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Broadcasting algorithms in radio networks with unknown topology ☆

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Abstract

In this paper we present new randomized and deterministic algorithms for the classical problem of broadcasting in radio networks with unknown topology. We consider directed n-node radio networks with specified eccentricity D (maximum distance from the source node to any other node). Bar-Yehuda et al. presented an algorithm that for any n-node radio network with eccentricity D completes the broadcasting in $\mathcal{O}(D \log n + \log^2 n)$ time, with high probability. This result is *almost* optimal, since as it has been shown by Kushilevitz and Mansour and Alon et al., every randomized algorithm requires $\Omega(D \log(n/D) + \log^2 n)$ expected time to complete broadcasting.

Our first main result closes the gap between the lower and upper bound: we describe an optimal randomized broadcasting algorithm whose running time complexity is $\mathcal{O}(D \log(n/D) + \log^2 n)$, with high probability. In particular, we obtain a randomized algorithm that completes broadcasting in any n-node radio network in time $\mathcal{O}(n)$, with high probability.

The main source of our improvement is a better "selecting sequence" used by the algorithm that brings some stronger property and improves the broadcasting time. Two types of "selecting sequences" are considered: randomized and deterministic ones. The algorithm with a randomized sequence is easier (more intuitive) to analyze but both randomized and deterministic sequences give algorithms of the same asymptotic complexity.

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Next, we demonstrate how to apply our approach to deterministic broadcasting, and describe a deterministic oblivious algorithm that completes broadcasting in time $\mathcal{O}(n \log^2 D)$, which improves upon best known algorithms in this case. The fastest previously known algorithm had the broadcasting time of $\mathcal{O}(n \log n \log D)$, it was non-oblivious and significantly more complicated; our algorithm can be seen as a natural extension of our randomized algorithm. In this part of the paper we assume that each node knows the *eccentricity* D.

Finally, we show how our randomized broadcasting algorithm can be used to improve the randomized complexity of the gossiping problem.

1. Introduction

In the *broadcasting problem*, one distinguished source node has a message that needs to be sent to all other nodes in the network. We consider *directed* networks with *unknown structure*. A *radio network* is modeled by a network (directed graph) $\mathcal{N} = (V, E)$, where the set of nodes corresponds to the set of transmitter–receiver stations. The nodes of the network are assigned different identifiers and we assume that $V = [n]$ with 1 being distinguished as the *source* node (where we use the standard notation $[n] = \{1, \ldots, n\}$). A directed edge $(v, u) \in E$ means that the node v can send a message to the node u.

We consider the standard model of unknown radio networks, called also sometimes the *ad-hoc network model* (for more elaborate discussion about our model, see, e.g., [1–4,6,8, 11,13,14]). We assume that a node does not have any prior knowledge about the topology of the network, its in-degree and out-degree, and the set of its neighbors. We assume that the only knowledge of each node is its own identifier and the size of the network, n. (We notice that one can relax the assumption about the knowledge about n to be either a linear upper bound for the number of nodes, or even to be unknown for the nodes in the network, see, e.g., [11] for more details. Similarly, in our randomized algorithms for broadcasting the nodes do not need to know their own IDs.) Additionally, in a part of the paper, we shall assume that each node knows the *eccentricity* of the network, which we denote by D, that is equal to the maximum distance from the source to any other node in $\mathcal{N}$.

The nodes are sending messages according to a *selecting sequence*, at each round this sequence specifies the probability with which all active nodes will transmit the message in this round, or in the deterministic algorithm, specifies which active nodes will transmit. It is convenient to assume that all nodes know the selecting sequence in advance, if not, then the source can include it together with its original starting message. (This is especially important if we consider a *randomized* selecting sequence, where one should understand that initially the source chooses the sequence at random and then distributes it to all other nodes together with the original starting message.) Our algorithms are *"oblivious,"* all nodes perform in the same way, according to the same selecting sequence. The nodes do not use any extra local memory.

To make the broadcasting problem feasible, we assume that every node in $\mathcal{N}$ is reachable by a path from the source node. We assume that all nodes have access to a global clock and work synchronously in discrete time steps called *rounds*. A node is *active* if it has already received a message from the source. If a node received the message at round r, then we say it is active at the end of round r; if it received the message at round $r' < r$, then we say it is active at the beginning of round r. The source message can be sent only through the edges of network. In each round each node can transmit the message to all its out-neighbors at once and can receive the messages from its in-neighbors. A node will receive a message at a given round if and only if *exactly one of its in-neighbors transmits* at that round. If more than one in-neighbor transmits simultaneously in a given round, then a *collision* occurs and none of the messages is received by the node. In that case, we assume that the node cannot distinguish such a collision from the situation when none of its neighbors is transmitting. Furthermore, we do not consider the possibility of spontaneous transmissions and we allow only active nodes to transmit. We say an algorithm *completes broadcasting in T rounds* if at the end of round T all nodes received the source message, or in other words, if all nodes are active at the end of round T.

1.1. Previous results

There has been a vast amount of research on broadcasting in unknown radio network models. For the randomized version of the problem, it has been shown by Alon et al. [1] that there exists a network of eccentricity $\mathcal{O}(1)$ for which broadcasting needs $\Omega(\log^2 n)$ expected time. Kushilevitz and Mansour [13] showed that any randomized broadcasting algorithm requires $\Omega(D \cdot \log(n/D))$ time for n-node networks of eccentricity D (see also [14]). Bar-Yehuda et al. [2] designed a broadcasting algorithm achieving the running time of $\mathcal{O}((D + \log n) \cdot \log n)$. By the lower bounds from [1,13], this algorithm is optimal for all $D \leqslant n^{1-\varepsilon}$, but it is by a logarithmic factor off from optimal for D close to n. In general n-node networks, when the bound on D is unknown, the algorithm due to Bar-Yehuda et al. [2] requires $\mathcal{O}(n \log n)$ time and no asymptotically faster algorithm has been known before. If all nodes have full knowledge of the network, then the lower bound of $\Omega(\log^2 n)$ is still valid and one can do broadcasting in $\mathcal{O}(D + \log^5 n)$ expected time [9].

The problem of deterministic broadcasting has been also intensively studied. The first sub-quadratic time algorithm has been given in [4], it runs in time $\mathcal{O}(n^{11/6})$, [5] gave the running time of $\mathcal{O}(n^{1.5})$ and the same bound was obtained by Peleg. Chrobak et al. [6] were the first who designed an algorithm that completes the broadcasting in $\mathcal{O}(n \log^2 n)$ time; very recently, Kowalski and Pelc [11] improved this bound to obtain a non-oblivious algorithm of complexity $\mathcal{O}(n \log n \log D)$. All known $\mathcal{O}(n\ \text{poly-log}(n))$ algorithms (including those in [6,11]) are probabilistic and non-constructive. The best constructive algorithm known up to date is by Indyk [10]. The best known lower bound is $\Omega(n \log D)$ due to Clementi et al. [8].

In this paper, we study also the problem of gossiping in unknown radio networks (for more details, see [6,7,14,15]). The fastest known randomized algorithms achieve the running time of $\mathcal{O}(n \log^4 n)$ [7] and $\mathcal{O}(n \log^3 n)$ [14]; the fastest deterministic algorithm has the running time of $\widetilde{\mathcal{O}}(n^{1.5})$ [6,15].

1.2. New contributions

In this paper we present several new algorithms which improve upon the complexity of the best known algorithms for the problems listed below. Let $\mathcal{N} = (V, E)$ be a directed n-node unknown radio network. We design several new algorithms:

- A randomized broadcasting algorithm which completes broadcasting in $\mathcal{N}$ in $\mathcal{O}(n)$ time with high probability; this is the *optimal running time* for this problem. (Using standard doubling technique for estimating the value of n, the algorithm does not have to know n.) We show algorithms using randomized and deterministic *selecting sequences*.
- A randomized broadcasting algorithm which completes broadcasting in $\mathcal{N}$ in $\mathcal{O}(D \log(n/D) + \log^2 n)$ time with high probability for networks of eccentricity bounded by D. Here we also show algorithms using randomized and deterministic selecting sequences. Our upper bound matches the lower bound given by Kushilevitz and Mansour [13] and Alon et al. [1] for all values of the eccentricity D.
- A deterministic oblivious broadcasting algorithm which completes broadcasting in $\mathcal{O}(n \log^2 D)$ time. The main source of our improvement is an appropriately chosen *structure* of new selecting sequences, extending our randomized algorithm.
- A randomized Las Vegas algorithm that performs gossiping in $\mathcal{N}$ in expected $\mathcal{O}(n \log^2 n)$ time. This improves upon the best previously known bound [14] by a logarithmic factor. The main source of our improvement is the use of our optimal randomized broadcasting algorithm.

1.2.1. Non-conflicting transmission from a set of nodes

In the broadcasting algorithms in unknown radio networks, the main challenging difficulty is to avoid conflicts when many nodes are transmitting to the same node at the same time.

Let $\mathcal{Q}$ be any set of nodes such that at least one node of $\mathcal{Q}$ is active (already contains the message from the source). The number of active nodes in $\mathcal{Q}$ can grow in each time step of the algorithm. We say that $\mathcal{Q}$ *fires* in a given round if exactly one active node of $\mathcal{Q}$ transmits. Our aim is to ensure that independently of the number of active nodes $\mathcal{Q}$ fires after a small number of rounds, with high probability. To achieve this goal we use carefully designed selecting sequences that, in randomized algorithms, determine the probability with which all active nodes in the network will transmit the message at a given round, and in deterministic algorithms, specifies which active nodes will transmit.

Assume that $\mathcal{Q}$ contains at least one active node. There are several possible scenarios in our algorithms, and they follow from suitable properties of corresponding types of selecting sequences:

- If the eccentricity is not bounded, then in the *randomized algorithm* with *randomized selecting sequence* $\mathcal{Q}$ *fires* with probability $\mathcal{O}(\min\{1, \max\{\frac{1}{|\mathcal{Q}|}, \frac{1}{\log n}\}\})$. This implies that $\mathcal{Q}$ *fires* with a positive constant probability after $\mathcal{O}(\min\{|\mathcal{Q}|, \log n\})$ rounds.
- Let D be the eccentricity of the network. In the *randomized algorithm* with *randomized selecting sequence* $\mathcal{Q}$ *fires* with probability $\mathcal{O}(\min\{1, \max\{\frac{n}{D \cdot |\mathcal{Q}|}, \frac{1}{\log n}\}\})$.

- In the *randomized algorithm* with *randomized selecting sequence* $\mathcal{Q}$ *fires* with a constant probability in any sequence of $\mathcal{O}(\max\{1, \min\{\frac{D\cdot|\mathcal{Q}|}{n}, \log n\}\})$ consecutive rounds.
- In the *deterministic algorithm* $\mathcal{Q}$ *fires* in any sequence of $\mathcal{O}(\max\{|\mathcal{Q}|, \frac{n}{D}\})$ consecutive rounds; in this case the role of selecting sequences is played by *selectors* [6,8], but their configuration is again led by the selecting sequences used for the randomized algorithms.

Our main technical contribution is the development of the aforementioned selecting sequences and their probabilistic analysis showing how they can be used in broadcasting algorithms.

1.2.2. Randomized vs. deterministic selecting sequences

We describe now the difference between randomized and deterministic selecting sequences that is essential to understand their use. In algorithms using deterministic selecting sequences, in each step of the algorithms all active nodes transmit the message with identical probability that depends on the given time step only. In algorithms using randomized selecting sequences, in each step of the algorithms all active nodes transmit the message with identical probability that is chosen at random according to certain probability distribution.

