A Regularty theory for Harmonic maps.

The goal:

→ A bounded minimizing map $u: M \to N$ is regular (in the interior) except for a closet set S of Hausdorf dimension at most $\underline{n-3}$

→ S is a discrete set of points for $n=3$.

→ When N has nonpositive sectional curvature $\Rightarrow S=\emptyset$, u is smooth

<u>To prove</u> • Find comparison maps which satisfy the constraints.

→ we need scaling inequality <u>Prop. 24</u>

<u>Motivation</u>

1) Suppose $u \in L^2_{1,loc}(\mathbb{R}^n, N)$ is a map s.t. $\frac{\partial u}{\partial r} = 0$ a.e

$\Rightarrow \exists$ $w: S^{n-1} \to N$ s.t. $u(x) = w\left(\frac{x}{|x|}\right)$. And u is harmonic iff w is harmonic. $E(u \mid B_\sigma(0)) = (n-2)^{-1} \sigma^{n-2} E(w)$.

2) If u_0 with an isolated singularity at $\underline{0}$ — a tangent map, and if it's $\hat{E}$-min. on comp. $K \subset \mathbb{R}^n$ is minimizing tangent map.

3) Let $u \in L^2_1(M,N)$, $u(x) \in N_0$, $N_0 \subseteq N$, and singular set

$S_u = M - \{x \in O : u(x) = const\}$. Define a density of u at x.

$$\rho_u(x) = \liminf_{r \to 0} \left(E_{B_r}(u) r^{2-n}\right)$$

$S = \{x \in M : \rho_u(x) > 0\} \Rightarrow N$ is simply connected, $\Rightarrow x \in M$, $\rho_u(x) \leq C(g,n)$.
(by Hardt, Kinderlehrer and Lin).

Notation

$$\|u\|_{1,2}^2 = E(u) + \int_M \sum_i (u^i(x))^2 \, dV$$

$$\tilde{E}(u) = E(u) + V(u) \quad , \quad V(u) = \int_M v(u)$$

$$v(u) = \left[\sum_i \sum_\alpha \gamma_i^\alpha (x, u(x)) \frac{\partial u^i}{\partial x^\alpha}(x) + \Gamma(x, u(x))\right] dV$$

$\gamma \in C^r(M \times \mathcal{O}, T^*M \otimes \mathbb{R}^k)$, $\Gamma \in C^r(M \times \mathcal{O}, \mathbb{R})$

- $\tilde{E}$-min. map: a map $u \in L_1^2(M,N)$ s.t. $\tilde{E}(u) \leq \tilde{E}(w)$ for $\forall w \in L_1^2(M,N)$ $(u-w) \in L_{1,0}^2$

  Euler-Lagrange equation:

$$\Delta_M u - A(du, du) + \sum_{i,\alpha} B_{i,\alpha}(x, u(x)) \frac{\partial u^i}{\partial x^\alpha} + C(x, u(x)) = 0. \quad (2.1)$$

$$\Delta_M u - A(du, du) = 0.$$

- For $\Lambda > 0$ lower order terms satisfying

$$\sum_{\alpha,\beta,\tau} \left| \frac{\partial}{\partial x^\tau} g_{\alpha\beta}(x) \right| + |\gamma(x,u)| + |d_u \gamma(x,u)| + |\Gamma(x,u)|^{1/2} + |d_u \Gamma(x,u)|^{1/2} \leq \Lambda \quad (2.2)$$

  $\tilde{E} \in \mathcal{F}_\Lambda$, $u(x) \in N_0$ a.e, we say $u \in \mathcal{H}_\Lambda$.

- $$\tilde{E}^{p,\sigma}(w) := \int_{B_1} \left(|dw|_{g_\sigma}^2 + \sigma (dw \cdot \gamma(y,w)) + \sigma^2 \Gamma(y,w) \right) g_\sigma^{1/2} \, dy \quad (2.3)$$

$$= \sigma^{2-n} \tilde{E}_{B_\sigma(p)}(u) \qquad - \tilde{E}^{p,\sigma} \text{ functional on } B_1$$

  $w(y) = u(\sigma y)$, $g_\sigma(y) = g(\sigma y)$

$\rightarrow \Lambda > 0$ $\exists \sigma_0 > 0$ s.t. for $0 < \sigma \leq \sigma_0$ $p \in M$ u is $\tilde{E}$-min.

