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Faster Longest Common Extension Queries in Strings over General Alphabets

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Abstract

Longest common extension queries (often called longest common prefix queries) constitute a fundamental building block in multiple string algorithms, for example computing runs and approximate pattern matching. We show that a sequence of q LCE queries for a string of size n over a general ordered alphabet can be realized in $\mathcal{O}(q \log\log n + n \log^* n)$ time making only $\mathcal{O}(q + n)$ symbol comparisons. Consequently, all runs in a string over a general ordered alphabet can be computed in $\mathcal{O}(n \log\log n)$ time making $\mathcal{O}(n)$ symbol comparisons. Our results improve upon a solution by Kosolobov (Information Processing Letters, 2016), who gave an algorithm with $\mathcal{O}(n \log^{2/3} n)$ running time and conjectured that $\mathcal{O}(n)$ time is possible. We make a significant progress towards resolving this conjecture. Our techniques extend to the case of general unordered alphabets, when the time increases to $\mathcal{O}(q \log n + n \log^* n)$. The main tools are difference covers and a variant of the disjoint-sets data structure by La Poutré (SODA 1990).

1 Introduction

While many text algorithms are designed under the assumption of integer alphabet sortable in linear time, in some cases it is enough to assume general alphabet. A general alphabet can be either ordered, meaning that one can check if one symbol is less than another, or unordered, meaning that only equality of two symbols can be checked. Many classical linear-time string-matching algorithms (e.g. Knuth-Morris-Pratt, Boyer-Moore) work for any unordered general alphabet. Recently, a linear-time algorithm for computing the leftmost critical factorization in such model was given [10]. On the other hand, algorithms related to detecting repetitions usually need $\Omega(n \log n)$ equality tests [18], and an on-line algorithm matching this bound is known [12].

In this paper we consider the longest common extension problem (LCE, in short) in case of general ordered and unordered alphabets. The goal is to preprocess a given word w of length n for queries $\mathrm{LCE}(i, j)$ returning the length of the longest common factor starting at position i and j in w. Such queries are often a basic building block in more complicated algorithms, for example in computing runs [1, 2] as well as in approximate string matching [15].

For integer alphabets of polynomial size, one can preprocess a given string in linear time and space to answer any LCE query in constant time. Preprocessing space can be traded for query time [4, 5] and generalizations to trees [3] and grammar-compressed strings [8, 9, 16, 19] are known. The situation is more complicated for general alphabets. If the alphabet is ordered, then of course we can reduce it to $[1..n]$ by sorting the characters in $\mathcal{O}(n \log n)$ time and preprocess the obtained string in linear time and space to

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answer any LCE query in constant time. However this increases the total preprocessing time to $\mathcal{O}(n \log n)$. For unordered alphabet the situation is even worse, because the reduction would take $\mathcal{O}(n^2)$ time. A natural question is hence how efficiently we can answer a collection of such queries given one by one (on-line), where we measure the preprocessing time plus the total time taken by all the queries.

It is known that if we can perform on-line $\mathcal{O}(n)$ LCE queries for a given word of length n in total time $T(n)$ making $\mathcal{O}(n)$ symbol comparisons, then we can compute all runs in $\mathcal{O}(n+T(n))$ time making only $\mathcal{O}(n)$ symbol comparisons. An algorithm with $T(n) = \mathcal{O}(n \log^{2/3} n)$ time was recently presented by Kosolobov [13], who posed the existence of a linear-time algorithm as an open question. In this paper we make a significant progress towards answering the question by giving a faster algorithm with $T(n) = \mathcal{O}(n \log \log n)$.

Our result For a given string of length n over a general ordered alphabet, we can answer on-line a sequence of q LCE queries in $\mathcal{O}(q \log \log n + n \log^* n)$ time making $\mathcal{O}(q + n)$ symbol comparisons. In particular, a sequence of $\mathcal{O}(n)$ queries can be answered in $\mathcal{O}(n \log \log n)$ time. Consequently, all runs in a string over a general ordered alphabet can be computed in $\mathcal{O}(n \log \log n)$ time making $\mathcal{O}(n)$ symbol comparisons. For a general unordered alphabet we answer q LCE queries in $\mathcal{O}(q \log n + n \log^* n)$ time, still making $\mathcal{O}(q + n)$ symbol comparisons.

Overview of the methods At a very high level, our approach is similar to the one used by Kosolobov. We first show how to calculate $\min(\mathrm{LCE}(i, j), t)$ efficiently, where $t = \mathrm{polylog}\, n$. Then we use a difference cover to sample some positions in the text. Using "short" queries, we can efficiently construct a sparse suffix array for these sampled positions, which in turn allows us to calculate any $\mathrm{LCE}(i, j)$ efficiently. The key difference is that instead of calculating $\min(\mathrm{LCE}(i, j), t)$ naively, we use a recursive approach. The main tool there is an efficient Union-Find structure. This is enough to answer $\mathcal{O}(n)$ short queries in $\mathcal{O}(n \log \log n \cdot \alpha(n))$ total time. We can remove the $\alpha(n)$ factor introducing another difference cover and using a Union-Find implementation by La Poutré [14]. Finally, we modify the algorithm to work faster when the number of queries q is smaller than n. The main insight allowing us to obtain $\mathcal{O}(q \log \log n + n \log^* n)$ total time is introducing multiple levels of difference covers with some additional properties. Such family of difference covers was implicitly provided in [7].