For the analysis and for understanding our method randomized selecting sequences are more natural and conceptually simpler. However, in order to be used by any distributed algorithm, the source node must first perform all random choices of the probabilities used in every step of the algorithm and then append them to the message to be broadcasted. In particular, this puts some additional overhead for the size of the message sent. Hence deterministic selecting sequences are slightly more complicated to analyze and are perhaps conceptually more difficult. However, they are more natural to be used by distributed broadcasting algorithms, because the random choice of each active node depends only on the time step.

In this paper, we will focus our analysis on randomized selecting sequences because we believe they are more natural for the analysis. The analysis of deterministic selecting sequences will be postponed to the end of the paper, Section 7. However our main results are first shown without deterministic sequences; the analysis for deterministic sequences is presented at the end of the paper, in Section 7, only for completeness.

1.2.3. Recent advances

After submitting the conference version of the paper, we have learned that a similar *randomized* broadcasting algorithm was obtained independently by Kowalski and Pelc [12]. The main model considered by Kowalski and Pelc is that of *undirected* networks, in which often the broadcasting can be performed more efficiently, but as it is claimed in [12], their randomized algorithm and its analysis can be applied to directed networks as well.

2. Preliminaries

We assume, without loss of generality, that n is a power of 2; otherwise, one should use $\lceil \log n \rceil$ instead of $\log n$. Similarly, we assume that D is a power of 2; otherwise, one should use $\lceil \log D \rceil$ instead of $\log D$. Moreover, each time we write an expression of the type $\log x$ or $\log(N/K)$, we mean $\max\{\log_2 x, 1\}$ and $\max\{\log_2(N/K), 1\}$, respectively; this is to avoid the case $N = K$, when we want "$\log(N/K)$" to be equal to $\mathcal{O}(1)$ rather than 0.

2.1. Previous approach to randomized broadcasting

Bar-Yehuda et al. [2] presented a randomized broadcasting algorithm that runs in time which is optimal within logarithmic factor. The following Simple Randomized Broadcasting Algorithm(T) is essentially identical to the algorithm presented in [2] and it completes the broadcasting in $\mathcal{O}((D + \log n)\log n)$ rounds with probability at least $1 - n^{-1}$, where D is the eccentricity of the input network.

Simple Randomized Broadcasting Algorithm(T).

Input: Network $\mathcal{N} = (V, E)$
$M_r = \log n - (r \bmod \log n)$ for every $r \in \mathbb{N}$

for $r = 1$ **to** T **do** {*round number* r}
  for each *active* node $v \in V$ *independently* **do**
    node v transmits with probability 2^{-M_r}

The main idea of this algorithm is that, informally, by choosing the "selecting sequence" $\langle M_1, M_2, \ldots \rangle$ we ensure that for any r, in the next $\mathcal{O}(\log n)$ rounds we expect to make one new node active, and thus, $\mathcal{O}(n \log n)$ rounds suffice to complete broadcasting.

Because of the lower bound for randomized broadcasting of $\Omega(D\log(n/D) + \log^2 n)$ [1,13], this algorithm is *almost* optimal—it achieves the optimal running time for all $D = n^{1-\varepsilon}$, but it is by a logarithmic factor off from optimal for D close to n.

3. Randomized broadcasting in linear time

In this section, we show how the Simple Randomized Broadcasting Algorithm(T) described in the previous section can be modified to complete broadcasting in $\mathcal{O}(n)$ time. The analysis that achieves the asymptotically optimal running time for all values of D is in Section 4.

In our new randomized algorithm we replace the selecting sequence $\langle M_1, M_2, \ldots \rangle$ by two types of selecting sequences $\mathfrak{I} = \langle I_1, I_2, \ldots \rangle$: a randomized sequence (each element is generated independently with some probability α_k), and a deterministic sequence defined constructively. All analyzes in this paper could be performed using either of the sequences, but for simplicity of the presentation we focus now our attention on the randomized sequence; the analysis of deterministic sequences is postponed to Section 7.

3.1. Randomized selecting sequences and broadcasting algorithms

We assume that n is a power of 2 and let $\mathcal{LL}(n) = \lceil \log\log n \rceil$. Intuitively, the main idea of our selecting sequences is to ensure two properties:

- for any k, $0 \leqslant k \leqslant \mathcal{LL}(n)$, the probability that at any given round all active nodes will transmit with probability 2^{-k} is roughly 2^{-k}, and
- for any k, $\mathcal{LL}(n) \leqslant k \leqslant \log n$, the probability that at any given round all active nodes will transmit with probability 2^{-k} is roughly $2^{-\mathcal{LL}(n)} \approx 1/\log n$.

More formally, for any $k \in \{0, 1, \ldots, \log n\}$, we define

$$\alpha_k = \begin{cases} 2^{-(k+1)} & \text{for } 1 \leqslant k \leqslant \mathcal{LL}(n), \\ \frac{1}{2\log n} & \text{for } \mathcal{LL}(n) \leqslant k \leqslant \log n, \\ 1 - \sum_{i=1}^{\log n} \alpha_i & \text{for } k = 0. \end{cases}$$

Then, in our first algorithm we use randomized sequence $\mathfrak{I} = \langle I_1, I_2, \ldots \rangle$ such that $\mathbf{Pr}[I_r = k] = \alpha_k$.

Linear Randomized Broadcasting Algorithm(T).

Input: Network $\mathcal{N} = (V, E)$
Randomized sequence $\mathfrak{I} = \langle I_1, I_2, \ldots \rangle$ such that
$\mathbf{Pr}[I_r = k] = \alpha_k$ $\forall r \in \mathbb{N}$, $\forall k \in \{0, 1, 2, \ldots, \log n\}$

for $r = 1$ **to** T **do** {*round number* r}
  for each *active* node $v \in V$ *independently* **do**
    node v transmits with probability 2^{-I_r}

Our first main result is the following theorem (the constant c does not depend on the network r on n, here and in the other theorems).

Theorem 1. *Let $\mathcal{N} = (V, E)$ be an n-node radio network. There is a constant c such that if $T \geqslant cn$, then the Linear Randomized Broadcasting Algorithm(T) completes broadcasting in $\mathcal{N}$ with probability at least $1 - n^{-1}$.*

Remark. It is tempting to run this algorithm with a simpler selection sequence, in which each I_i is chosen independently at random according to the following simpler distribution:

$$\mathbf{Pr}[I_i = k] = 2^{-k} \quad \text{for any integer } k \geqslant 1.$$

However, for such a sequence it is *not true* that the Linear Randomized Broadcasting Algorithm completes broadcasting in $\mathcal{O}(n)$ rounds with probability at least $1 - n^{-1}$. To see this, let us consider a complete layered n-node network with three layers L_0, L_1, L_2, with $|L_0| = |L_2| = 1$ and $|L_1| = n - 2$, where the only edges are from all nodes in L_i to all nodes in L_{i+1} for $i = 0, 1$. The Linear Randomized Broadcasting Algorithm with the modified sequence for such networks requires $\Omega(n \log(1/\beta))$ rounds to complete the broadcasting with the probability of at least $1-\beta$. With our setting of $\beta = n^{-1}$, this implies $\Omega(n \log n)$ time.

3.2. Notational conventions and basic properties

Before we provide a formal analysis of the Linear Randomized Broadcasting Algorithm(T) and prove Theorem 1, let us first introduce some further notation and discuss some basic properties of the algorithm.

Let $\Gamma_v[r]$ be the set of nodes $u \in V$ that are active at the end of round r and for which $(u, v) \in E$. Let $\gamma_v[r] = |\Gamma_v[r]|$. Let Act_v be the random variable denoting the round in which node v becomes active (for the source node, which has index 1, we define $\mathsf{Act}_1 = 0$). Notice that a node v will make its first attempt to transmit at round $\mathsf{Act}_v + 1$, one round after it becomes active. Observe also that the broadcasting time of the Linear Randomized Broadcasting Algorithm(T) is equal to $\max\{\mathsf{Act}_v\colon v \in V\}$.

Eager nodes and successful rounds. For any node $v \in V$ and a sequence $\mathfrak{I}$, we say v is *eager at round r* if $\gamma_v[r-1] > 0$ and $\frac{1}{2}\gamma_v[r-1] < 2^{I_r} \leqslant \gamma_v[r-1]$, where I_r is the rth element in sequence $\mathfrak{I}$. (Additionally, if $\gamma_v[r-1] = 1$ then v is eager at round r if $I_r \leqslant 1$.)

Lemma 3.1. *For any node $v \in V$ and any round $r \in \mathbb{N}$, if $\gamma_v[r-1] > 0$, then the probability that v is eager at round r is at least* $\frac{1}{2}\max\{1/\gamma_v[r-1], 1/\log n\}$.

Proof. It follows immediately from the definition of the α_k's that for any $k \in \{1, \ldots, \log n\}$ and any $r \in \mathbb{N}$, we have $\mathbf{Pr}[I_r = k] = \frac{1}{2}\max\{2^{-k}, 1/\log n\}$. □

Lemma 3.2. *For any node $v \in V$ and any round $r \in \mathbb{N}$, if v is eager at round r, then the probability that v is active at the end of round r is at least* 0.1.

Proof. Since v is eager at round r, we have $0 < \frac{1}{2}\gamma_v[r-1] < 2^{I_r} \leqslant \gamma_v[r-1]$.

Let us consider round r in the Linear Randomized Broadcasting Algorithm(T). At the beginning of that round there are $s = \gamma_v[r-1] \geqslant 1$ nodes in $\mathcal{N}$ that are active and that can send a message to node v. In order for message to be successfully received, there must be exactly one node in $\Gamma_v[r-1]$ that will transmit in round r. Since each active node transmits the message in round r independently with probability 2^{-I_r}, and since $\frac{1}{2}s < 2^{I_r} \leqslant s$, the probability that one message will be successfully transmitted to v is equal to $\binom{s}{1} \cdot 2^{-I_r} \cdot (1-2^{-I_r})^{s-1}$, which is at least 0.1 for all $s \geqslant 2$ (for $s = 1$, this probability is also greater than 0.1). This clearly means that v is active at the end of round r with probability at least 0.1. □

3.3. Proof of Theorem 1: analysis of the Linear Randomized Broadcasting Algorithm

In our analysis we concentrate ourselves on a single path from the source to an arbitrary node and show that with high probability the end node of the path will be active after $\mathcal{O}(n)$ rounds. Since there are n paths whose union ends at all the vertices in the network, this will

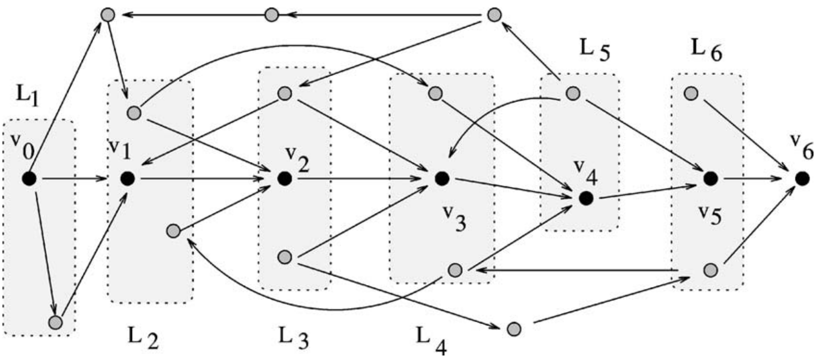

Fig. 1. Let $P = (v_0, v_1, \ldots, v_6)$. Then $L_i = \mathsf{Layer}_P(i)$ for the example graph.

suffice to prove that the algorithm will complete broadcasting in $\mathcal{O}(n)$ rounds, with high probability.