$\Rightarrow$ $w(y) = u(\exp_p \sigma y)$ is $\tilde{E}^{p,\sigma}$-minimizing, $\tilde{E}^{p,\sigma} \in \mathcal{F}_\Lambda$

- Let E be the energy on B_1 (Eucl. metric). E_σ, $\tilde{E}$ energies over B_σ

$0 < \sigma \leq 1$

$$|E_\alpha(u) - \tilde{E}_\sigma(u)| \leq C\Lambda(\sigma E_\alpha(u) + \sigma^{n/2} E_\alpha(u)^{1/2} + \Lambda \sigma^n) \leq$$

$$\leq \frac{3}{2} C\Lambda (\sigma E_\alpha(u) + \sigma^{n-1})$$

$\Rightarrow \Lambda\sigma \leq 1$, for $\sigma \in (0,1]$

$$\tilde{E}_\sigma(u) \leq (1 + \tilde{C}\Lambda\sigma) E_\alpha(u) + \bar{C}\Lambda\sigma^{n-1} \quad (*)$$

$$E_\alpha(u) \leq (1 + \bar{C}\Lambda\sigma) \tilde{E}_\sigma(u) + \bar{C}\Lambda\sigma^{n-1} \quad (**)$$

Lemma 2.3 Λ suff. small, $E \in \mathcal{F}_\Lambda \Rightarrow \exists\, C(n) > 0$ s.t. for $\sigma \in (0,1]$

$$E_\alpha(u) \leq (1 + C\Lambda\sigma) E_\alpha(w) + C\Lambda\sigma^{n-1}$$

$\forall w \in L^2_1(B_1, N)$. $w = u$ on $B_1 \sim B_\sigma$.

Notation
$$E^x_\sigma(u) = \int_{B_\sigma(x)} |du|^2(y)\, dy$$

<u>Prop 2.4.</u> Let $u \in \mathcal{H}_\Lambda$ for Λ suff. small. Then

$$\sigma^{2-n} E^x_\sigma(u) \leq c\left[\rho^{2-n} E^x_\rho(u) + \Lambda\rho\right]$$

$x \in B_{1/2}$, $0 < \sigma \leq \rho \leq \frac{1}{2}$

<u>Proof</u> $\dashv$ B_1. We have $\int_{|x|=\sigma} |du|^2 d\xi < \infty$, $\forall \sigma \in (0,1]$

Comparison map

$$v_\sigma(x) = u(x), \quad |x| \geq \sigma$$
$$v_\sigma(x) = u(\sigma x/|x|), \quad |x| \leq \sigma.$$

$$|du|^2 = |d_\xi u|^2 + \left|\frac{\partial u}{\partial r}\right|^2$$

$$E_\alpha(v_\sigma) = (n-2)^{-1}\sigma \int_{|x|=\sigma} |d_\xi u|^2 d\xi = (n-2)^{-1}\sigma\left[\frac{d}{d\sigma} E_\alpha(u) - \int_{|x|=\sigma} \left|\frac{\partial u}{\partial r}\right|^2 d\xi\right]$$

From L.2.3:

$$E_\alpha(u) \leq (1 + \bar{c}\sigma) E_\alpha(v_\alpha) + \bar{c}\sigma^{n-1}$$
$$\leq (n-2)^{-1}\sigma(1 + \bar{c}\sigma)\left[\frac{d}{d\sigma} E_\alpha(u) - \int_{|x|=\sigma}\left|\frac{\partial u}{\partial r}\right|^2 d\xi\right] + \bar{c}\sigma^{n-1}$$

$$0 \leq \sigma^{2-n}\int_{|x|=\sigma}\left|\frac{\partial u}{\partial r}\right|^2 d\xi \leq \frac{d}{d\sigma}\left[(1+\bar{c}\sigma)^{n-2}\sigma^{2-n}E_\alpha(u)\right] + \bar{\bar{c}} \qquad (2.4)$$

By integration from σ to ρ: $\quad \bar{c} = c\Lambda \leq \frac{1}{2}$

$$(1+\bar{c}\sigma)^{n-2}\sigma^{2-n}E_\alpha(u) \leq (1+\bar{c}\rho)^{n-2}\rho^{2-n}E_\rho(u) + \bar{\bar{c}}(\rho - \sigma) \qquad (2.5)$$

$\blacksquare$

Note that, $u_\lambda(x) = u(\lambda x)$ for $\lambda \in (0,1]$.

$u_\lambda \in \mathcal{H}_{\lambda\Lambda}$, $u \in \mathcal{H}_\Lambda$

$$E_1(u_\lambda) = \lambda^{2-n} E_\lambda(u) \qquad (2.6)$$

Lemma 2.5 $\exists\, \lambda(i) \to 0$, $\lambda(i) \in (0,1]$ s.t. $u_{\lambda(i)} \xrightarrow{w} u_0$ in $L^2_1(B_1, N)$

u_0 is a harmonic map $\quad \frac{\partial u_0}{\partial r} = 0$ a.e. in B_1

Proof we know that $E_1(u_\lambda)$ is bdd $\quad \lambda \in (0,1]$

$\Rightarrow \quad u_{\lambda(i)} \xrightarrow{w} u_0 \in L^2_1(B_1, N)$

$u_{\lambda(i)} \in \mathcal{H}_{\lambda(i)\Lambda}$ it satisfy Euler eq $\Rightarrow u_0$ satisfies E-L eq, is harmonic

From (2.5) $\exists L_0$: $\quad L_0 = \lim_{\sigma \to 0} \sigma^{2-n} E_\sigma(u) = \lim_{\sigma\to 0} E_1(u_\sigma) \qquad (2.7)$

Integrate (2.4) from 0 to λ.