2 Preliminaries

2.1 t-covers

A difference cover is a number-theoretic tool used throughout the paper. A set $\mathbf{D} \subseteq [0..t-1]$ is said to be a *t-difference-cover* if $[0..t-1] = \{ (x-y) \bmod t \ : \ x, y \in \mathbf{D} \}$.

Lemma 2.1 (Maekawa [17]). *For every integer t there is t-difference-cover of size $\mathcal{O}(\sqrt{t})$, which can be constructed in $\mathcal{O}(\sqrt{t})$ time.*

A subset X of $[1..n]$ is *t-periodic* if for each $i \in [1..n-t]$ we have: $i \in X \Leftrightarrow i+t \in X$. A set $\mathbf{S} \subseteq [1..n]$ is called a *t-cover* of $[1..n]$ if $\mathbf{S}$ is t-periodic and there is a constant-time computable function h such that for $1 \le i, j \le n-t$ we have $0 \le h(i,j) \le t$ and $i + h(i,j), j + h(i,j) \in \mathbf{S}(t)$.

A t-cover can be obtained by taking a t-difference-cover $\mathbf{D}$ and setting $\mathbf{S}(t) = \{i \in [1..n] : i \bmod t \in \mathbf{D}\}$. This is a well-known construction implicitly used in [6], for example.

Lemma 2.2. *For each $t \le n$ there is a t-cover $\mathbf{S}(t)$ of size $\mathcal{O}(\frac{n}{\sqrt{t}})$ which can be constructed in $\mathcal{O}(\frac{n}{\sqrt{t}})$ time.*

2.2 Union-Find structure

Our another tool is a disjoint-sets data structure. In this problem we maintain a family of disjoint subsets of $[1..n]$, initially consisting of singleton sets. We perform Find queries asking for a subset containing a given element, and Union operations which merge two subsets; see [14, 20].

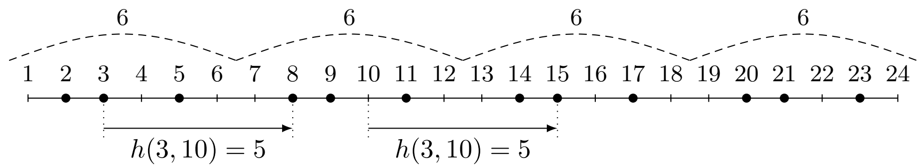

Figure 1: An example of a 6-cover $\mathbf{S}(6) = \{2,3,5,8,9,11,14,15,19,20,21,23\}$ (for $\mathbf{D} = \{2,3,5\}$), with the elements marked as black circles. For example, we have $h(3,10) = 5$, since $3+5,\ 10+5 \in \mathbf{S}(6)$.

Lemma 2.3 (Tarjan [20], La Poutré [14])**.** *A sequence of up to n* Union *and m* Find *operations on an n-element set can be executed on-line (a) in $\mathcal{O}(n + m \cdot \alpha(m,n))$, or (b) in $\mathcal{O}(m + n \log^* n)$ total time.*

3 Generic LCE algorithm for general ordered alphabets

We define t-short LCE queries by restricting the answer to at most t:

$$\mathrm{ShortLCE}_t(i,j) \;=\; \min(\mathrm{LCE}(i,j),\, t).$$

We define a t-block as a fragment of the input text w which starts in $\mathbf{S}(t)$ and has length t. If a position in $\mathbf{S}(t)$ lies near the end of w, we form a t-block from a suffix of w and enough dummy symbols to reach length t. We also introduce t-coarse LCE queries, which are LCE queries restricted to positions from $\mathbf{S}(t)$ returning the number of matching t-blocks:

$$\mathrm{CoarseLCE}_t(i,j) \;=\; \begin{cases} \lfloor \mathrm{LCE}(i,j)/t \rfloor & \text{if } i,j \in \mathbf{S}(t), \\ \perp & \text{otherwise.} \end{cases}$$

We now describe how to use ShortLCE and CoarseLCE queries for general LCE queries.