Let $P = \langle v_0, v_1, \ldots, v_\ell \rangle$ be any path that begins at the source. For any node $v \in V$, let us define

$$\mathrm{last}_P(v) = \begin{cases} v_i & \text{if } (v, v_i) \in E \text{ and } \forall_j ((v, v_j) \in E \Rightarrow j \leqslant i), \\ \emptyset & \text{if } \forall_j (v, v_j) \notin E. \end{cases}$$

For any node i, $1 \leqslant i \leqslant \ell$, let $\mathsf{Layer}_P(i) = \{v \in V\colon \mathrm{last}_P(v) = v_i\}$, see Fig. 1.

In other words, $\mathrm{last}_P(v)$ is the last node on the path P for which v is an in-neighbor and $\mathsf{Layer}_P(i)$ is the set of all in-neighbors of v_i which are not in-neighbors of any v_j for $j > i$.

The set $\mathsf{Layer}_P(i)$ is called the *layer* of rank i with respect to P. We say a layer $\mathsf{Layer}_P(i)$ is *leading* (with respect to P) at round r of the algorithm if i is the highest rank layer containing an active node at the beginning of round r (that is, $i = \max\{j\colon \exists v \in \mathsf{Layer}_P(j)$ with $\mathsf{Act}_v < r\}$) *and* node v_ℓ is not active at the beginning of round r. Define:

$$Z_P(i) = \left|\left\{r\colon \mathsf{Layer}_P(i) \text{ is leading at round } r\right\}\right|.$$

The following lemma describes some basic properties of the definitions above.

Lemma 3.3. *Consider the layers corresponding to some shortest path $P = \langle v_0, v_1, \ldots, v_\ell \rangle$, where $v_0 = 1$.*

- *The layers are disjoint, consequently $\sum_{i=0}^{\ell} |\mathsf{Layer}_P(i)| \leqslant n$.*
- *The last node of P, v_ℓ, receives the message after $\sum_{i=1}^{\ell} Z_P(i)$ rounds, that is, $\mathsf{Act}_{v_\ell} = \sum_{i=1}^{\ell} Z_P(i)$.*
- *For every $r \in \mathbb{N}$ and any i, $1 \leqslant i \leqslant \ell$, if $\mathsf{Layer}_P(i)$ is leading at round r, then $\gamma_{v_i}[r-1] \leqslant |\mathsf{Layer}_P(i)|$.*

Proof. The first two claims follow immediately from the definitions above. We show the third claim by contradiction. Let us suppose that $\gamma_{v_i}[r-1] > |\mathsf{Layer}_P(i)|$. Since $\gamma_{v_i}[r-1] = |\Gamma_{v_i}[r-1]| = |\{v \in V\colon (v, v_i) \in E \text{ and } \mathsf{Act}_v \leqslant r-1\}|$, $\gamma_{v_i}[r-1] > |\mathsf{Layer}_P(i)|$ implies that there must be some node x with $(x, v_i) \in E$ that is active in round $r - 1$ and which is not in $\mathsf{Layer}_P(i)$. Since it is easy to see that $\{v \in V\colon\ (v, v_i) \in E\} \subseteq \bigcup_{j=i}^{\ell} \mathsf{Layer}_P(j)$, this yields $x \in \mathsf{Layer}_P(j)$ for some $j > i$. Therefore, if $\gamma_{v_i}[r-1] > |\mathsf{Layer}_P(i)|$ then $\mathsf{Layer}_P(i)$ is not leading at round r, what implies the third claim. □

Our next *key* lemma gives a majorization result for the round in which the last node from P becomes active. (Let us recall that $\mathfrak{Y}$ has *geometric distribution with parameter* ϱ if $\mathbf{Pr}[\mathfrak{Y} = k] = (1-\varrho)^{k-1} \cdot \varrho$ for any $k \in \mathbb{N}$.)

Lemma 3.4 *(Key majorization lemma). Let $P = \langle v_0, v_1, \ldots, v_\ell \rangle$ be any shortest path from the source to a node v_ℓ. Let $Y_1, \ldots, Y_\ell$ be a sequence of independent random variables, each Y_ℓ has geometric distribution with parameter $\rho_i = \frac{1}{20\cdot\min\{\log n, |\mathsf{Layer}_P(i)|\}}$.*

Then the random variable Act_{v_ℓ} is stochastically majorized by the sum $\sum_{i=1}^{\ell} Y_i$, that is, for $r \in \mathbb{N}$,

$$\mathbf{Pr}[v_\ell \textit{ is active at the end of round } r] \geqslant \mathbf{Pr}\left[\sum_{i=1}^{\ell} Y_i \leqslant r\right].$$

Proof. Since from Lemma 3.3 we know that $\mathsf{Act}_{v_\ell} = \sum_{i=1}^{\ell} Z_P(i)$, it is enough to prove that for every i, $1 \leqslant i \leqslant \ell$, the random variable $Z_P(i)$ is majorized by Y_i.

Let us fix any i and we show that $Z_P(i)$ is majorized by Y_i. Observe first that if $\mathsf{Layer}_P(i)$ is never a leading layer, then $Z_P(i)$ is majorized by Y_i. Otherwise, let $r_{\langle i \rangle}$ be the first round in which $\mathsf{Layer}_P(i)$ is leading. Let us consider any round $r \geqslant r_{\langle i \rangle}$ in which $\mathsf{Layer}_P(i)$ is leading. Then, there is at least one node in $\Gamma_{v_i}[r-1]$, and therefore $\gamma_{v_i}[r-1]$ nodes will try to transmit in round r to node v_i. By Lemma 3.1, the probability that node v_i is eager is at least $\frac{1}{2}\max\{1/\gamma_{v_i}[r-1], 1/\log n\}$, and by Lemma 3.2, if v_i is eager in round r, then the probability that it will become active (and hence $\mathsf{Layer}_P(i)$ will be not leading at round $r+1$) is at least 0.1. Therefore, since by Lemma 3.3 we know that $\gamma_{v_i}[r-1] \leqslant |\mathsf{Layer}_P(i)|$, we can conclude that the probability that node v_i will become active at round r, conditioned on it not being active before, is at least $\frac{1}{20\cdot\min\{|\mathsf{Layer}_P(i)|, \log n\}}$. Since this bound is true in every round r in which $\mathsf{Layer}_P(i)$ is leading, we clearly have

$$\begin{aligned}\mathbf{Pr}\big[Z_P(i) = k\big] &\geqslant \frac{1}{20\cdot\min\{|\mathsf{Layer}_P(i)|, \log n\}}\left(1 - \frac{1}{20\cdot\min\{|\mathsf{Layer}_P(i)|, \log n\}}\right)^{k-1}\\ &= \mathbf{Pr}[Y_i = k],\end{aligned}$$

what concludes the proof of the lemma. □

Next, we show a concentration result for sums of geometric random variables (the proof of this technical lemma is in Appendix A).

Lemma 3.5. *Let $X_1, \ldots, X_\ell$ be a sequence of independent integer-valued random variables, each X_i being geometrically distributed with a parameter p_i, $0 < p_i < 1$. For* *every i, $1 \leqslant i \leqslant \ell$, let $\mu_i = 1/p_i$, and assume that all μ_i are from a set Δ, that is, $\Delta = \{\mu_i\colon 1 \leqslant i \leqslant \ell\}$. If $\sum_{i=1}^{\ell} \mu_i \leqslant N$, then for every positive real number β,*

$$\mathbf{Pr}\left[\sum_{i=1}^{\ell} X_i \leqslant 2 \cdot N + 8 \cdot \ln\bigl(|\Delta|/\beta\bigr) \cdot \sum_{z \in \Delta} z\right] \geqslant 1 - \beta.$$

Equipped with Lemmas 3.4 and 3.5, we are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. Our goal is to show that for every node $v \in V$ we have $\mathsf{Act}_v \leqslant cn$ with probability at least $1 - n^{-2}$. Since there are n nodes to be considered, the union bound will imply the theorem:

$$\mathbf{Pr}[\forall_{v \in V} \mathsf{Act}_v \leqslant cn] \geqslant 1 - \sum_{v \in V} \mathbf{Pr}[\mathsf{Act}_v > cn] \geqslant 1 - n \cdot n^{-2} = 1 - n^{-1}.$$

Let us fix any node $v \in V$. Let us consider a shortest path from the source to v and denote it by $P = \langle v_0, v_1, \ldots, v_\ell \rangle$. By analyzing the algorithm on this path we show that $\mathbf{Pr}[\mathsf{Act}_v \leqslant cn] \geqslant 1 - n^{-2}$.

We apply Lemma 3.4 to show that $\mathbf{Pr}[\mathsf{Act}_v \leqslant cn] \geqslant 1 - n^{-2}$. Let us define the random variable $Y_1, Y_2, \ldots, Y_\ell$ as in Lemma 3.4. Now, we can estimate the expected value Act_v as follows:

$$\begin{aligned} \mathbf{E}[\mathsf{Act}_v] &= \sum_{i=1}^{\ell} \mathbf{E}[Y_i] = \sum_{i=1}^{\ell} \frac{1}{\rho_i} = \sum_{i=1}^{\ell} 20 \cdot \min\bigl\{\log n, \bigl|\mathsf{Layer}_P(i)\bigr|\bigr\} \\ &\leqslant 20 \cdot \sum_{i=1}^{\ell} \bigl|\mathsf{Layer}_P(i)\bigr| \leqslant 20 \cdot n, \end{aligned}$$

where the last inequality follows from Lemma 3.3.

Next, we want to prove that the sum of random variables $\sum_{i=1}^{\ell} Y_i$ is highly concentrated around its mean. For this, we use Lemma 3.5, from which we obtain (with $X_i \equiv Y_i$, $\Delta = \{1, 2, \ldots, 20 \log n\}$, $\beta = n^{-2}$, and $N = 20 \cdot n$):

$$\mathbf{Pr}\left[\sum_{i=1}^{\ell} Y_i \leqslant 40n + 8 \ln\bigl(20n^2 \log n\bigr)(20 \log n)^2\right] \geqslant 1 - n^{-2}.$$

This immediately implies that for a certain positive constant c we obtain

$$\mathbf{Pr}[\mathsf{Act}_v \leqslant cn] = \mathbf{Pr}\left[\sum_{i=1}^{\ell} Y_i \leqslant cn\right] \geqslant 1 - n^{-2}.$$

This inequality, combined with our arguments at the beginning of the proof, concludes the proof. $\square$

4. Improved randomized algorithm for shallow networks

In this section we present an algorithm that achieves the optimal broadcasting time in "shallow networks": we extend our approach from Section 3 to show how to complete broadcasting in n-node networks of eccentricity D in $\Theta(D\log(n/D) + \log^2 n)$ rounds.

4.1. Randomized selecting sequence for shallow networks

Similarly as in Section 3, we consider the algorithm using a randomized selecting sequence; later, in Section 7.2, we discuss in details deterministic selecting sequences.

The Randomized Broadcasting Algorithm for Shallow Networks(T) is identical to the Linear Randomized Broadcasting Algorithm(T) with the sequence $\mathfrak{J}$ replaced by $\mathfrak{J}^D$ which is defined below using a new distribution α'.