$$0 \le \int_{B_\lambda} r^{2-n} \left|\frac{\partial u}{\partial r}\right|^2 dx \le \Big[(1+\bar{c}\lambda)^{n-2} \underbrace{\lambda^{2-n} E_\lambda(u) - L_0}_{\to 0}\Big] + \bar{c}\lambda$$

$$\int_{B_1} r^{2-n} \left|\frac{\partial u_\lambda}{\partial r}\right|^2 dx = \int_{B_\lambda} r^{2-n} \left|\frac{\partial u}{\partial r}\right|^2 dx$$

$$\Rightarrow \quad \lim_{\lambda \to 0} \int_{B_1} r^{2-n} \left|\frac{\partial u_\lambda}{\partial r}\right|^2 dx = 0 \quad \Rightarrow \quad \frac{\partial u_0}{\partial r} = 0 \text{ a.e.} \qquad \blacksquare$$

Regularity Estimate 2.6 $\exists\, \bar{\varepsilon}(N_0, n) > 0$ s.t. if $u \in \mathcal{H}_\Lambda$, $\Lambda \le \bar{\varepsilon}$, $E_1(u) \le \bar{\varepsilon}$ then u is Hölder continuous on $B_{1/2}$, $|u(x) - u(y)| \le c|x-y|^\alpha$ $x, y \in B_{1/2}$

$c, \alpha(n) > 0$ depend only on n, N_0.

Corollary 2.7 If $u \in L^2_1(B_1, N)$ is in $\mathcal{H}_\Lambda$, and $\mathcal{S}$ is singular set of u $\Rightarrow \mathcal{H}^{n-2}(\mathcal{S} \cap B_{1/2}) = 0$. If $u \in L^2_1(M,N)$ is $\tilde{E}$-min., then $\mathcal{H}^{n-2}(\mathcal{S} \cap \operatorname{int} M) = 0$.

Proof $x \in \mathcal{S} \cap B_{1/2} \quad u_{x,\lambda}(y) = u(\exp_x \lambda y) \Rightarrow u_{x,\lambda} \in \mathcal{H}_{\lambda\Lambda}$

By regul. est.: $\quad \bar{\varepsilon} \le E_1(u_{x,\lambda}) = \lambda^{2-n} \int_{B_\lambda(x)} |du|^2 dx \quad, \forall x \in \mathcal{S} \cap B_{1/2} \quad (2.8)$

$\lambda\Lambda \le \bar{\varepsilon}$.

$\delta \in (0, \bar{\varepsilon}/\Lambda)$, $\{B_\delta(x_1), \dots, B_\delta(x_\ell)\}$ max. family of $\ell = \ell(\delta)$ disjoint balls

$\Rightarrow \quad \mathcal{S} \cap B_{1/2} \subseteq \bigcup_{j=1}^{\ell} B_{2\delta}(x_i)$

By (2.8) and summing

$$\ell \delta^{n-2} \leq \bar{\varepsilon}^{-1} \int_{\bigcup_i B(x_i)} |du|^2 dx \leq \bar{\varepsilon}^{-1} E(u) \qquad (2.9)$$

$$\mathcal{H}^{n-2}(S \cap B_{1/2}) \leq c E(u) \quad , \mathcal{H}^n(\bigcup_i B_\delta(x_i)) \leq \bar{c} \delta^2 E(u)$$

by DCT $\lim_{\delta \to 0} \int_{\bigcup_i B_\delta(x_i)} |du|^2 dx = 0$ - using this in (2.9)

$$\Rightarrow \quad \mathcal{H}^{n-2}(S \cap B_{1/2}) = 0 \qquad \blacksquare$$

<u>Poincare inequality</u>

$$\int_{B_1} |u - u^*|^2 dx \leq c_1 E_1(u) \leq c_1 \bar{\varepsilon}$$

$$dist(u^*, N) \leq c_2 (\bar{\varepsilon})^{1/2}$$

$$W_\sigma(u) = \int_{B_\sigma} |u - u^*|^2 dx$$

$C_\sigma = B_\sigma^{n-1} \times [-\sigma, \sigma]$ - the cylinder $h = d = 2\sigma$.

<u>Lemma 4.1</u> $u \in L_1^2(\partial C_\sigma^n, N)$: $u(x, -\sigma) = u_1(x)$, $u(x, \sigma) = u_2(x)$ for $x \in B_\sigma^{n-1}$

$u_1, u_2 \in L_1^2(B_\sigma^{n-1}, N)$. Also $u(x,t) = u'(x)$ for $x \in S_\sigma^{n-2} \times [-\sigma, \sigma]$.