Algorithm 1: GenericLCE(i,j)

$\ell_1 \;=\; \mathrm{ShortLCE}_t(i,j)$
if $\ell_1 < t$ **then return** ℓ_1
$\Delta \;=\; h_t(i,j)$ $\qquad\qquad \triangleright\ i+\Delta, j+\Delta \in \mathbf{S}(t)$
$\ell_2 \;=\; t \cdot \mathrm{CoarseLCE}_t(i+\Delta, j+\Delta)$
$\ell_3 \;=\; \mathrm{ShortLCE}_t(i+\Delta+\ell_2, j+\Delta+\ell_2)$
return $\Delta + \ell_2 + \ell_3$

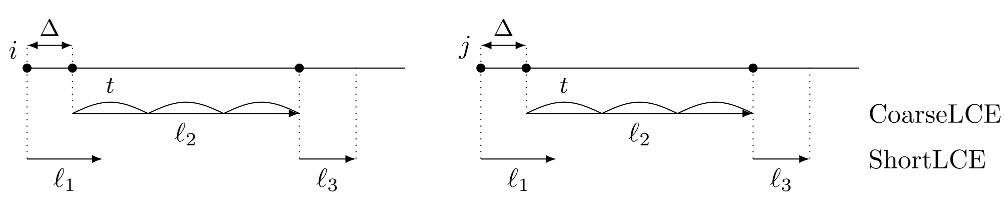

Figure 2: Illustration of Algorithm 1 for the case $\ell_1 \geq \Delta$.

Lemma 3.1. *If every sequence of q* $\mathrm{ShortLCE}_t$ *queries and* $\mathrm{CoarseLCE}_t$ *queries can be executed on-line in total time $T(n,q)$, then every sequence of q* LCE *queries can be executed on-line in total time $T(n,\mathcal{O}(q)) + \mathcal{O}(n+q)$.*

Proof. To calculate $\text{LCE}(i,j)$ we first check if $\text{LCE}(i,j) < t$ by calling $\text{ShortLCE}_t(i,j)$. If so, we are done. Otherwise, we can reduce computing $\text{LCE}(i,j)$ to computing $\text{LCE}(i+\Delta, j+\Delta)$ for any $\Delta \le t$. In particular, we can choose $\Delta = h_t(i,j)$, so that $i+\Delta, j+\Delta \in \mathbf{S}(t)$. Then we call $\text{CoarseLCE}_t(i+\Delta, j+\Delta)$ which gives us the value $\lfloor \frac{1}{t}(\text{LCE}(i,j)-\Delta)\rfloor$. Computing the exact value of $\text{LCE}(i,j)$ requires another ShortLCE_t query; see Algorithm 1. The whole process is illustrated in Figure 2. ☐

4 ShortLCE_t queries in $\mathcal{O}(\log t)$ amortized time

In this section we show how to implement fast on-line ShortLCE_t queries. We assume that $t = 2^k$ and set $t' = \Theta(\log t)$ to be a smaller power of two. The amortized running time is $\mathcal{O}(\log t + \sqrt{\log t}\log^* n)$, which in particular is $\mathcal{O}(\log t)$ for $t = \log^{\Omega(1)} n$. The key components are Union-Find structures and t'-covers. We start with a simpler (and slightly slower) algorithm without t'-covers.

4.1 ShortLCE_t queries in $\mathcal{O}(\log t \cdot \alpha(n))$ amortized time

Lemma 4.1. *A sequence of q $\text{ShortLCE}_{2^k}(i,j)$ queries can be executed on-line in total time $\mathcal{O}((q+n)k\cdot\alpha(n))$.*

Proof. We compute $\text{ShortLCE}_{2^k}(i,j)$ using a recursive procedure; see Algorithm 2. The procedure first checks if $w[i..i+2^k-1]$ is already known to be equal to $w[j..j+2^k-1]$ using a Union-Find structure. If so, we are done. Otherwise, if $k=0$, we simply compare $w[i]$ and $w[j]$. If $k>0$, we recursively calculate $\text{ShortLCE}_{2^{k-1}}(i,j)$ and, if the call returns 2^{k-1}, also $\text{ShortLCE}_{2^{k-1}}(i,j)$. Finally, if both calls return 2^{k-1}, we update the Union-Find structure to store that $w[i..i+2^k-1] = w[j..j+2^k-1]$.

To analyze the complexity of the procedure, we first observe that the total number of calls to Union is $\mathcal{O}(nk)$, because each such call discovers that $w[i..i+2^k-1] = w[j..j+2^k-1]$ (which was not known before). We will argue that the time taken by $\text{ShortLCE}_{2^k}(i,j)$ is proportional to the number of calls to Union plus $\mathcal{O}(k+1)$, which implies the lemma. In the analysis we omit the time necessary to actually execute all these calls, because we already know that this sums up to $\mathcal{O}(nk\cdot\alpha(n))$. For the sake of conciseness, $\#\mathsf{union}$ denotes the number of calls to Union triggered by the considered call to ShortLCE (including itself).

We inductively bound the number of recursive calls triggered by $\text{ShortLCE}_{2^k}(i,j)$:

$$\begin{array}{ll} 2k+1+2\#\mathsf{union} & \text{if } w[i..i+2^k-1] \neq w[j..j+2^k-1], \\ 1+2\#\mathsf{union} & \text{if } w[i..i+2^k-1] = w[j..j+2^k-1]. \end{array}$$

ShortLCE_1 terminates immediately, so this holds for $k=0$. For $k>0$ we have four cases.

