very simple instances of the PATH PAIRS PROBLEM, in each of which the compact trees T and T' are single paths corresponding to pairs of heavy paths from the original compact trees. We call such an instance *special*. The total number of path pairs across the special instances is $\mathcal{O}(M \log N)$. In the lemma below, we use the assumption that the depth of the initial compact tree is at most N in order to efficiently sort the depths of path endpoints.

Lemma 7.12. *The answers to up to K special instances of the PATH PAIRS PROBLEM containing compact trees of depth at most N and at most K paths in total can be computed in $\mathcal{O}(N + K)$ time.*

Proof. For convenience let us reverse the order of edges in the tree T' of each instance so that both paths in each path pair lead downwards. Let us number the (explicit and implicit) nodes of trees T and T' top-down as $0, 1, \dots, \mathcal{O}(N)$ in every instance. Then a path pair (π, π') such that π connects nodes with numbers i and j and π' connects nodes with numbers i' and j' , with $j - i = j' - i'$, can be viewed as a diagonal segment that connects points (i, i') and (j, j') in a 2D grid. Thus, each instance reduces to counting the number of grid points that are covered by the segments. Again for convenience we can rotate each grid by 45 degrees to make the segments horizontal.

This problem can easily be solved by a top-down, and then left-to-right sweep. We only need the segment endpoints to be ordered first by the vertical, and then by the horizontal coordinate. This ordering can be achieved using radix sort in $\mathcal{O}(N + K)$ time across all instances. \square

This concludes the proof of the following lemma.

Lemma 7.13. *PATH PAIRS PROBLEM can be solved in $\mathcal{O}(N + M \log N)$ time.*

7.5. Counting different weak powers with $i = 0$

We say that word v is a *cyclic shift* of word u if there exist words x and y such that $u = xy$ and $v = yx$. For a word s , by $\text{minrot}(s)$ we denote a position $i \in [0..|s|)$ such that $s[i..|s|]s[0..i)$ is the lexicographically minimum cyclic shift of s . If there is more than one such position (i.e., that s is a power of a shorter word), we select as $\text{minrot}(s)$ the *leftmost* such position.

If $i = 0$, we partition every interval $A = \text{WeakP}_{k,0,j}(d, \gamma)$ into four (possibly empty) intervals. Let

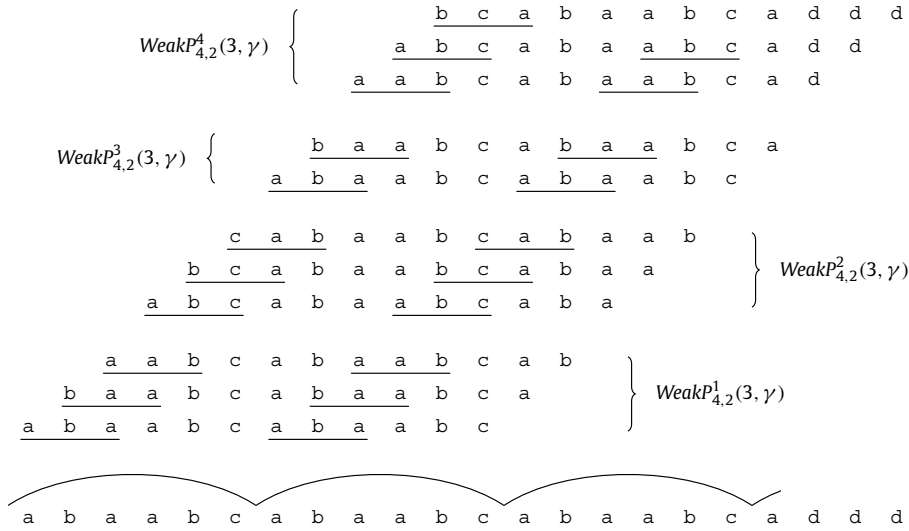
$$J_1 = [\text{start}(\gamma) \dots \text{end}(\gamma) - kd + 1], \quad J_2 = J_1 \cap [0 \dots \text{start}(\gamma) + \text{per}(\gamma)).$$

We denote $\text{pos}(\gamma) = \text{minrot}(\gamma[0 \dots \text{per}(\gamma)))$ and

$$I_1 = J_2 \cap [0 \dots \text{start}(\gamma) + \text{pos}(\gamma)], \quad I_2 = J_2 \setminus I_1, \quad I_3 = J_1 \setminus J_2, \quad I_4 = [\text{start}(\gamma) \dots \text{end}(\gamma)] \setminus J_1.$$

Example 7.14. If $\text{per}(\gamma) + kd \leq |\gamma|$, then

$$\begin{aligned} I_1 &= [\text{start}(\gamma) \dots \text{pos}(\gamma)], & I_2 &= (\text{pos}(\gamma) \dots \text{start}(\gamma) + \text{per}(\gamma)), \\ I_3 &= [\text{start}(\gamma) + \text{per}(\gamma) \dots \text{end}(\gamma) - kd + 1], & I_4 &= [\text{end}(\gamma) - kd + 2 \dots \text{end}(\gamma)]. \end{aligned}$$



Lemma 7.18. *Computing the number of different weak $(k, 0, j, d)$ -powers for given k and all $0 < j < k$, $d \leq \frac{n}{k}$ in a word of length n reduces in $\mathcal{O}(nk^3 \log k)$ time to an instance of the PATH PAIRS PROBLEM with $M, N = \mathcal{O}(nk^3 \log k)$.*

We finally arrive at the main result of this section.

Theorem 7.19. *The number of different k -antipower factors in a word of length n can be computed in $\mathcal{O}(nk^4 \log k \log n)$ time.*

Proof. Let w be a word of length n . We reduce counting different k -antipower factors of w to counting the numbers of different factors of w of length that is divisible by k and of different weak k -power factors of w . As in the proof of Proposition 7.1, the former can be computed in $\mathcal{O}(n)$ time using the suffix tree of w . By Observation 7.3, every weak (k, d) -power is a weak (k, i, j, d) -power for exactly one pair of indices $0 \leq i < j < k$. We reduce counting the number of different weak (k, i, j, d) -power factors of w to instances of the PATH PAIRS PROBLEM with $N, M = \mathcal{O}(nk^4 \log k)$ using Lemmas 7.11 and 7.18 for $i > 0$ and $i = 0$, respectively, and solve these instances in $\mathcal{O}(nk^4 \log k \log n)$ time using Lemma 7.13. \square

Declaration of competing interest

None declared.

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