We have $u^1 = u^2 = u'$ on $\partial B_\sigma^{n-1} = S_\sigma^{n-2}$, $u' \in L_1^2(S_\sigma^{n-2}, N)$

$$\Rightarrow \quad \exists\, \bar{u} \in L_1^2(C_\sigma^n, N) : \quad \bar{u} = u \quad \text{on } \partial C_\sigma^n$$

$$E(\bar{u}) \leq c\sigma (E_\sigma(u_1) + E_\sigma(u_2) + \sigma E(u'))$$

$$W(\bar{u}) \leq c\sigma (W_\sigma(u_1) + W_\sigma(u_2) + \sigma W(u'))$$

<u>Proof</u>: Assume $\sigma = 1$

$\exists$ bi-Lip. homeom. $f : \partial B_1^n \to \partial C_1^n$

extends to $\bar{f} : B_1^n \to C_1^n \qquad \bar{f}(x) = |x| f(\frac{x}{|x|})$

$$\Pi : B_1^n \sim \{0\} \to \partial B_1^n \qquad \Pi(x) = \frac{x}{|x|}$$

$$\hat{\Pi} : C_1^n \sim \{(0,0)\} \to \partial C_1^n \qquad \hat{\Pi} = f \circ \Pi \circ \bar{f}^{-1}$$

$$\bar{u} = u \circ \hat{\Pi}$$

Prop 2.4 $$E(\bar{u} \circ \bar{f}) \leq (n-2)^{-1} E(u \circ f)$$

Due to Lipschitz equivalence

$$E(\bar{u}) \leq K E(\bar{u} \circ \bar{f}) \qquad E(u \circ f) \leq K E(u)$$

$\blacksquare$

<u>Lemma 4.2</u> If $u \in L_1^2(S_\sigma^1, N)$ $\quad E(u) W(u) \leq \delta_1^2$ for $\delta_1 = \delta_1(N_0)$

$\Rightarrow \exists\ \bar{u} \in L_1^2(B_\sigma^2, N)$ $\quad \bar{u}|_{\partial B_\sigma^2} = u$ $\quad (n=2)$

$$E_\sigma(\bar{u}) \leq c_1 (E(u) W(u))^{1/2}, \quad W_\sigma(\bar{u}) \leq c_1 \sigma W(u).$$

<u>Proof</u> $\sigma = 1$, $\delta^2 = E(u) W(u)$ $\quad \delta^2 \leq \delta_1^2$ $\quad S^1$ param. by $\theta \in [0, 2\pi)$

$$\Rightarrow \quad |u(\theta) - u^*|^2 \leq 2 \int_0^{2\pi} |u(\theta) - u^*| \, |u'(\theta)| \, d\theta \leq 2\delta$$

δ_1 is small $\quad \bar{u}$ lie in convex ball $\quad$ by Hildebrandt, Kaul

$$\Rightarrow \quad \|u - u^*\|_\infty \leq c_2 \delta^{1/2}$$

E-L eq

$$\Delta \bar{u} = A_{\bar{u}}(d\bar{u}, d\bar{u})$$

$$\frac{1}{2} \Delta |\bar{u} - u^*|^2 - |d\bar{u}|^2 = \langle \bar{u} - u^*, A_{\bar{u}}(d\bar{u}, d\bar{u}) \rangle \geq -\|\bar{u} - u^*\|_\infty \|A\|_\infty |d\bar{u}|^2$$

we have $$\Delta |\bar{u} - u^*|^2 \geq 0$$

by mean value ineq $$W(\bar{u}) \leq \frac{1}{2} W(u)$$

$v : B_1^2 \to R^k$

$$\Delta v = 0$$
$$v|_{\partial B_1} = u$$

$\Pi : \mathcal{D} \to N$

$$E(v) \leq c \delta^{1/2}$$

$\bar{u}$ min. map $$E(\bar{u}) \leq E(\Pi \circ v) \leq c_3 E(v) \leq c_4 \delta^{1/2}$$ $\blacksquare$

Lemma 4.3 For $n \geq 2$ $\exists\, \delta = \delta(n, N_0)$, $q = q(n)$ s.t. if $\varepsilon \in (0,1)$

$u \in L_1^2(\partial B_\sigma^n, N_0)$ satisfies $\sigma^{4-2n} E(u)\, W(u) \leq \delta^2 \varepsilon^q$

$\Rightarrow$ $\exists\, \bar{u} \in L_1^2(B_\sigma^n, N)$, $\bar{u}|_{\partial B_\sigma^n} = u$ s.t.

$$E(\bar{u}) \leq C_5(\varepsilon \sigma E(u) + \varepsilon^{-q} \sigma^{-1} W(u))$$

$$W(\bar{u}) \leq c_5 \varepsilon^{-q} \sigma W(u).$$

Lemma 4.4 Let $\sigma \in (0; \frac{1}{2})$, $A_\sigma = S^{n-1} \times [-\sigma, \sigma]$, Assume L 4.3 is true $n-1$

$v \in L_1^2(S^{n-1}, N)$, $E(v)\, W(v) \leq \sigma^{2n-4} (\delta')^2$ where $\delta' = \delta'(n-1, N_0)$

$\Rightarrow$ $\exists\, \alpha = \alpha(n) < 1$, $K = K(n)$, $\bar{v} \in L_1^2(A_\sigma, N)$