1. $w[i..i+2^k-1]$ is already known to be equal to $w[j..j+2^k-1]$. Then we terminate immediately.

Algorithm 2: $\text{ShortLCE}_{2^k}(i,j)$: compute $\text{LCE}(i,j)$ up to length 2^k

```
if Find_k(i) = Find_k(j) then return 2^k
if k = 0 then
    if w[i] = w[j] then ℓ = 1 else ℓ = 0
else
    ℓ = ShortLCE_{2^{k-1}}(i, j)
    if ℓ = 2^{k-1} then
        ℓ = 2^{k-1} + ShortLCE_{2^{k-1}}(i + 2^{k-1}, j + 2^{k-1})
if ℓ = 2^k then Union_k(i, j)
return ℓ
```

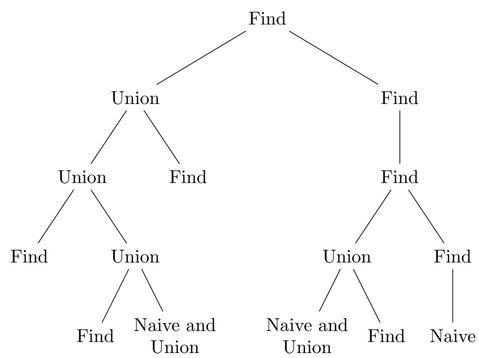

Figure 3: A recursion tree of $\text{SparseShortLCE}_{t,t'}(i,j)$ for some example parameters such that $t = 2^4 t'$. The calls terminating with Union, Find and naive tests (in a segment of size t') are shown as nodes in the figure. The naive tests are only at the bottom of the tree and they are accompanied by Unions (except the last one).

2. $w[i..i+2^{k-1}-1] \neq w[j..j+2^{k-1}-1]$. Then the number of recursive calls triggered by $\text{ShortLCE}_{2^{k-1}}(i,j)$ is $2k-1+2\#\mathsf{union}$ so the number of recursive calls triggered by $\text{ShortLCE}_{2^k}(i,j)$ is $2k+2\#\mathsf{union}$.
3. $w[i..i+2^{k-1}-1] = w[j..j+2^{k-1}-1]$ but $w[i+2^{k-1}..i+2^k-1] \neq w[j+2^{k-1}..j+2^k-1]$. The number of recursive calls triggered by $\text{ShortLCE}_{2^{k-1}}(i,j)$ and $\text{ShortLCE}_{2^{k-1}}(i+2^{k-1}, j+2^{k-1})$ is $1+2\#\mathsf{union}$ and $2k-1+2\#\mathsf{union}$, respectively. The total number of triggered recursive calls is hence $2k+1+2\#\mathsf{union}$.
4. $w[i..i+2^{k-1}-1] = w[j..j+2^{k-1}-1]$ and $w[i+2^{k-1}..i+2^k-1] = w[j+2^{k-1}..j+2^k-1]$. The number of recursive calls triggered by both $\text{ShortLCE}_{2^{k-1}}(i,j)$ and $\text{ShortLCE}_{2^{k-1}}(i+2^{k-1}, j+2^{k-1})$ is $1+2\#\mathsf{union}$. However, $w[i..i+2^{k-1}]$ was not known to be equal to $w[j..j+2^k-1]$, so we then execute $\text{Union}_k(i,j)$. Hence the total number of recursive calls is $1+2\#\mathsf{union}$ (rather than of $3+2\#\mathsf{union}$).

Consequently the total running time follows from Lemma 2.3(a). ◻

4.2 Faster ShortLCE_t queries

Assume $t = 2^k = \Omega(\log n)$. We show how to reduce the factor $\alpha(n)$ introducing a t'-cover, for $t' = 2^{k'}$. We define a sparse version of ShortLCE queries, which are ShortLCE queries restricted to positions from $\mathbf{S}(t')$:

$$\text{SparseShortLCE}_{t,t'}(i,j) \;=\; \begin{cases} \text{ShortLCE}_t(i,j) & \text{if } i,j \in \mathbf{S}(t') \\ \perp & \text{otherwise} \end{cases}$$

We slightly modify Algorithm 2 to obtain Algorithm 3, which computes $\min(\text{LCE}(i,j), 2^k)$ for positions $i,j \in \mathbf{S}(t')$.

Lemma 4.2. *A sequence of q $\text{SparseShortLCE}_{2^k,2^{k'}}$ queries can be executed on-line in total time $\mathcal{O}(q(k + 2^{k'}) + n\sqrt{2^{k'}} + \frac{nk}{\sqrt{2^{k'}}}\log^* n)$.*

Proof. The analysis is similar to the proof of Lemma 4.1. The total number of calls to Union is now only $\mathcal{O}(\frac{nk}{2^{k'/2}})$ because we always have that $i,j \in \mathbf{S}(2^{k'})$. Hence, excluding the cost of computing $\ell = \text{ShortLCE}_{2^{k'}}(i,j)$, the total time complexity is $\mathcal{O}(qk + \frac{nk}{2^{k'/2}}\log^* n)$ by the same reasoning as in Lemma 4.1, except that we use Lemma 2.3(b) instead of Lemma 2.3(a).