Let us fix D and n. Let $\lambda = \log(n/D)$. Let us remind that $\mathcal{LL}(n) = \lceil \log\log n\rceil$ and that we assumed that both, n and D are powers of 2 (hence, λ is an integer). In the previous section, while discussing a randomized $\mathcal{O}(n)$-time broadcasting algorithm we were using a distribution α depending on n. Now, we use a similar distribution, denoted by α', that is defined for a given D as follows (notice that $\alpha_k = \alpha'_k$ for $D = n$),

$$\alpha'_k = \begin{cases} \frac{1}{2\lambda} & \text{for } 1 \leqslant k \leqslant \lambda, \\ \frac{1}{2\lambda}\cdot 2^{-(k-\lambda)} & \text{for } \lambda < k \leqslant \lambda + \mathcal{LL}(n),\ k \leqslant \log n, \\ \frac{1}{2\lambda}\cdot\frac{1}{\log n} & \text{for } \lambda + \mathcal{LL}(n) < k \leqslant \log n, \\ 1 - \sum_{i=1}^{\log n}\alpha'_i & \text{for } k = 0. \end{cases}$$

(Comparing to the definition of the probability distribution α, we notice that essentially (besides normalization), the only change is that all values of α'_k are identical for all $k \leqslant \lambda \equiv \log(n/D)$.)

Define a randomized sequence $\mathfrak{J}^D = \langle J_1, J_2, \ldots\rangle$ as the sequence of elements chosen independently at random with $\mathbf{Pr}[J_r = k] = \alpha'_k$.

Randomized Broadcasting Algorithm for Shallow Networks(T).

Input: Network $\mathcal{N} = (V, E)$, $n = |V|$, and the eccentricity D
Randomized sequence $\mathfrak{J}^D = \langle J_1, J_2, \ldots\rangle$ such that
$\mathbf{Pr}[J_r = k] = \alpha'_k$ for each $r \in \mathbb{N}$, $k \in \{0, 1, 2, \ldots, \log n\}$

for $r = 1$ **to** T **do** {*round number r*}
  for each *active* node $v \in V$ *independently* **do**
    node v transmits with probability 2^{-J_r}

4.2. Analysis of optimal randomized algorithm for shallow networks

In this section we analyze the Randomized Broadcasting Algorithm for Shallow Networks(T) in a way similar to the analysis in Section 3. (We could apply our analysis for any D to achieve the broadcasting time of $\mathcal{O}(D\log(n/D) + \log^2 n)$ with high probability.

However, since such a result is already known for small $D = \mathcal{O}(\log^3 n)$, we concentrate our analysis on the most interesting case.)

Theorem 2. *Let $\mathcal{N} = (V, E)$ be an n-node radio network of eccentricity $D = \Omega(\log^3 n)$. There exists a constant c such that if $T \geqslant cD\log(n/D)$, then the Randomized Broadcasting Algorithm for Shallow Networks(T) completes broadcasting in $\mathcal{N}$ with probability at least $1 - n^{-1}$.*

The proof of Theorem 2 mimics the analysis from Section 3. We start with a sketch of the proof. We consider the layers with respect to a specified path starting at the source node and separately analyze the time spent in *small* layers ($|\mathsf{Layer}_P(i)| \leqslant n/D$) and *large* layers ($|\mathsf{Layer}_P(i)| > n/D$). We can show that the expected number of rounds spent in any small layer is $\mathcal{O}(\lambda) \equiv \mathcal{O}(\log(n/D))$, and hence, since there are at most D layers and by applying an appropriate concentration bound, we can show that the total number of rounds spent in small layers is $\mathcal{O}(D \cdot \log(n/D))$, with high probability. For large layers, we can show that the expected number of rounds spent in a large layer of size $s \cdot (n/D)$ is $\mathcal{O}(s \cdot \lambda) \equiv \mathcal{O}(s \cdot \log(n/D))$, and consequently, the time spent in all larger layers is upper bounded by $\sum_{i=1}^{\ell} \mathcal{O}(|\mathsf{Layer}_P(i)| \cdot (D/n) \cdot \log(n/D))$, with high probability, which is $\mathcal{O}(D \cdot \log(n/D))$. Summarizing, the algorithm completes broadcasting in $\mathcal{O}(D \cdot \log(n/D))$ rounds, with high probability.

Now, we present all details of our analysis. For conciseness of presentation, we consider small and large layers separately only implicitly. We begin with a modification of Lemma 3.1.

Lemma 4.1. *For any node $v \in V$ and any round $r \in \mathbb{N}$, if $\gamma_v[r-1] > 0$, then*

$$\mathbf{Pr}[v \textit{ is eager at round } r] \geqslant \frac{1}{2\log(n/D)} \cdot \min\left\{1, \max\left\{\frac{n}{D \cdot \gamma_v[r-1]}, \frac{1}{\log n}\right\}\right\}.$$

Proof. The proof is essentially identical to the proof of Lemma 3.1 and follows directly from the fact that for every k, $1 \leqslant k \leqslant \log n$, we have

$$\begin{aligned}\mathbf{Pr}[J_r = k] &= \frac{1}{2\lambda} \cdot \min\left\{1, \max\left\{2^{\lambda-k}, \frac{1}{\log n}\right\}\right\} \\ &= \frac{1}{2\log(n/D)} \cdot \min\left\{1, \max\left\{\frac{n}{D} \cdot 2^{-k}, \frac{1}{\log n}\right\}\right\},\end{aligned}$$

from which the lemma follows. $\square$

Lemma 4.1 together with Lemma 3.2 lead to the following modification of our key Lemma 3.4.

Lemma 4.2. *Let $P = \langle v_0, v_1, \ldots, v_\ell \rangle$ be any shortest path from the source to a node v_ℓ. Let $Y_1, \ldots, Y_\ell$ be a sequence of independent random variable having geometric distribution with parameter $\rho_i^{\langle D \rangle}$ each, where $\rho_i^{\langle D \rangle}$ equals*

$$\frac{1}{20\log(n/D)} \cdot \min\left\{1, \max\left\{\frac{n}{D \cdot 2^{\lceil \log |\mathsf{Layer}_P(i)| \rceil}}, \frac{1}{\log n}\right\}\right\}.$$

Then the random variable Act_{v_ℓ} *is stochastically majorized by the sum* $\sum_{i=1}^{\ell} Y_i$*, that is, for any* $r \in \mathbb{N}$,

$$\mathbf{Pr}[v_\ell \textit{ is active at the end of round } r] \geqslant \mathbf{Pr}\left[\sum_{i=1}^{\ell} Y_i \leqslant r\right].$$

Proof. The proof of the lemma is essentially identical to the proof of Lemma 3.4. The only difference is the use of Lemma 4.1 instead of Lemma 3.1; all other details are identical. □

Equipped with Lemma 4.2, we can now prove Theorem 2.

Proof of Theorem 2. As in the proof of Theorem 1, we only have to show that for any single node $v \in V$, if we take a shortest path $P = \langle v_0, v_1, \dots, v_\ell \rangle$ from the source node to v, then node v will be active after at most $cD\log(n/D)$ rounds with probability at least $1 - n^{-2}$. Since we consider only shortest paths from the source to v, we can assume that $\ell \leqslant D$.

Let us fix a node $v \in V$ and consider a shortest path $P = \langle v_0, v_1, \dots, v_\ell \rangle$ that begins at the source node and ends at v. We use the notation from Lemma 4.2. By Lemma 4.2, to prove a good upper bound for the value of Act_v it is enough to analyze the distribution of the sum of random variables $\sum_{i=1}^{\ell} Y_i$. We apply Lemma 3.5 with $X_i \equiv Y_i$ and $\beta = n^{-2}$. To estimate the right value of N, Δ, and $\sum_{z \in \Delta} z$, we observe that,

$$\begin{aligned}
\sum_{i=1}^{\ell} \mu_i &= \sum_{i=1}^{\ell} 20\log(n/D) \cdot \max\left\{1, \min\left\{\frac{D \cdot 2^{\lceil \log |\mathsf{Layer}_P(i)| \rceil}}{n}, \log n\right\}\right\} \\
&\leqslant \sum_{i=1}^{\ell} 20\log(n/D) \cdot \left(1 + \frac{D \cdot 2 \cdot |\mathsf{Layer}_P(i)|}{n}\right) \\
&\leqslant 20\log(n/D) \cdot \left(\ell + \frac{2 \cdot D}{n} \cdot \sum_{i=1}^{\ell} |\mathsf{Layer}_P(i)|\right) \\
&\leqslant 20\log(n/D) \cdot \left(D + \frac{2 \cdot D}{n} \cdot n\right) \\
&= 60 \cdot D \cdot \log(n/D).
\end{aligned}$$

Hence, we will set $N = 60 \cdot D \cdot \log(n/D)$. To analyze the size of the set Δ, we observe that we can set

$$\Delta = \{20\log(n/D)\} \cup \left\{\frac{20\log(n/D) \cdot D}{n} \cdot 2^k \colon k = \lceil \log(n/D) \rceil, \dots, \lceil \log(n/D) + \log\log n \rceil\right\}.$$

Therefore, in particular, $|\Delta| \leqslant 2 + \lceil \log\log n \rceil$ and $\sum_{z \in \Delta} z \leqslant 80 \cdot \log^3 n$. So now, we apply Lemma 3.5 to obtain that for $D = \Omega(\log^3 n)$ there is some positive constant c for which

$$\mathbf{Pr}\big[\mathsf{Act}_v \leqslant cD\log(n/D)\big] \geqslant \mathbf{Pr}\left[\sum_{i=1}^{\ell} Y_i \leqslant cD\log(n/D)\right] \geqslant 1 - n^{-2},$$

what concludes the proof of the theorem. □

5. Oblivious deterministic broadcasting in shallow networks

In this section, we discuss the problem of deterministic broadcasting in an unknown n-node radio network $\mathcal{N}$ with eccentricity D. It has been shown non-constructively, using complicated counting arguments, that there is a deterministic protocol for such networks that completes broadcasting in time $\mathcal{O}(n \log n \log D)$, see [11]. In this section, we present a more natural approach to this problem and obtain an improvement to $\mathcal{O}(n \log^2 D)$. Moreover our construction and its analysis are simpler than those in [11].

Our improvement comes from two main sources:

- appropriately chosen *structure* of the selecting sequence and
- in the selecting sequence only indices corresponding to powers of two between n/D and n are considered; there are only $\log D$ such powers.

We consider an infinite sequence $\mathfrak{I}$ of elements from $\{0, 1, 2, \ldots, \log D\}$. First we introduce some measures of *sparseness* and *density* of $\mathfrak{I}$ with respect to an integer k.

Sparseness and density. Define $\mathsf{sparseness}(\mathfrak{I}, k)$ to be the minimal distance between two distinct positions in $\mathfrak{I}$ containing k and define $\mathsf{density}(\mathfrak{I}, k)$ to be the smallest integer ℓ such that in every subsequence of ℓ consecutive elements of $\mathfrak{I}$ there is at least one with value k.

Let us define two properties of sequences $\mathfrak{I}$.

***D*-sparseness property.** $\mathsf{sparseness}(\mathfrak{I}, k) = \Omega(2^k)$ for each $0 \leqslant k \leqslant \log D$.

***D*-density property.** $\mathsf{density}(\mathfrak{I}, k) = \mathcal{O}(2^k)$ for each $0 \leqslant k \leqslant \log D$.

Lemma 5.1. *There exists a sequence $\mathfrak{I}$ satisfying both the D-sparseness and the D-density properties.*

Proof. Define a finite sequence $\mathfrak{T}$ of length $2 \cdot D - 1$ such that for any j, $1 \leqslant j \leqslant 2 \cdot D - 1$, if $(j \bmod 2^{k+1}) = 2^k$ then the jth element of sequence $\mathfrak{T}$ is equal to k. In other words k is the position of the lowest 1 in the binary representation of j.