$\bar{v}|_{S^{n-1} \times \{-\sigma\}} = v'$, $\bar{v}|_{S^{n-1} \times \{\sigma\}} = v$ $\quad v' \in L_1^2(S^{n-1}, N)$ s.t

$$E(\bar{v}) \leq K \sigma E(v) + K \sigma^{-1} W(v)$$

$$W(\bar{v}) \leq K \sigma W(v)$$

$$E(v') \leq \sigma E(v) + K \sigma^{-2} W(v)$$

$$W(v') \leq K W(v)$$

Proposition 4.5 Given $B > 0$ $\exists\, \varepsilon_0 = \varepsilon_0(n, N_0, B)$ s.t. if $u \in \mathcal{H}_\Lambda$

$\Lambda \leq \varepsilon_0$, $E_1(u) \leq B$ $\quad W_1(u) \leq \varepsilon_0$

$\Rightarrow$ u is Hölder cont. on $B_{1/2}$ $\quad |u(x) - u(y)| \leq c|x-y|^\alpha$, $x, y \in B_{1/2}$, $\alpha = \alpha(n) \geq 0$

$c = c(n, N_0)$.

Proof by Fubini's thm. $\exists\, \sigma \in [\frac{3}{4}, 1]$ s.t

$$W(u | \partial B_\sigma) = \int_{\partial B_\sigma} |u - u^*|^2 d\xi \leq 8 W_1(u) \leq 8 \varepsilon_0$$

$$E(u | \partial B_\sigma) \leq 8 E_1(u) \leq 8B.$$

App. L. 4.3 on B_σ $\quad \exists\, \bar{u} \in L_1^2(B_\sigma, N_0)$ $\quad \bar{u}|\partial B_\sigma = u$ s.t.

if $W(u | \partial B_\sigma) \leq 8^{-1} \sigma^{2n-4} \delta^2 \varepsilon^q B^{-1}$

$$E_\sigma(\bar{u}) \leq 8 C_5 (\varepsilon \sigma B + \varepsilon^{-q} \sigma^{-1} \varepsilon_0)$$

$u \in \mathcal{H}_\Lambda$ app L 2.3.

$$E_\sigma(u) \leq (1 + c\Lambda\sigma) E_\sigma(\bar{u}) + c\Lambda \sigma^{n-1} \leq C_{18} (\varepsilon B + \varepsilon^{-q} \varepsilon_0)$$

Fix ε so small

$$C_{18}\,\varepsilon B \le \frac{1}{2}\bar{\varepsilon}\,\sigma^{n-2}$$

ε_0 so small that

$$C_{18}\,\varepsilon^{-9}\varepsilon_0 \le \frac{1}{2}\bar{\varepsilon}\,\sigma^{n-2}, \qquad 8\varepsilon_0 B \le \frac{1}{8}\sigma^{2n-4}\gamma^2\varepsilon^9$$

$$\Rightarrow \quad \sigma^{2-n}E_\sigma(u) \le \bar{\varepsilon}$$

$$\underline{\sigma\Lambda \le \bar{\varepsilon}}$$

By app. thm 3.1 that u is Hölder cont. on $B_{\sigma/2}$ ∎

<u>Proposition 4.6</u> Let $\{u_i\} \subseteq \mathcal{H}_\Lambda$ $\quad u_i \xrightarrow{w} u_0$ in L^2_1 s.t

$$E_1(u_i) \le c_{19}$$

$\Rightarrow$ u_0 is locally Hölder cont. outside a closed set $\mathcal{S}$, $\mathcal{H}^{n-2}(\mathcal{S}_0)=0$

$u_i \to u_0$ in L^2_1-norm on $B_{1/2}$

$u_i \rightrightarrows u_0$ on compact subsets of $\bar{B}_{1/2} \sim \mathcal{S}$

<u>Proof</u> Since $E_1(u_i) \le c_{19}$ uniformly

$$\Rightarrow \quad u_i \to u_0 \text{ in } L^2$$

by lower-semi cont.

$$E_1(u_0) \le c_{19} \quad , \mathcal{S}_0 \text{ is sing. set of } u_0$$

We need to show that $\mathcal{S}_0$ is small

$$W^x_\sigma(u_0) < \varepsilon_0\sigma^n \qquad u^* \in \mathbb{R}^k,\ \varepsilon_0 > 0$$

$$W^x_\sigma(u_i) < \varepsilon_0\sigma^n \qquad \text{for large } i$$

By Prop 2.4 $\quad \exists B: \quad \sigma^{2-n}E^x_\sigma(u_i) \le B \quad \forall i$

By Prop 4.5 $\quad \Lambda \le \varepsilon_0\sigma^{-1} \quad$ for σ small

$$\Rightarrow \quad u_i \rightrightarrows u_0 \text{ on } B_{\sigma/2}(x),\ u_0 \text{ is Hölder cont.}$$

Thus by Cord. 2.7 we have

$$\mathcal{H}^{n-2}(\mathcal{S}_0) = 0.$$

Also we have $\quad u_i \rightrightarrows u_0$ on compact subsets of $B_{1/2}$

We cover $\mathcal{S}_0 \cap B_{1/2}$ by $\{B_{r_i}(x_i)\}$ s.t. $\sum_i r_i^{n-2} < \varepsilon$, $\varepsilon > 0$