Now we analyze the cost of computing $\ell = \text{ShortLCE}_{2^{k'}}(i,j)$. First, observe that for every original call to $\text{SparseShortLCE}_{2^k,2^{k'}}(i,j)$ we have at most one such computation with $\ell < 2^{k'}$ (because it means that we have found a mismatch and no further recursive calls are necessary). On the other hand, if $\ell = 2^{k'}$, then we

Algorithm 3: SparseShortLCE$_{2^k,2^{k'}}(i,j)$: compute $\min(\text{LCE}(i,j),2^k)$ for $i,j \in \mathbf{S}(2^{k'})$

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if $\text{Find}_k(i) = \text{Find}_k(j)$ **then return** 2^k

if $k = k'$ **then**
  Compute naively $\ell = \text{ShortLCE}_{2^{k'}}(i,j)$
else
  $\ell = \text{SparseShortLCE}_{2^{k-1},2^{k'}}(i,j)$
  if $\ell = 2^{k-1}$ **then**
    $\ell = 2^{k-1} + \text{SparseShortLCE}_{2^{k-1},2^{k'}}(i+2^{k-1}, j+2^{k-1})$

if $\ell = 2^k$ **then** $\text{Union}_k(i,j)$

return ℓ

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call $\text{Union}_{k'}(i,j)$, which may happen at most $\frac{n}{2^{k'/2}}$ times. Therefore, the total complexity of all these naive computations is $\mathcal{O}(n2^{k'/2} + q \cdot 2^{k'})$. □

Algorithm 4: FasterShortLCE$_{2^k,2^{k'}}(i,j)$

Compute naively $\ell = \text{ShortLCE}_{2^{k'}}(i,j)$

if $\ell < 2^{k'}$ **then return** l

$\Delta = h_{2^{k'}}(i,j)$
$\ell = \Delta + \text{SparseShortLCE}_{2^k,2^{k'}}(i+\Delta, j+\Delta)$
return $\min(\ell, 2^k)$

The next lemma is a direct consequence of Lemma 4.2 and Algorithm 4 with $2^{k'} = \Theta(k)$.

Lemma 4.3. *A sequence of q ShortLCE_{2^k} queries can be executed on-line in total time $\mathcal{O}(qk + n\sqrt{k}\log^* n)$.*

5 CoarseLCE$_t$ queries

Let $t = \Omega(\log^2 n)$. Recall that we defined a t-block of w as a factor of size t starting in $\mathbf{S}(t)$. We want to show how to preprocess w in $\mathcal{O}(n \log\log n)$ time, so that any CoarseLCE$_t$ query can be answered in constant time. To this end we proceed as follows:

1. sort all t-blocks in lexicographic order and remove duplicates,
2. encode every t-block with its rank on the sorted list,
3. construct a new string code(w) of length $\mathcal{O}(n)$ over alphabet $[1..n]$, such that any CoarseLCE$_t$ query can be reduced to an LCE query on code(w),
4. preprocess code(w) for LCE queries.

Lemma 5.1. *For $t = \Omega(\log^2 n)$ we can lexicographically sort all t-blocks of w in $\mathcal{O}(n \log t)$ time.*

Proof. Two t-blocks can be lexicographically compared with a ShortLCE$_t$ query. We have $\mathcal{O}(\frac{n}{\sqrt{t}})$ such blocks, hence one of the classical sorting algorithms they can be all sorted using $\mathcal{O}(\frac{n}{\sqrt{t}} \log n) = \mathcal{O}(n)$ queries. By Lemma 4.3, the total time to execute these queries and sort all t-blocks is therefore $\mathcal{O}(n \log t)$. □

We can use the lexicographic order of t-blocks to assign ranks to all t-blocks. Then we reduce CoarseLCE queries to LCE queries in a word code(w) over an integer alphabet; see Figure 4.

Lemma 5.2. *For $t = \Omega(\log^2 n)$ we can preprocess w in $\mathcal{O}(n \log t)$ time, so that any* $\mathrm{CoarseLCE}_t$ *query can be answered in constant time.*

Proof. Using Lemma 5.1 we assign a number to each t-block, so that two t-blocks are identical if and only if their numbers are equal. The number assigned to the block starting at position $p \in \mathbf{S}(t)$ is denoted $\mathrm{rank}(p)$. These numbers are ranks on a sorted list of length $|\mathbf{S}(t)|$, so $\mathrm{rank}(p) \in [1..|\mathbf{S}(t)|]$. Then we construct a new string $\mathrm{code}(w)$ as follows. Let

$$\{\, i_1, i_2, \ldots i_k \,\} \;=\; [1, t] \cap \mathbf{S}(t)$$

and z_s be the word obtained from w by concatenating the numbers assigned to all t-blocks starting at positions $i_s, i_s + t, i_s + 2t, i_s + 3t, \ldots$:

$$z_s = \mathrm{rank}(i_s)\mathrm{rank}(i_s + t)\mathrm{rank}(i_s + 2t)\mathrm{rank}(i_s + 3t)\ldots.$$