Then $\mathfrak{I}$ results by iterating the sequence $\mathfrak{T}$. For example, for $D = 16$, we have

$$\mathfrak{I} = \langle 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2,\\ 0, 1, 0, 0, 1, 0, 2, \ldots \rangle.$$

It is easy to see that such sequence $\mathfrak{I}$ satisfies both the D-sparseness and the D-density properties. □

Following [6], for a given n, a family $\mathcal{F}$ of subsets of $[n]$ is called a *j-selector* if for any pair of disjoint sets $X \subseteq [n]$ and $Y \subseteq [n]$ with $j/2 \leqslant |X| \leqslant j$ and $|Y| \leqslant j$ there exists a set $F \in \mathcal{F}$ such that $|X \cap F| = 1$ and $|Y \cap F| = 0$.

In [8], using probabilistic arguments, j-selectors of asymptotically optimal size are constructed.

Lemma 5.2. [8] *For every $1 \leqslant j \leqslant n$, there exists a j-selector $\mathcal{F}$ of size $\mathcal{O}(j \log(n/j))$.*

For any j, $0 \leqslant j \leqslant \log n$, let

$$\mathcal{F}^j = \{\mathcal{F}_1^j, \mathcal{F}_2^j, \mathcal{F}_3^j, \ldots, \mathcal{F}_{s_j}^j\}$$

be a 2^j-selector of size $s_j = \mathcal{O}(2^j \cdot \log(n/(2^j)))$, as promised by Lemma 5.2.

Now, we are ready to describe our new deterministic broadcasting algorithm.

Oblivious Deterministic Broadcasting Algorithm(M).

Input: Network $\mathcal{N} = (V, E)$ of eccentricity D
  Sequence $\mathfrak{I} = \langle \mathfrak{I}_1, \mathfrak{I}_2, \ldots \rangle$ satisfying the D-sparseness
    and the D-density properties
  $(2^k \cdot (n/D))$-selectors $\mathcal{F}^k$, for all $0 \leqslant k \leqslant \log D$

for $t = 1$ **to** M **do**
  $k = \mathfrak{I}_t$ {beginning of **phase** *number t* of rank k}
  for $j = 1$ **to** $\log(2^k\, n/D)$ **do** {*SubPhase*(k, j)}
    for $r = 1$ **to** s_j **do**
      for each *active* node $v \in V$ **do**
        if $v \in \mathcal{F}_r^j$ **then** v transmits

Let us begin with a simple fact that follows from the definition of selector families, using the same arguments as in [6, Theorem 1]. We say a *phase number t is of rank k* if $\mathfrak{I}_t = k$.

Lemma 5.3. *If* $|\mathsf{Layer}_P(i)| \leqslant 2^k \cdot (n/D)$ *and* $\mathsf{Layer}_P(i)$ *is* leading, *then after a phase of rank k it is no longer* leading.

Proof. Denote by *SubPhase*(k, j) the part of a phase of rank k corresponding to a given j, which is a subphase in which we use all sets from the selecting family $\mathcal{F}^j$.

Let X be the number of active nodes in $\mathsf{Layer}_P(i)$ at the beginning of *SubPhase*(k, j) and Y be the set of newly activated nodes during this subphase in $\mathsf{Layer}_P(i)$. Let j be an integer such that $2^{j-1} \leqslant |X| \leqslant 2^j$. Consider first the case when $|Y| \leqslant 2^j$. Then, due to the properties of 2^j-*selector* there is a round in which only one node of X transmits. Hence, afterwards $\mathsf{Layer}_P(i)$ is no longer the leading layer.

Otherwise $|Y| > 2^j$ and therefore after *SubPhase*(k, j) the number of active nodes in $\mathsf{Layer}_P(i)$ doubles. It cannot double in every subphase, due to the size of the layer. Hence in some subphase $\mathsf{Layer}_P(i)$ stops to be leading. □

Next, we can proceed to the first main result of this section. First we analyze the number M of phases which are sufficient in the algorithm. Observe that each phase consists of many rounds.

Lemma 5.4. *There exists a constant c such that if $M \geqslant cD$, then the Oblivious Deterministic Broadcasting Algorithm(M) completes broadcasting in $\mathcal{N}$.*

Proof. The proof is similar to the proof of Theorem 1. It is enough to show that for any path $P = \langle v_0, v_1, \ldots, v_\ell \rangle$ that begins at the source all the nodes will be active after M phases. We consider layers with respect to P. For each layer we can estimate the number of *phases* in which $\mathsf{Layer}_P(i)$ is *leading* using Claim 5.3. Due to the D-density property and Claim 5.3, $\mathsf{Layer}_P(i)$ is leading $\mathcal{O}(|\mathsf{Layer}_P(i)| \cdot (D/n) + 1)$ phases, independently in which position of the selecting sequence $\mathfrak{I}$ we start. Consequently, the number of phases is $\mathcal{O}(\sum_{i=1}^{\ell} (|\mathsf{Layer}_P(i)| \cdot (D/n) + 1)) = \mathcal{O}(D)$, what completes the proof. □

Our next main result is the following theorem.

Theorem 3. *Let $\mathcal{N} = (V, E)$ be an n-node radio network of eccentricity D. There exists a constant c such that if $T \geqslant cn \log^2 D$, then the Oblivious Deterministic Broadcasting Algorithm completes broadcasting in $\mathcal{N}$ after T rounds.*

Proof. Denote $k' = \log(2^k \cdot (n/D))$. Due to Lemma 5.2, the number of rounds performed by each phase of rank k is

$$\begin{aligned} s_1 + s_2 + s_3 + \cdots + s_{k'} &= \mathcal{O}(s_{k'}) = \mathcal{O}\big(2^k \cdot (n/D) \cdot \log\big(n / \big(2^k \cdot (n/D)\big)\big)\big) \\ &= \mathcal{O}\big(2^k \cdot (n/D) \cdot \log D\big). \end{aligned}$$

Due to the D-sparseness property of the sequence $\mathfrak{I}$ and Lemma 5.4, we perform $\tau_k = \mathcal{O}(D/2^k)$ phases of rank k. Hence, the total number of rounds is

$$\begin{aligned} \mathcal{O}\left(\sum_{k=0}^{\log D} s_{k'} \tau_k\right) &= \mathcal{O}\left(\sum_{k=0}^{\log D} \big(2^k (n/D) \log D\big)\big(D/2^k\big)\right) = \mathcal{O}\left(\sum_{k=0}^{\log D} n \log D\right) \\ &= \mathcal{O}\big(n \log^2 D\big). \qquad \square \end{aligned}$$

6. Application to randomized gossiping

Our randomized algorithm from Section 3 can be used as a building block to improve the complexity of randomized broadcasting in n-node radio networks. In the gossiping problem we are to distribute to every node the messages from all other nodes. We assume that at the beginning each node has its own single message. In the course of the algorithm a node can contain many messages from other nodes (including its own message). The main idea is to duplicate each message by distributing it to other nodes. Denote by $\#copies(v)$ the number of nodes which contain the original message from the node v. Initially only v contains its own message, hence initially $\#copies(v) = 1$ for each node v.

Chrobak et al. [7] were the first who designed a randomized algorithm for gossiping. The algorithm from [7] has a logarithmic number of phases for $i = 0, 1, 2, \ldots, \log n - 1$. The general structure of one phase in [7] is:

Phase(i).

Let $K = 2^i$. Assume that before this phase, $\#copies(v) \geqslant K$ for each node v
repeat $\mathcal{O}((n/K)\log n)$ **times**
each node v with probability $1/n$ performs $\text{LTDBROADCAST}_v(2K)$
After completing the phase, $\#copies(v) \geqslant 2K$ for each node v, with high probability

The main part of the algorithm is the function $\text{LTDBROADCAST}_v(2K)$ which broadcasts all messages contained now in v (as a single package) to all nodes at the distance at most $2K$ from v.

We are making $\mathcal{O}((n/K)\log n)$ iterations on one phase. Since $\#copies(v) \geqslant K$ for each node v, then with high probability, for each node v there will be a node in one phase which contains original message from v and which will perform LTDBROADCAST as a unique node performing LTDBROADCAST in this iteration. Then the message from v will be distributed at distance at least 2^K and $\#copies(v) \geqslant 2^{i+1}$. The probabilistic analysis has been already done in [7]. The basic improvement here is to change the implementation of LTDBROADCAST.

In [7], $\text{LTDBROADCAST}_v(2K)$ has been done deterministically in $\mathcal{O}(K\log^2 n)$ time, then in [14], the deterministic broadcast has been replaced by a randomized one working in time $\mathcal{O}(K\log n)$.

Theorem 4. *Let $\mathcal{N} = (V, E)$ be an n-node radio network. There exists a randomized Las Vegas algorithm that completes gossiping in $\mathcal{N}$ in $\mathcal{O}(n\log^2 n)$ rounds with probability at least* $1 - n^{-1}$.

Proof. We base on the previous works [7,14]. Comparing to the algorithm from [7], we first begin with $\mathcal{O}(\log^2 n)$ rounds of the round robin algorithm to ensure that every message is received by at least $\mathcal{O}(\log^2 n)$ nodes. Then, we run algorithm RANDGOSSIP, but only with the values of $i = 2\log\log n + \Theta(1), \ldots, \log n - 1$, and each execution of algorithm $\text{LTDBROADCAST}_v(2K)$ is replaced by our Linear Randomized Broadcasting Algorithm(T) that takes v as the source node and set $T = c \cdot 2K$ for certain large constant c, where $K = 2^i$. Similar arguments as in [14] can be used to show that an improvement in the complexity of LTDBROADCAST gives an improvement of the algorithm of [7], in which LTDBROADCAST can be treated as a separate function. In [14], the improvement was logarithmic with respect to [7]: $\mathcal{O}(K\log^2 n)$-time deterministic broadcasting function from [6] has been replaced by a corresponding $\mathcal{O}(K \log n)$-time randomized function, using the randomized broadcasting of [2]. We can reduce it further by a single logarithmic factor using our broadcasting in randomized linear time. The same analysis as in [14] works here. $\square$

7. Randomized broadcasting with deterministic selecting sequence

As we already discussed in Section 1, the use randomized selecting sequence in our randomized broadcasting algorithm requires some significant overhead in the size of the message sent by the source. Therefore, from algorithmic point of view it would be more natural to require that the broadcasting algorithm is using a deterministic selecting sequence. In this section we discuss how randomized selecting sequences can be replaced by deterministic ones without any loss in the algorithm's performance. Our deterministic selecting sequences are in a stochastic sense almost identical to the randomized sequences used in Sections 3 and 4.

We use notation from Sections 3, 4, and 5. Let σ be a positive constant, $\sigma \geqslant 2$.

Definition 7.1 *(Strong deterministic density property).* An infinite sequence $\mathfrak{I}$ satisfies the *strong deterministic density property* if the following two conditions are satisfied:

- for every k, $k \leqslant \log(n) + \mathcal{LL}(n)$, $\mathsf{density}(\mathfrak{I}, k) \leqslant \sigma \cdot 2^k$, and
- every subsequence of $\sigma \cdot \log n$ consecutive elements in $\mathfrak{I}$ contains a subsequence $\langle \mathcal{LL}(n)+1, \ldots, \log n \rangle$.

It is easy to construct deterministically selecting sequences with strong deterministic density property with $\sigma = 2$. We present here a simple and very natural construction.