$\mathcal{O} = \bigcup_i B_{r_i}(x_i)$, by Prop. 2.4

$$E_{\mathcal{O}}(u_j) \le \sum_i E^{x_i}_{r_i}(u_j) \le C_{20}\sum_i r_i^{n-2} < C_{20}\varepsilon \tag{4.7}$$

we have $u_i \rightrightarrows u_0$ on $B_{1/2} \sim \mathcal{O}$

the Euler eq by $u_j - u_k$:

$$\int_{B_{1/2}\sim\mathcal{O}} |d(u_j - u_k)|^2 dx \le C(\mathcal{O}, \Lambda) \sup_{\bar{B}_{1/2}\sim\mathcal{O}} |u_j - u_k|$$

From (4.7):

$$\int_{B_{1/2}} |d(u_j - u_k)|^2 dx \le c_{21}\varepsilon + C(\mathcal{O}, \Lambda) \sup_{\bar{B}_{1/2}\sim\mathcal{O}} |u_j - u_k|$$

$\Rightarrow \{u_j\}$ is Cauchy seq. in $L_1^2(B_{1/2}, N_0)$

$\Rightarrow \|u_0 - u_j\|_{1,2} \le \varepsilon$. $\quad u_j \to u_0$ in L_1^2 norm. ∎

<u>Proposition 4.7</u> $u \in \overline{\mathcal{H}}_{\Lambda,B}$, $x_0 \in B_{1/2}$ $\exists\, \lambda(i) \to 0$, $\lambda(i) \in (0, \frac{1}{2}]$ s.t

$u_{x_0,\lambda(i)} \in L_1^2(B_1, N_0)$, $u_{x_0,\lambda(i)}(x) = u(\lambda(i)(x - x_0))$ conv. in L_1^2-norm on B_1^n to a harmonic map $u_0 \in \overline{\mathcal{H}}_{0,B'}$ $\quad \frac{\partial u_0}{\partial r} = 0$ a.e on B_1

$u_{x_0,\lambda(i)} \rightrightarrows u_0$ on comp. subsets of $\bar{B}_1 \sim S_0$.

<u>Proof</u> From L2.5. $\exists\, \lambda(i) \to 0$ s.t $u_{x_0,\lambda(i)} \xrightarrow{w} u_0$, $\frac{\partial u_0}{\partial r} = 0$ a.e

Since $u_{x_0,\lambda(i)} \in \overline{\mathcal{H}}_{\lambda(i)\Lambda,B} \Rightarrow \exists\, \tilde{u}_i \in \mathcal{H}_{\lambda(i)\Lambda,B}$

$\|\tilde{u}_i - u_{x_0,\lambda(i)}\|_{1,2} \le i^{-1}$. $\Rightarrow \tilde{u}_i \xrightarrow{w} u_0$

By prop. 4.6 $\quad \tilde{u}_i \to u_0 \Rightarrow u_{x_0,\lambda(i)} \to u_0$ in $L_1^2(B_1^n)$ ∎

<u>Lemma 5.1</u> $l \ge 3$ $\quad u \in L_{1,loc}^2(\mathbb{R}^l, N)$ $\quad \frac{\partial u}{\partial x^l} = 0$ a.e

$\Rightarrow \exists\, u_0 \in L_{1,loc}^2(\mathbb{R}^{l-1}, N)$ s.t $u(x', x^l) = u_0(x')$ a.e. $x' \in \mathbb{R}^{l-1}$

If u is E min. on each comp. subset of $\mathbb{R}^l$, then u_0 is E-minimizing on each comp. subset of $\mathbb{R}^{l-1}$.

<u>Proof</u> $\quad v : B_\sigma^{l-1} \to N_0 \quad v = u_0$ on ∂B_σ^{l-1}

$E_\sigma(v) \le E_\sigma(u_0) - \eta \qquad \eta > 0.$

$\lambda \gg 0 \qquad \bar{v} : B_\sigma^{l-1} \times [-\lambda - 2\sigma, \lambda + 2\sigma] \to N_0$

$\bar{v}(x', x^l) = v(x') \qquad |x^l| \le \lambda$

Constructed by L.41 on

$$B_\sigma^{l-1}\times[-\lambda-2\sigma,-\lambda] \qquad B_\sigma^{l-1}\times[\lambda,\lambda+2\sigma]$$

$$\bar v = u \qquad \text{on } \partial(B_\sigma^{l-1}\times[-\lambda-2\sigma,\lambda+2\sigma])$$

$$E(\bar v) \le 2\lambda E_\sigma(v)+c \qquad c(\sigma,u,v).$$

By minimizing prop of u in $R^l \sim S$

$$(2\lambda+4\sigma)E_\sigma(u) \le 2\lambda E_\sigma(v)+c$$

Choosing λ large it contradicts to $E_\sigma(v)\le E_\sigma(u)-\eta$ $\quad\lightning\ \blacksquare$

<u>Lemma 5.2</u> Let $u_0\in L^2_{1,loc}(\mathbb{R}^l,N_0)$ $l\ge 3$ be a harmonic map with an isolated singularity at 0 s.t. $\frac{\partial u_0}{\partial r}=0$ a.e.