Finally, we introduce k new distinct letters $\#_1, \#_2, \ldots, \#_s$ and construct $\mathrm{code}(w)$:

$$\mathit{code}(w) \;=\; z_1 \cdot \#_1 \cdot z_2 \cdot \#_2 \cdot z_3 \cdot \#_3 \cdots z_k \cdot \#_k.$$

Next, $\mathrm{code}(w)$ is preprocessed to answer LCE queries in constant time. A $\mathrm{CoarseLCE}_t(p, q)$ query for positions $p, q \in \mathbf{S}(t)$ is answered by first computing positions p', q' corresponding to p, q in $\mathrm{code}(w)$. Formally, if $p = i_s \bmod t$, then $p' = |z_1\#_1 z_2\#_2 \ldots z_{s-1}\#_{s-1}| + \frac{p - i_s}{t} + 1$; q' is computed similarly. Then an $\mathrm{LCE}(p', q')$ query on $\mathrm{code}(w)$ returns $\mathrm{CoarseLCE}_t(p, q)$. The positions p' and q' can be computed in constant time, so the total query time is constant. Preprocessing $\mathrm{code}(w)$ requires constructing its suffix array, which takes linear time for integer alphabets of polynomial size, and preprocessing it for range minimum queries, which also takes linear time. Hence the total preprocessing time is $\mathcal{O}(n \log t)$. ☐

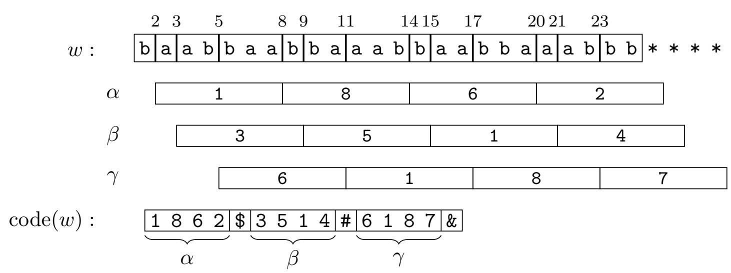

Figure 4: 6-blocks of w are lexicographically sorted (using $\mathrm{ShortLCE}_t$) and ranked. Then $\mathrm{CoarseLCE}_6(2, 11)$ in w is reduced to $\mathrm{LCE}(1, 12)$ in $\mathrm{code}(w)$.

Theorem 5.3. *A sequence of $\mathcal{O}(n)$ LCE queries for a string over a general ordered alphabet can be executed on-line in total time $\mathcal{O}(n \log \log n)$ making only $\mathcal{O}(n)$ symbol comparisons.*

Proof. We set $t = \Theta(\log^2 n)$ and reduce each LCE query to constant number of $\mathrm{CoarseLCE}_t$ queries and $\mathrm{ShortLCE}_t$ queries as described in Lemma 3.1. Thus together with Lemma 4.3 and Lemma 5.2 we obtain that any sequence of q LCE queries for a string over a general ordered alphabet can be realized in $\mathcal{O}(n \log \log n)$ time. However, the total number of symbol comparisons used by the algorithm might be $\Omega(n \log \log n)$. This can be decreased to $\mathcal{O}(n)$ with yet another Union-Find data structure, where we maintain sets of positions already known to store the same letter. This is essentially the idea used in Lemma 7 of [11]. ☐

6 Faster solution for sublinear number of queries

The algorithm presented in the previous section is not efficient when the number of queries q is significantly smaller than the length of the string n. In this section we show that this can be avoided, and we present an $\mathcal{O}(q \log\log n + n \log^* n)$-time algorithm. This requires some nontrivial changes in our approach. In particular, we need a stronger notion of t-covers, which form a *monotone family.*

$\mathbf{S}(4^0), \mathbf{S}(4^1), \mathbf{S}(4^2), \ldots \subseteq [1, n]$ is a monotone family of covers if the following conditions hold for every k:

1. $\mathbf{S}(4^k)$ is a 4^k-cover (except that h_{4^k} is computable in $\mathcal{O}(k)$ instead of constant time).
2. $\mathbf{S}(4^{k+1}) \subseteq \mathbf{S}(4^k)$.
3. For any $i, j \in \mathbf{S}(4^k)$ we have that $h_{4^{k+1}}(i, j) \in \{0, 4^k, 2 \cdot 4^k\}$, and furthermore for such arguments $h_{4^{k+1}}$ can be evaluated in constant time.
4. $|\mathbf{S}(4^k)| \leq (\frac{3}{4})^k n$.