Let us first define a sequence $\mathfrak{T}$ of length $2^{\mathcal{LL}(n)} - 1$ such that for any i, $1 \leqslant i \leqslant 2^{\mathcal{LL}(n)} - 1$:

$$\text{if} \quad \left(i \bmod 2^k\right) = 2^{k-1} \quad \text{then} \quad \text{the } i\text{th element of } \mathfrak{T} \text{ is equal to } k.$$

For example, for $\mathcal{LL}(n) = 3$ we have $\mathfrak{T} = \langle 1, 2, 1, 3, 1, 2, 1 \rangle$. Next, let us take two infinite sequences $\mathfrak{I}^{\langle 1 \rangle}$ and $\mathfrak{I}^{\langle 2 \rangle}$ which result by iterating the sequence $\mathfrak{T}$ and the sequence $\langle \mathcal{LL}(n)+1, \mathcal{LL}(n)+2, \ldots, \log n \rangle$, respectively. The deterministic selecting sequence $\mathfrak{I}$ is constructed by interleaving the sequences $\mathfrak{I}^{\langle 1 \rangle}$ and $\mathfrak{I}^{\langle 2 \rangle}$. For example, if $n = 2^7$ (and hence, $\mathcal{LL}(n) = 3$ and $\log n = 7$), then

$$\mathfrak{I} = \langle 1, 4, 2, 5, 1, 6, 3, 7, 1, 4, 2, 5, 1, 6, 1, 7, 2, 4, 1, 5, 3, 6, 1, 7, 2, 4, 1, 5, 1, 6, 2, 7, \ldots \rangle.$$

The following lemma follows in a straightforward way.

Lemma 7.2. *The sequence $\mathfrak{I}$ constructed above satisfies the strong deterministic density property* (*for any $\sigma \geqslant 2$*).

7.1. Deterministic selecting sequences and randomized broadcasting

Assuming we have a sequence $\mathfrak{I}$ satisfying the strong deterministic density property, we can replace it in the Linear Randomized Broadcasting Algorithm to obtain our first main theorem in this section.

Theorem 5. *There exists $T = \Theta(n)$ such that if one replaces the randomized sequence $\mathfrak{I}$ by any deterministic sequence satisfying the strong deterministic density property, then the Linear Randomized Broadcasting Algorithm(T) completes broadcasting in any n-node radio network with probability at least $1 - n^{-1}$.*

The proof of Theorem 5 follows the analysis from Section 3. It is easy to see that Lemma 3.2 holds for sequences satisfying the strong deterministic density property (actually, it holds for any infinite sequences). Similarly, Lemma 3.3 holds for sequences satisfying the strong deterministic density property too. However, it is also easy to see that Lemma 3.1 does not hold for deterministic sequences satisfying the strong deterministic density property. Fortunately, we can replace it by a similar lemma that holds for a consecutive subsequence of $\mathfrak{I}$.

Lemma 7.3. *Let $P = \langle v_0, v_1, \ldots, v_\ell \rangle$ be any path that begins at the source. Let us consider the Linear Randomized Broadcasting Algorithm with an infinite sequence $\mathfrak{I}$ satisfying the strong deterministic density property. Then for any round $r \in \mathbb{N}$, if $\mathsf{Layer}_P(i)$ is leading at round r, then in one of the rounds*

$$r, r+1, \ldots, r + 5 \cdot \sigma \cdot \min\bigl\{\bigl|\mathsf{Layer}_P(i)\bigr|, \log n\bigr\} - 1,$$

- *either $\mathsf{Layer}_P(i)$ is not leading anymore,*
- *or v_i is eager.*

Proof. Let $\Xi = 5 \cdot \sigma \cdot \min\{|\mathsf{Layer}_P(i)|, \log n\} - 1$ and let us suppose that $\mathsf{Layer}_P(i)$ is leading in all the rounds $r, r+1, \ldots, r + \Xi$. To prove the lemma it is sufficient to show that v_i will be eager in one of the rounds $r, r+1, \ldots, r + \Xi$.

Notice that by Lemma 3.3 our first assumption (that $\mathsf{Layer}_P(i)$ is leading) implies that for all r^*, $r \leqslant r^* < r + \Xi$, we have $0 < \gamma_{v_i}[r^* - 1] \leqslant |\mathsf{Layer}_P(i)|$.

Let $r_0 = r$. Let r_1 be the smallest round number $r^* \geqslant r_0$ for which $\frac{1}{2}\gamma_{v_i}[r_0 - 1] < 2^{I_{r^*}} \leqslant \gamma_{v_i}[r_0 - 1]$. Notice that by the deterministic density property we must have $r_0 \leqslant r_1 < r_0 + \sigma \cdot \min\{2^{\lceil \log \gamma_{v_i}[r_0 - 1] \rceil}, \log n\}$. Clearly, if $\frac{1}{2}\gamma_{v_i}[r_1 - 1] < 2^{I_{r_1}} \leqslant \gamma_{v_i}[r_1 - 1]$ then v_i is eager at round r_1. Otherwise, since the sequence $\gamma_{v_i}[r^*]$ is non-decreasing with r^*, we must have $\gamma_{v_i}[r_1 - 1] < 2^{I_{r_1}}$.

Let us consider the case when v_i is not eager at round r_1. Then, similarly as above, let r_2 be the smallest round number $r^* > r_1$ for which $\frac{1}{2}\gamma_{v_i}[r_1 - 1] < 2^{I_{r^*}} \leqslant \gamma_{v_i}[r_1 - 1]$. By the deterministic density property we have $r_2 - r_1 \leqslant \sigma \cdot \min\{2^{\lceil \log \gamma_{v_i}[r_1 - 1] \rceil}, \log n\}$. Similarly as above, if $\frac{1}{2}\gamma_{v_i}[r_2 - 1] < 2^{I_{r_2}} \leqslant \gamma_{v_i}[r_2 - 1]$ then v_i is eager at round r_2. Otherwise, we must have $\gamma_{v_i}[r_2 - 1] < 2^{I_{r_2}}$.

Continuing in this way, if v_i is not eager at round r_j, then let r_{j+1} be the smallest round number $r^* > r_j$ for which $\frac{1}{2}\gamma_{v_i}[r_j - 1] < 2^{I_{r^*}} \leqslant \gamma_{v_i}[r_j - 1]$.

Now, we want to show that the assumption that $\gamma_{v_i}[r^* - 1] \leqslant |\mathsf{Layer}_P(i)|$ for all $r_0 \leqslant r^* < r_0 + \Xi$ implies that v_i will be eager at some round r', $r_0 \leqslant r' < r_0 + \Xi$. Indeed, by our arguments above, we have $r_1 - r_0 \leqslant \sigma \cdot \min\{2^{\lceil \log \gamma_{v_i}[r_0-1]\rceil}, \log n\}$, then, $\lceil \log \gamma_{v_i}[r_0 - 1]\rceil < \lceil \log \gamma_{v_i}[r_1 - 1]\rceil$ and $r_2 - r_1 \leqslant \sigma \cdot \min\{2^{\lceil \log \gamma_{v_i}[r_1-1]\rceil}, \log n\}$, and so, for arbitrary j we have $\lceil \log \gamma_{v_i}[r_j - 1]\rceil < \lceil \log \gamma_{v_i}[r_{j+1} - 1]\rceil$ and $r_{j+1} - r_j \leqslant \sigma \cdot \min\{2^{\lceil \log \gamma_{v_i}[r_j-1]\rceil}, \log n\}$. Furthermore, the strong deterministic density property implies that if for some j^* we have $2^{\lceil \log \gamma_{v_i}[r_j^*-1]\rceil} > \mathcal{LL}(n)$, then for every $j \geqslant j^*$ we will have $r_j - r_j^* \leqslant \sigma \cdot \log n$.

Notice that since the sequence $\lceil \log \gamma_{v_i}[r_j - 1]\rceil$ is strictly increasing with j and since $\gamma_{v_i}[r^* - 1] \leqslant |\mathsf{Layer}_P(i)|$ for all $r_0 \leqslant r^* < r_0 + \Xi$, we have $j \leqslant \lceil \log |\mathsf{Layer}_P(i)|\rceil$. Therefore, the sequence of inequalities $r_{j+1} - r_j \leqslant \sigma \cdot \min\{2^{\lceil \log \gamma_{v_i}[r_j-1]\rceil}, \log n\}$, the inequality $r_j - r_j^* \leqslant \sigma \cdot \log n$ that holds for every $j \geqslant j^*$, and the upper bound for j above imply that

- either $2^{\lceil \log |\mathsf{Layer}_P(i)|\rceil} \leqslant \mathcal{LL}(n)$ and
$$\begin{aligned} r_j - r_0 &\leqslant \sum_{t=0}^{\lceil \log |\mathsf{Layer}_P(i)|\rceil} \sigma \cdot \min\bigl\{2^t, \log n\bigr\} < \sigma \cdot 2^{\lceil \log |\mathsf{Layer}_P(i)|\rceil + 1} \\ &\leqslant 4 \cdot \sigma \cdot \bigl|\mathsf{Layer}_P(i)\bigr|, \end{aligned}$$
- or $2^{\lceil \log |\mathsf{Layer}_P(i)|\rceil} > \mathcal{LL}(n)$ and
$$\begin{aligned} r_j - r_0 &\leqslant \sigma \cdot \sum_{t=0}^{\mathcal{LL}(n)} \min\bigl\{2^t, \log n\bigr\} + \sigma \cdot \log n \leqslant \sigma \cdot \sum_{t=0}^{\mathcal{LL}(n)} 2^t + \sigma \cdot \log n \\ &< 5 \cdot \sigma \cdot \log n. \end{aligned}$$

These two inequalities imply that for every j we have
$$r_j - r \leqslant 5 \cdot \sigma \cdot \min\bigl\{\bigl|\mathsf{Layer}_P(i)\bigr|, \log n\bigr\} - 1 = \Xi.$$
Therefore, since we consider the rounds $r, r+1, \ldots, r+\Xi$, this implies that there must be j for which v_i is eager at round r_j. This concludes the proof of the lemma. □

Notice now that the following lemma follows immediately from Lemma 7.3. (Let us remind that $Z_P(i)$ is the random variable denoting the number of rounds in which $\mathsf{Layer}_P(i)$ is leading.)

Lemma 7.4. *The random variable* $Z_P(i)$ *is majorized by* $5 \cdot \sigma \cdot \min\{|\mathsf{Layer}_P(i)|, \log n\}$ *times the geometric random variable with the parameter* 0.1. *In particular,* $\mathbf{E}[Z_P(i)] \leqslant 50 \cdot \sigma \cdot \min\{|\mathsf{Layer}_P(i)|, \log n\}$.

Proof. Indeed, if $\mathsf{Layer}_P(i)$ is leading at certain round r then by Lemma 7.3, in the next $5 \cdot \sigma \cdot \min\{|\mathsf{Layer}_P(i)|, \log n\}$ rounds it will either be not leading anymore or v_i will be eager in at least one round. Moreover, if v_i is eager then it will become active with probability at least 0.1. These two claims imply the lemma. □

Now, we could show a variant of Lemma 3.5 to conclude with the proof of Theorem 5, but since such a variant of Lemma 3.5 is rather complicated, we prefer a different, (we believe) a simpler approach.

We say an algorithm is in *stage s* if it is in a round r, where $(s-1)\cdot 112\cdot\sigma\cdot\log n+1\leqslant r\leqslant s\cdot 112\cdot\sigma\cdot\log n$. Let $P=\langle v_0,v_1,\ldots,v_\ell\rangle$ be any path in $\mathcal{N}$ that begins at the source node. If $\mathsf{Layer}_P(i)$ is leading at round r, then we define

$$\mathsf{Leftover}_P(r)=\sum_{j=i}^{\ell}\bigl|\mathsf{Layer}_P(j)\bigr|.$$

For any $s\in\mathbb{N}$, we define a function $\Delta_P(s)$, which represent the "gain" in stage s:

$$\Delta_P(s)=\mathsf{Leftover}_P\bigl((s-1)\cdot 112\cdot\sigma\cdot\log n\bigr)-\mathsf{Leftover}_P(s\cdot 112\cdot\sigma\cdot\log n).$$

Now, we can use Lemma 7.4 to prove the following lemma.