$u\in L^2_{1,loc}(\mathbb{R}^n,N_0)$, $n\ge l$ $\qquad u(x',x'')=u_0(x')$, $x'\in\mathbb{R}^l$, $x''\in\mathbb{R}^{n-l}$.

$\exists\, u_i\in\mathcal{H}_{\Lambda_i,B}$ s.t. $u_i\to u$ in $L^2_1(B_1,N_0)$. $\Lambda_i\to 0$

$\Rightarrow$ u, u_0 are E-min. on compact subsets of $\mathbb{R}^n, \mathbb{R}^l$.

u_0 is a MTM. $\quad$ We need to show that u min. on $B_1^l\times B_1^{n-l}$

<u>Proof</u> Suppose $v_0\in L^2_1(B_1^l,N_0)$ s.t. $v_0=u_0$ on ∂B_1^l

define a map agree with u near the origin $\delta>0$. $v_\delta\in L^2_1(B_1^l,N)$

$$v_\delta=\begin{cases} v(x) & ; |x|\ge\delta \\ v(\delta\frac{x}{|x|}) & , |x|\le\delta\end{cases}$$

$$E_\delta(v_\delta)\le\delta\int_{\partial B_\delta^l}|dv_0|^2dx \qquad (5.1)$$

$\varepsilon\in(0,2^{-1}\delta)$, $v_{\delta,\varepsilon}: B_1^l\to N$

$$v_{\delta,\varepsilon}(r,\xi)=\begin{cases} v_\delta(r,\xi) & , r\ge 2\varepsilon \\ u_0(r,\xi) & , r\le\varepsilon \\ v_\delta(\rho(r),\xi) & , \varepsilon<r<2\varepsilon.\end{cases}$$

$$v_0=u_0 \quad \text{on } \partial B_1^l$$

$$v_{\delta,\varepsilon}=u_0 \text{ on } \partial B_1^l \qquad v_{\delta,\varepsilon}=u_0 \text{ in } B_\varepsilon$$

$$\int_{B_1^l\times B_1^{n-l}}|dv_{\delta,\varepsilon}|^2dx \le \int_{B_1^l\times B_1^{n-l}}|dv|^2+\eta$$

From above construction $v_{\delta,\varepsilon}=u$ on $\partial(B_1^l\times B_1^{n-l})$

$v_{\delta,\varepsilon}=u$ on $B_\delta^l\times B_1^{n-1}$ a neigh. of $S=\{0\}\times B_1^{n-1}$

by Prop 4.6 $v_i \rightrightarrows u$ away from $\mathcal{S}$

$\Rightarrow$ u min. on each comp set of $\mathbb{R}^n$

$$\int_{B_1^\ell \times B_1^{n-\ell}} |du|^2 \leq \int_{B_1^\ell \times B_1^{n-\ell}} |dv_{\delta,\varepsilon}|^2$$

$$\int_{B_1^\ell \times B_1^{n-\ell}} |du|^2 \leq \int_{B_1^\ell \times B^{n-\ell}} |dv|^2$$

$\Rightarrow$ u is minimizing. by L 5.1 $\Rightarrow$ u_0 is minimizing. ∎

Hausdorff measure for $E \subseteq \mathbb{R}^n$, $s \geq 0$.

$$\varphi^s(E) = \inf \{ \sum r_i^s : E \subseteq \bigcup_i B_{r_i}(x_i) \} \qquad (5.8)$$

$$\varphi^s(E) = 0 \iff \mathcal{H}^s(E) = 0. \qquad (5.9)$$

$$\overline{\lim}\ \lambda^{-s} \varphi^s(E \cap B_\lambda^n(x)) \geq c_6 > 0. \qquad x \in E$$

for φ^s a.e.

<u>Lemma 5.3</u> $\{u_i\} \in \mathcal{H}_\Lambda$ $u_i \xrightarrow{w} u$. in $L_1^2(B_1^n, N)$

$\mathcal{S}_i, \mathcal{S}$ denote the singular sets of u_i, u resp., then

$$\varphi^s(\mathcal{S} \cap B_{1/2}^n) \geq \overline{\lim_{i \to 0}}\ \varphi^s(\mathcal{S}_i \cap B_{1/2}^n), \text{ for any } s \geq 0.$$

<u>Proof</u> $\forall \varepsilon > 0$, $\{B_{r_i}(x_i)\}$ be a covering of $\mathcal{S} \cap B_{1/2}^n$

$$\sum r_i^s \leq \varphi^s(\mathcal{S} \cap B_{1/2}^n) + \varepsilon$$

$K = \bar{B}_{1/2}^n \sim \bigcup_i B_{r_i}^n(x_i)$ compact subset of $\bar{B}_{1/2}^n \sim \mathcal{S}$.