The existence of such a family is not completely trivial, in particular plugging in the standard construction of $\mathbf{S}(4^k)$ from Lemma 2.1 does not guarantee that $\mathbf{S}(4^{k+1}) \subseteq \mathbf{S}(4^k)$. The following lemma, implicitly shown in [7], provides an efficient construction.

Lemma 6.1 (Gawrychowski et al. [7], Section 4.1)**.** *Let $\mathbf{S}(4^k)$ be the set of non-negative integers $i \in [1, n]$ such that none of the k least significant digits of the base-4 representation of i is zero. Then $\mathbf{S}(4^0), \mathbf{S}(4^1), \mathbf{S}(4^2), \ldots$ is a monotone family of covers, which can be constructed in $\mathcal{O}(n)$ total time.*

6.1 $\mathrm{ShortLCE}_t$ queries with monotone family of covers

Similarly as in the proof of Lemma 4.3, we reduce ShortLCE queries to SparseShortLCE queries. However, now we slightly change the definition of SparseShortLCE queries so that there is only one parameter as follows:

$$\mathrm{SparseShortLCE}_t(i, j) = \begin{cases} \mathrm{ShortLCE}_t(i, j) & \text{if } i, j \in \mathbf{S}(t) \\ \perp & \text{otherwise} \end{cases}$$

Lemma 6.2. *For a non-negative integer K consider a sequence of q $\mathrm{SparseShortLCE}_{4^k}$ queries with $k \leq K$. These queries can be answered online in $\mathcal{O}(q + q'K + n \log^* n)$ total time where q' is the number of $\mathrm{SparseShortLCE}_{4^k}$ queries in the sequence* not *returning 4^k.*

Proof. We maintain a separate Union-Find structure for every $k \in [0, K]$. We check if $\mathrm{Find}_k(i) = \mathrm{Find}_k(j)$ and if so, return 4^k. Otherwise, we calculate the answer with at most four calls to $\mathrm{SparseShortLCE}_{4^{k-1}}$. This is possible because $\mathbf{S}(4^{k-1}) \subseteq \mathbf{S}(4^k)$ and $\mathbf{S}(4^{k-1})$ is 4^{k-1}-periodic. Finally, we call $\mathrm{Union}_k(i, j)$ if the answer is 4^k; see Algorithm 5.

We again analyze the number of recursive calls to $\mathrm{SparseShortLCE}_{4^k}$ counting Union operations. The total number of unions is $\mathcal{O}(n)$. A call to $\mathrm{SparseShortLCE}_{4^k}$ takes amortized constant time if $\mathrm{LCE}(i, j) = 4^k$, and otherwise amortized $\mathcal{O}(k + 1)$ time. Hence by Lemma 2.3(a) the total time is as claimed. ◻

Lemma 6.3. *A sequence of q $\mathrm{ShortLCE}_{4^k}$ queries can be answered online in total time $\mathcal{O}(qk + n \log^* n)$.*

Proof. We calculate $\mathrm{ShortLCE}_{4^k}(i, j)$ using $\mathcal{O}(k)$ SparseShortLCE queries; see Algorithm 6. We iterate through $k' = 0, 1, \ldots, k - 1$ maintaining Δ such that $0 \leq \Delta \leq \mathrm{LCE}(i, j)$ and $i + \Delta, j + \Delta \in \mathbf{S}(4^{k'})$. Before incrementing k', we keep increasing Δ by $4^{k'}$ until $i + \Delta, j + \Delta \in \mathbf{S}(4^{k'})$ or $\Delta > \mathrm{LCE}(i, j)$. The latter condition is checked by calling $\mathrm{SparseShortLCE}_{4^{k'}}(i + \Delta, j + \Delta)$ and terminating if it returns less than $4^{k'}$. The while loop iterates at most twice, because $h_{4^{k'+1}} \in \{0, 4^{k'}, 2 \cdot 4^{k'}\}$. Eventually, we either terminate having found the answer, or we can obtain it with a single call to $\mathrm{SparseShortLCE}_{4^k}(i + \Delta, j + \Delta)$.

Let us analyze the total time complexity. Each call to $\mathrm{ShortLCE}_{4^k}$ performs up to k $\mathrm{SparseShortLCE}_{4^{k'}}$ queries, but we terminate as soon as we obtain an answer other than $4^{k'}$. Hence, by Lemma 6.2, the total time used by all these calls is only $\mathcal{O}(qk + n \log^* n)$. Note that checking whether $i + \Delta$ and $j + \Delta$ belong to $\mathbf{S}(4^{k'+1})$ takes constant time, because we know that these indices are in $\mathbf{S}(4^{k'})$. ◻

Algorithm 5: SparseShortLCE$_{4^k}(i,j)$: compute $\min(\mathrm{LCE}(i,j), 4^k)$ for $i,j \in \mathbf{S}(4^k)$

```
if Find_k(i) = Find_k(j) then return 4^k
if k = 0 then
    if w[i] = w[j] then ℓ = 1 else ℓ = 0
else
    ℓ = 0
    for p = 0 to 3 do
        ℓ = ℓ + SparseShortLCE_{4^{k-1}}(i + p·4^{k-1}, j + p·4^{k-1})
        if ℓ < (p + 1)·4^{k-1} then break
if ℓ = 4^k then Union_k(i, j)
return ℓ
```