Lemma 7.5. *Let $P=\langle v_0,v_1,\ldots,v_\ell\rangle$ be any shortest path from the source node to a node v_ℓ. Then, for any $s\in\mathbb{N}$, with probability at least* 0.1*,*

- *either $\Delta_P(s)\geqslant\log n$,*
- *or $\mathsf{Leftover}_P(s\cdot 112\cdot\sigma\cdot\log n)=0$; in which case the node v_ℓ is active at the end of stage s.*

Proof. Let $\mathsf{Layer}_P(i)$ be leading at round $(s-1)\cdot 112\cdot\sigma\cdot\log n$. Let $j\geqslant i$ be the smallest j such that

$$\sum_{t=i}^{j}\bigl|\mathsf{Layer}_P(t)\bigr|\geqslant\log n;$$

if no such an j exists then let $j=\ell$. Notice that, in particular, we have $\sum_{t=i}^{j-1}|\mathsf{Layer}_P(t)|<\log n$.

In order to prove the lemma it is enough to show that

$$\mathbf{Pr}\Biggl[\sum_{t=i}^{j}Z_P(t)\leqslant 112\cdot\sigma\cdot\log n\Biggr]\geqslant 0.1.$$

Observe that by Lemma 7.4 and the inequality $\sum_{t=i}^{j-1}|\mathsf{Layer}_P(t)|<\log n$ we have

$$\begin{aligned}\mathbf{E}\Biggl[\sum_{t=i}^{j}Z_P(t)\Biggr]&=\mathbf{E}\Biggl[\sum_{t=i}^{j-1}Z_P(t)\Biggr]+\mathbf{E}\bigl[Z_P(j)\bigr]\\&\leqslant 50\cdot\sigma\cdot\sum_{t=i}^{j-1}\bigl|\mathsf{Layer}_P(t)\bigr|+50\cdot\sigma\cdot\min\bigl\{\bigl|\mathsf{Layer}_P(j)\bigr|,\log n\bigr\}\\&<100\cdot\sigma\cdot\log n.\end{aligned}$$

Therefore, by Markov inequality we obtain

$$\mathbf{Pr}\left[\sum_{t=i}^{j} Z_P(t) \geqslant 112 \cdot \sigma \cdot \log n\right] \leqslant \frac{100 \cdot \sigma \cdot \log n}{112 \cdot \sigma \cdot \log n} < 0.9,$$

what yields the lemma. □

The way one can understand this lemma is that for the deterministic sequence in every $112\sigma \log n$ consecutive rounds (a stage) we have at least a "logarithmic" progress with probability at least 0.1.

Proof of Theorem 5. Once we have Lemma 7.5, similar approach as that used in the proof of Theorem 1 together with the fact that $\mathsf{Leftover}_P(0) = n$, can be used to obtain the proof of Theorem 5. Indeed, by Lemma 7.5 and because $\mathsf{Leftover}_P(0) = n$, in order to prove the theorem it suffices to show that

$$\mathbf{Pr}\left[\sum_{t=1}^{\lceil n/\log n\rceil} Y_t \leqslant 20 \cdot \lceil n/\log n\rceil\right] \geqslant 1 - n^{-2},$$

where $Y_1, \ldots, Y_{\lceil n/\log n\rceil}$ are independent random variables with geometric distribution with the parameter 0.1 each. And this follows directly from Chernoff-like bound in Lemma 3.5. □

7.2. Deterministic selecting sequence for shallow networks

Now, we discuss how our analysis can be extended to deterministic selecting sequences in networks of eccentricity D. Let us consider an extension of the strong deterministic density property that takes also the parameter D into account. Our analysis works fine with all $\sigma \geqslant 3$, and we will use $\sigma = 3$.

Definition 7.6 *(D-modified strong deterministic density property).* A sequence $\mathfrak{I}^D$ satisfies the *D-modified strong deterministic density property* if it satisfies the following three conditions:

- every subsequence of $\sigma \cdot \log(n/D)$ consecutive elements in $\mathfrak{I}^D$ contains a subsequence $\langle 1, 2, \ldots, \log(n/D)\rangle$,
- for every k, $\log(n/D) < k \leqslant \log(n/D) + \mathcal{LL}(n)$, $\mathsf{density}(\mathfrak{I}^D, k) \leqslant \sigma \cdot \log(n/D) \cdot 2^k \cdot (D/n)$,
- every subsequence of $\sigma \cdot \log n \cdot \log(n/D)$ consecutive elements in $\mathfrak{I}^D$ contains a subsequence $\langle \log(n/D) + \mathcal{LL}(n) + 1, \ldots, \log n\rangle$.

It is easy to construct an infinite sequence $\mathfrak{I}^D$ satisfying the D-modified strong deterministic density property (and actually, in the last condition above we have $\sigma \log n$ instead of $\sigma \log n \log(n/D)$).

Lemma 7.7. *There exists an infinite sequence satisfying the D-modified strong deterministic density property.*

Proof. We take three sequences and will interleave them to satisfy the required property (with $\sigma = 3$). The first infinite sequence $\mathfrak{I}^{\langle 0\rangle}$ contains iterated sequence $\langle 1, 2, \ldots, \log(n/D)\rangle$, that is,

$$\mathfrak{I}_i^{\langle 0\rangle} = \big((i-1) \bmod \log(n/D)\big) + 1.$$

The second sequence $\mathfrak{I}^{\langle 1\rangle}$ corresponds to the sequence $\mathfrak{I}^{\langle 1\rangle}$ defined in Lemma 7.2, and is obtained by repeating the sequence of length $2^{\mathcal{LL}(n)} - 1$ defined such that the ith element in the sequence is equal to $\log(n/D) + j$ for j such that $(i \bmod 2^j) = 2^{j-1}$. The third sequence $\mathfrak{I}^{\langle 2\rangle}$ is obtained by iterating the sequence $\langle \log(n/D) + \mathcal{LL}(n) + 1, \log(n/D) + \mathcal{LL}(n) + 2, \ldots, \log n - 1, \log n\rangle$.

Then, the final sequence $\mathfrak{I}^D$ is defined by interleaving the sequences $\mathfrak{I}^{\langle 0\rangle}$, $\mathfrak{I}^{\langle 1\rangle}$, and $\mathfrak{I}^{\langle 2\rangle}$, that is, such that for any $i \geqslant 1$ and $j \in \{0, 1, 2\}$, we have $\mathfrak{I}^D_{3i-j} = \mathfrak{I}_i^{\langle 2-j\rangle}$. It is easy to see that such sequence $\mathfrak{I}^D$ satisfies the D-modified strong deterministic density property. □

Using modification of the proof of Theorem 2 similar to those done in the proof of Theorem 5, we can show:

Theorem 6. *Let $\mathcal{N} = (V, E)$ be an n-node radio network of eccentricity $D = \Omega(\log^3 n)$. There exists $T = \Theta(D \log(n/D))$, such that if one replaces the randomized sequence $\mathfrak{I}^D$ by any deterministic sequence satisfying the D-modified deterministic density property, then the Randomized Broadcasting Algorithm for Shallow Networks(T) completes broadcasting in any n-node radio network of eccentricity D with probability at least $1 - n^{-1}$.*

We now show how the proof of Theorem 5 should be modified to get the proof of Theorem 6. The key difference is a variant of Lemma 7.3.

Lemma 7.8. *Let $P = \langle v_0, v_1, \ldots, v_\ell\rangle$ be any path that begins at the source. Let us consider the Linear Randomized Broadcasting Algorithm with the infinite sequence $\mathfrak{I}^D$ satisfying the D-modified strong deterministic density property. Then for any round $r \in \mathbb{N}$, if $\mathsf{Layer}_P(i)$ is leading at round r, then in one of the rounds*

$$r, r+1, \ldots, r + \Xi(i) - 1,$$

- *either $\mathsf{Layer}_P(i)$ is not leading anymore,*
- *or v_i is eager,*

where $\Xi(i)$ is defined as

$$\Xi(i) = 5 \cdot \sigma \cdot \log(n/D) \cdot \max\left\{1, \min\left\{\big|\mathsf{Layer}_P(i)\big| \cdot \frac{D}{n}, \log n\right\}\right\}.$$

Proof. The proof is essentially identical to the proof of Lemma 7.3. The only change is a different upper bound for $r_j - r_0$, which is now as follows:

- either $|\mathsf{Layer}_P(i)| \leqslant n/D$ and

$$r_j - r_0 \leqslant \sigma \cdot \log(n/D),$$

- or $\log(n/D) < \lceil \log|\mathsf{Layer}_P(i)|\rceil \leqslant \log(n/D) + \mathcal{LL}(n)$ and

$$\begin{aligned}
r_j - r_0 &\leqslant \sigma \cdot \log(n/D) + \sum_{t=1+\log(n/D)}^{\lceil \log|\mathsf{Layer}_P(i)|\rceil} \sigma \cdot \log(n/D) \cdot 2^t \cdot (n/D) \\
&< \sigma \cdot \log(n/D) \cdot (n/D) \cdot 2^{1+\lceil \log|\mathsf{Layer}_P(i)|\rceil} \\
&\leqslant 4 \cdot \sigma \cdot \log(n/D) \cdot (n/D) \cdot \bigl|\mathsf{Layer}_P(i)\bigr|,
\end{aligned}$$

- or $\log(n/D) + \mathcal{LL}(n) < \lceil \log|\mathsf{Layer}_P(i)|\rceil \leqslant \log n$ and

$$r_j - r_0 < 2 \cdot \sigma \cdot \log(n/D) \cdot 2^{\mathcal{LL}(n)} + \sigma \cdot \log n \leqslant 5 \cdot \sigma \cdot \log(n/D) \cdot \log n.$$

This immediately yields the lemma. □

With Lemma 7.8 replacing Lemma 7.3, we immediately obtain the following lemma.

Lemma 7.9. *The random variable $Z_P(i)$ is majorized by $\Xi(i)$ times the geometric random variable with the parameter* 0.1*, where $\Xi(i)$ is defined as in Lemma* 7.8*. In particular,* $\mathbf{E}[Z_P(i)] \leqslant 10 \cdot \Xi(i) = 50 \cdot \sigma \cdot \log(n/D) \cdot \max\{1, \min\{|\mathsf{Layer}_P(i)| \cdot \frac{D}{n}, \log n\}\}$.

Next, similarly as we did in Section 7, we combine the rounds into stages, but this time the definition of stages is slightly more complicated and the stages are not disjoint.

Let us consider any $P = \langle v_0, v_1, \ldots, v_\ell\rangle$ shortest path from the source node to a node v_ℓ. Let us consider a round $r \in \mathbb{N}$, and let $\mathsf{Layer}_P(i)$ be leading at round r. Let $j \leqslant \ell$ be the largest j such that $\sum_{t=i}^{j-1} |\mathsf{Layer}_P(t)| < \frac{n\cdot\log n}{D}$. Define $\vartheta(r) = r + 112 \cdot \sigma \cdot \log(n/D) \cdot ((j - i + 1) + \log n)$. We define a *stage with respect to r* to consist of all rounds r^* such that $r < r^* \leqslant \vartheta(r)$.

With this and using the definition of $\mathsf{Leftover}_P(\cdot)$ from Section 7, we can prove the following lemma.