Algorithm 6: ShortLCE$_{4^k}(i,j)$

```
ℓ = Δ = 0
for k' = 0 to k − 1 do
    while i + Δ ∉ S(4^{k'+1}) or j + Δ ∉ S(4^{k'+1}) do
        ℓ = ℓ + SparseShortLCE_{4^{k'}}(i + Δ, j + Δ)        ▷ i + Δ, j + Δ ∈ S(4^{k'})
        Δ = Δ + 4^{k'}
        if ℓ < Δ then return min(4^k, ℓ)
return min(4^k, Δ + SparseShortLCE_{4^k}(i + Δ, j + Δ))       ▷ i + Δ, j + Δ ∈ S(4^k)
```

6.2 Final algorithm

We first modify the implementation details for CoarseLCE to reduce the preprocessing time.

Lemma 6.4. *For $t = \Omega(\log^6 n)$ we can preprocess a string of length n in $\mathcal{O}(n \log^* n)$ time, so that each* CoarseLCE$_t$ *query can be answered in constant time.*

Proof. We set $k = \lceil \frac{1}{2} \log t \rceil$ and lexicographically sort all 4^k-blocks using ShortLCE$_{4^k}$ queries. The number of blocks is at most $(\frac{3}{4})^k n \le \frac{n}{t^{0.5 \log 0.75}} \le \frac{n}{t^{0.2}}$. By Lemma 6.3 the sorting time is:

$$\mathcal{O}\left(\frac{n}{t^{0.2}} \log n \log t + n \log^* n\right) = \mathcal{O}\left(n \frac{\log n \log \log n}{\log^{1.2} n} + n \log^* n\right) = \mathcal{O}(n \log^* n).$$

Then we proceed as in the proof of Lemma 5.2. ☐

By combining Lemma 6.4 and Lemma 6.3, we obtain the final theorem.

k'	SparseShortLCE calls
0	SparseShortLCE$_{4^0}(1013\mathbf{0}_4, 00101_4) \to \Delta = 00001_4$
1	SparseShortLCE$_{4^1}(10131_4, 001\mathbf{0}2_4) \to \Delta = 00011_4$
1	SparseShortLCE$_{4^1}(102\mathbf{0}1_4, 00112_4) \to \Delta = 00021_4$
3	SparseShortLCE$_{4^3}(1\mathbf{0}211_4, 0\mathbf{0}122_4) \to \Delta = 01021_4$
return call	SparseShortLCE$_{4^4}(11211_4, 01122_4)$

Figure 5: An execution of ShortLCE$_{4^4}(i = (10130)_4, j = (00101)_4)$ (assuming $\mathrm{LCE}(i,j) > 4^4$). The numbers are given in base-4 representation. Note that there is no SparseShortLCE$_{4^2}$ call.

Theorem 6.5. *A sequence of q LCE queries for a string over a general ordered alphabet can be executed on-line in total time $\mathcal{O}(q \log\log n + n \log^* n)$ making $\mathcal{O}(q + n)$ symbol comparisons.*

7 Final remarks

We gave an $\mathcal{O}(n \log\log n)$-time algorithm for answering on-line $\mathcal{O}(n)$ LCE queries for general ordered alphabet. It is known (see [13]) that the runs of the string can be computed in $\mathcal{O}(T(n))$ time, where $T(n)$ is the time to execute on-line $\mathcal{O}(n)$ LCE queries. Hence our algorithm implies the following result:

Corollary 7.1. *The runs of a string over general ordered alphabet can be computed in $\mathcal{O}(n \log\log n)$ time.*

Our algorithm is a major step towards a positive answer for a question posed by Kosolobov [13], who asked if $\mathcal{O}(n)$ time algorithm is possible.

It is also natural to consider general unordered alphabets, that is, strings where the only allowed operation is checking equality of two characters.

Theorem 7.2. *A sequence of q LCE queries for a string over a general unordered alphabet can be executed in $\mathcal{O}(q \log n + n \log^* n)$ time making $O(n + q)$ symbol equality-tests.*

Proof. We can use the faster ShortLCE_{4^k} algorithm described in Section 6.1 with $k = \lceil \frac{1}{2} \log n \rceil$. Observe that in this approach we did not use the order of the characters, and thus it still works for unordered alphabets. $\square$

Note that for unordered alphabets the reduction by Kosolobov [13] (see also [2]) from computing runs to LCE queries no longer works. Actually, deciding whether a given string is square-free already requires $\Omega(n \log n)$ comparisons, as shown by Main and Lorentz [18]. On the other hand for $\mathcal{O}(n)$ LCE queries $\mathcal{O}(n)$ equality tests always suffice.

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