Lemma 7.10. *Let $P = \langle v_0, v_1, \ldots, v_\ell\rangle$ be any shortest path from the source node to a node v_ℓ. Let $r \in \mathbb{N}$ be any round. Then with probability at least* 0.1*,*

- *either* $\mathsf{Leftover}_P(\vartheta(r)) - \mathsf{Leftover}_P(r) \geqslant \frac{n\cdot\log n}{D}$,
- *or* $\mathsf{Leftover}_P(\vartheta(r)) = 0$; *in which case the node v_ℓ is active at the end of the stage with respect to r.*

Proof. The proof mimics the proof of Lemma 7.5. Let $\mathsf{Layer}_P(i)$ be leading at round r. Let $j \leqslant \ell$ be the largest j such that $\sum_{t=i}^{j-1} |\mathsf{Layer}_P(t)| < \frac{n\cdot\log n}{D}$. This inequality and Lemma 7.9 yield the following

$$\begin{aligned}
\mathbf{E}\left[\sum_{t=i}^{j} Z_P(t)\right] &= \mathbf{E}\left[\sum_{t=i}^{j-1} Z_P(t)\right] + \mathbf{E}\bigl[Z_P(j)\bigr] \\
&\leqslant \sum_{t=i}^{j-1} 50 \cdot \sigma \cdot \log(n/D) \cdot \max\left\{1, \min\left\{\bigl|\mathsf{Layer}_P(t)\bigr| \cdot \frac{D}{n}, \log n\right\}\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ 50 \cdot \sigma \cdot \log(n/D) \cdot \max\left\{1, \min\left\{\left|\mathsf{Layer}_P(j)\right| \cdot \frac{D}{n}, \log n\right\}\right\} \\
&\leqslant 50 \cdot \sigma \cdot \log(n/D) \cdot \sum_{t=i}^{j-1}\left(1 + \left|\mathsf{Layer}_P(t)\right| \cdot \frac{D}{n}\right) \\
&\quad + 50 \cdot \sigma \cdot \log(n/D) \cdot (1 + \log n) \\
&\leqslant 50 \cdot \sigma \cdot \log(n/D) \cdot \left((j-i) + 1 + \log n + \frac{D}{n} \cdot \sum_{t=i}^{j-1}\left|\mathsf{Layer}_P(t)\right|\right) \\
&< 50 \cdot \sigma \cdot \log(n/D) \cdot (j - i + 1 + 2 \cdot \log n).
\end{aligned}
$$

Therefore, by Markov inequality we obtain

$$
\mathbf{Pr}\left[\sum_{t=i}^{j} Z_P(t) \geqslant 56 \cdot \sigma \cdot \log(n/D) \cdot (j - i + 1 + 2 \cdot \log n)\right] \leqslant 0.9,
$$

and hence

$$
\begin{aligned}
&\mathbf{Pr}\left[\sum_{t=i}^{j} Z_P(t) \leqslant \vartheta(r) - r\right] \\
&\equiv \mathbf{Pr}\left[\sum_{t=i}^{j} Z_P(t) \leqslant 112 \cdot \sigma \cdot \log(n/D) \cdot (j - i + 1 + \log n)\right] \geqslant 0.1.
\end{aligned}
$$

This implies that with probability at least 0.1, at round $\vartheta(r)$ all nodes $v_i, v_{i+1}, \ldots, v_j$ will be active. Thus, either $\mathsf{Leftover}_P(\vartheta(r)) - \mathsf{Leftover}_P(r) \geqslant \frac{n \cdot \log n}{D}$ or $\mathsf{Leftover}_P(\vartheta(r)) = 0$. □

With Lemma 7.10, Theorem 6 follows easily by applying the Chernoff-like bound from Lemma 3.5.

Proof of Theorem 6. As before, it is sufficient to prove the high probability bound for any single shortest path $P = \langle v_0, v_1, \ldots, v_\ell \rangle$ from the source node to a node v_ℓ. Let us fix such a path.

Let us consider partition of the running time of the algorithm into phases such that the 1st phase is the stage with respect to round 0, and recursively, if the τth phase ends at round r_τ, then the $(\tau + 1)$st phase is the stage with respect to the round r_τ. We call a phase τ successful if either node v_ℓ is active at the end of the phase or $\mathsf{Leftover}_P(\vartheta(r_\tau)) - \mathsf{Leftover}_P(r_\tau) \geqslant \frac{n \cdot \log n}{D}$. Notice that since $\mathsf{Leftover}_P(0) = n$, $\mathsf{Leftover}_P(r)$ is non-increasing with r and is non-negative, after $D/\log n$ successful phases node v_ℓ must be active. Let Y_t be the indicator random variable that is 1 if and only if the tth phase is successful. Notice that by Lemma 7.10, each Y_i is stochastically majorized by the random variable with the geometric distribution with parameter 0.1. Therefore, we can use Lemma 3.5 to conclude that with probability at least $1 - n^{-2}$, after $\frac{20 \cdot D}{\log n}$ phase at least $\frac{D}{\log n}$ phases will be successful, that is,

$$\mathbf{Pr}\left[\sum_{t=1}^{20 \cdot D/\log n} Y_i \geqslant D/\log n\right] \geqslant 1 - n^{-2}.$$

Therefore to conclude the proof of the theorem, we must only show how many rounds can be in the first $20 \cdot D/\log n$ phases. And one can easily see that the number of rounds is upper bounded by

$$112 \cdot \sigma \cdot \log(n/D) \cdot \ell + \frac{20 \cdot D}{\log n} \cdot \bigl(112 \cdot \sigma \cdot \log(n/D) \cdot \log n\bigr) = \mathcal{O}\bigl(D \cdot \log(n/D)\bigr),$$

what yields the proof of the theorem. □

Appendix A. Proof of Lemma 3.5

Since $\mathbf{E}[X_i] = 1/p_i = \mu_i$, we have $\mathbf{E}[\sum_{i=1}^{\ell} X_i] = \sum_{i=1}^{\ell} 1/p_i = \sum_{i=1}^{\ell} \mu_i \leqslant N$. Next, for any $z \in \Delta$, let $K_z = \{i\colon\ \mu_i = z\}$. Then, clearly, we have

$$\sum_{i=1}^{\ell} X_i = \sum_{z \in \Delta} \sum_{i \in K_z} X_i.$$

Next, our goal is to study $\sum_{i \in K_z} X_i$ for all $z \in \Delta$. Since all X_i with $i \in K_z$ are identically distributed with geometric distribution with the parameter $1/z$, the standard relation between geometric and binomial distributions implies the following inequality that holds for every $z \in \Delta$ and for every $M \in \mathbb{N}$:

$$\mathbf{Pr}\left[\sum_{i \in K_z} X_i > M\right] \leqslant \mathbf{Pr}\bigl[\mathbb{B}(M, 1/z) < |K_z|\bigr] \leqslant \mathbf{Pr}\bigl[\mathbb{B}(M, 1/z) \leqslant |K_z|\bigr]. \tag{A.1}$$

Therefore, we focus our analysis on the study of the binomial distribution. Let us recall a standard variant of Chernoff bound: for any $M \in \mathbb{N}$ and any $0 \leqslant p \leqslant 1$: $\mathbf{Pr}[\mathbb{B}(M, p) \leqslant \frac{1}{2}\mathbf{E}[\mathbb{B}(M, p)]] \leqslant \exp(-\mathbf{E}[\mathbb{B}(M, p)]/8) = \exp(-Mp/8)$.

For simplicity, let us consider two cases separately:

- $|K_z| \geqslant 4\ln(|\Delta|/\beta)$: we apply the Chernoff bound above with $M = 2z|K_z|$ to obtain
$$\mathbf{Pr}\bigl[\mathbb{B}\bigl(2z|K_z|, 1/z\bigr) \leqslant |K_z|\bigr] \leqslant \exp\bigl(-2|K_z|/8\bigr) \leqslant \exp\bigl(-\ln\bigl(|\Delta|/\beta\bigr)\bigr) \leqslant \beta/|\Delta|.$$
- $|K_z| < 4\ln(|\Delta|/\beta)$: we apply the Chernoff bound above with $M = 8z\ln(|\Delta|/\beta)$ to obtain
$$\begin{aligned}\mathbf{Pr}\bigl[\mathbb{B}\bigl(8z\ln\bigl(|\Delta|/\beta\bigr), 1/z\bigr) \leqslant |K_z|\bigr] &\leqslant \mathbf{Pr}\bigl[\mathbb{B}\bigl(8z\ln\bigl(|\Delta|/\beta\bigr), 1/z\bigr) \leqslant 4\ln\bigl(|\Delta|/\beta\bigr)\bigr] \\ &\leqslant \exp\bigl(-\ln\bigl(|\Delta|/\beta\bigr)\bigr) = \beta/|\Delta|.\end{aligned}$$

Thus, we can summarize these two cases in a single bound:

$$\mathbf{Pr}\big[\mathbb{B}\big(\max\big\{2z|K_z|, 8z\ln\big(|\Delta|/\beta\big)\big\}, 1/z\big) \leqslant |K_z|\big] \leqslant \beta/|\Delta|.$$

Now, we can combine this inequality with (A.1) to obtain for every $z \in \Delta$:

$$\begin{aligned}&\mathbf{Pr}\Bigg[\sum_{i\in K_z} X_i > \max\big\{2z|K_z|, 8z\ln\big(|\Delta|/\beta\big)\big\}\Bigg]\\ &\quad\leqslant \mathbf{Pr}\big[\mathbb{B}\big(\max\big\{2z|K_z|, 8z\ln\big(|\Delta|/\beta\big)\big\}, 1/z\big) < |K_z|\big] \leqslant \beta/|\Delta|.\end{aligned}$$

Therefore, we can apply the union bound to obtain

$$\begin{aligned}&\mathbf{Pr}\Bigg[\exists z \in \Delta \text{ such that } \Bigg(\sum_{i\in K_z} X_i > \max\big\{2z|K_z|, 8z\ln\big(|\Delta|/\beta\big)\big\}\Bigg)\Bigg]\\ &\quad\leqslant \sum_{z\in\Delta}\mathbf{Pr}\Bigg[\sum_{i\in K_z} X_i < \max\big\{2z|K_z|, 8z\ln\big(|\Delta|/\beta\big)\big\}\Bigg] \leqslant |\Delta|\cdot\big(\beta/|\Delta|\big) = \beta.\end{aligned}$$

From this, we see that for all $z \in \Delta$, $\sum_{i\in K_z} X_i \leqslant \max\{2z|K_z|, 8z\ln(|\Delta|/\beta)\}$ with probability at least $1-\beta$. Therefore, with probability at least $1-\beta$ we obtain

$$\begin{aligned}\sum_{i=1}^{\ell} X_i &= \sum_{z\in\Delta}\sum_{i\in K_z} X_i \leqslant \sum_{z\in\Delta}\max\big\{2z|K_z|, 8z\ln\big(|\Delta|/\beta\big)\big\}\\ &\leqslant \sum_{z\in\Delta}\big(2z|K_z| + 8z\ln\big(|\Delta|/\beta\big)\big)\\ &= 2\cdot\sum_{z\in\Delta} z\cdot|K_z| + 8\cdot\ln\big(|\Delta|/\beta\big)\cdot\sum_{z\in\Delta} z\\ &= 2\cdot\sum_{i=1}^{\ell}\mu_i + 8\cdot\ln\big(|\Delta|/\beta\big)\cdot\sum_{z\in\Delta} z\\ &\leqslant 2\cdot N + 8\cdot\ln\big(|\Delta|/\beta\big)\cdot\sum_{z\in\Delta} z.\end{aligned}$$

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