by prop 4.6 for $j \gg 0$ u_j is smooth on K

$\Rightarrow$
$$\mathcal{S} \cap B_{1/2}^n \subseteq \bigcup_i B_{r_i}^n(x_i)$$

$$\varphi^s(\mathcal{S}_j \cap B_{1/2}^n) \leq \varphi^s(\mathcal{S} \cap B_{1/2}^n) + \varepsilon \qquad \forall \varepsilon > 0, j \text{ large} \quad ∎$$

<u>Theorem II</u> Let $u \in L_1^2(M,N)$ be $\tilde{E}$-min. $u(x) \in N_0$ a.e. $N_0 \subseteq N$

$\Rightarrow \dim(\mathfrak{S} \cap \operatorname{int} M) \leq n-3$, $n = \dim M$, $\dim A$ - Haus. dim.

If $n = 3 \Rightarrow \mathfrak{S}$ is a discrete set of points.

<u>Theorem IV</u> Suppose $\exists\, \ell \geq 3$ s.t. every MTM from $\mathbb{R}^j \to N$ is trivial, $3 \leq j < \ell$.

$\Rightarrow$ if $u \in L_1^2(M,N)$ is $\tilde{E}$-min. with $u(x) \in N_0$ a.e.
then $\dim(\mathfrak{S} \cap \operatorname{int} M) \leq n-\ell-1$

If $n = \ell+1$, then $\mathfrak{S}$ is a discrete set of points

if $n < \ell+1$, $\mathfrak{S} = \emptyset$.

<u>Proof</u> $u \in L_1^2(M,N)$ $\tilde{E}$-min., $\mathfrak{S} \subset \operatorname{int} M$.

$0 \leq s < n-2$ s.t. $\varphi^s(\mathfrak{S}) > 0$.

By (5.9) $p_0 \in \mathfrak{S}$ s.t

$$\lim \lambda_i^{-s} \varphi^s(\mathfrak{S} \cap B^n_{\lambda_i/2}) > 0 \qquad (5.10)$$

for $\lambda_i \to 0$, B_λ is in normal coordinates x

assume $u_\lambda(x) = u(\lambda x)$ scaled maps

By Prop 4.7 $\exists\, \lambda_i$: $u_{\lambda_i} \xrightarrow{w} u_0$ in $L_1^2(B_1^n, N)$

$u_{\lambda_i} \to u_0$ in $L_1^2(B^n_{1/2}, N)$, $\frac{\partial u_0}{\partial r} = 0$ a.e

$$\mathfrak{S}_\lambda \cap B^n_{1/2} = \{\tfrac{x}{\lambda} : x \in \mathfrak{S} \cap B^n_{1/2}\}$$

$$\Rightarrow \varphi^s(\mathfrak{S}_\lambda \cap B^n_{1/2}) = \lambda^{-s} \varphi^s(\mathfrak{S} \cap B^n_{\lambda/2})$$

(5.10) $\Rightarrow$

$$\lim_{\lambda_i \to 0} \varphi^s(\mathfrak{S}_{\lambda_i} \cap B^n_{1/2}) > 0$$

By L 5.3.

$$\varphi^s(\mathfrak{S}_0 \cap B^n_{1/2}) > 0.$$

$\frac{\partial u_0}{\partial r} = 0$ a.e $\Rightarrow \lambda \mathfrak{S}_0 \subseteq \mathfrak{S}_0 \quad \forall \lambda \geq 0$.

Choose a point $x_1 \in \mathcal{S}_0 \cap \partial B_1^n$ by (5.9)

$$\overline{\lim_{\lambda \to 0}}\, \lambda^{-s} \varphi^s(\mathcal{S}_0 \cap B_\lambda^n(x_1)) > 0$$

we get

$$u_1 \in L^2_{1,loc}(\mathbb{R}^n, N_0) \quad ; \varphi^s(\mathcal{S}_0 \cap B_1^n) > 0$$

$$\Rightarrow \frac{\partial u_1}{\partial x^1} = 0 \ \ a.e. \qquad s-1 \leq 0 \text{ we stop}$$

$\exists x_2 \in \mathcal{S}_1 \cap \partial B_1^{n-1}$, $\mathbb{R}^{n-1} = \{(0, x^2, \dots, x^n)\}$

If we repeat the procedure m times

$$u_j \in L^2_{1,loc}(\mathbb{R}^n, N_0) \quad j = 1 \dots m \qquad u_j|B_1^n \in \overline{\mathcal{H}}_{\Lambda,B}$$

Prop 4.7 $\quad \frac{\partial u_j}{\partial r} = \frac{\partial u^j}{\partial x^\alpha} = 0$ a.e $\alpha = 1, \dots j$.

$$\varphi^s(\mathcal{S}_j \cap B_1^n) > 0 \qquad \text{until} \quad s-m \leq 0.$$

In order to construct u_m $\quad s-m+1 > 0$.

$$s < n-2 \Rightarrow m \leq n-2.$$

If $m = n-2 \Rightarrow \mathcal{S}_m \supseteq \mathbb{R}^{n-2} = \{(x^1, \dots, x^{n-2}, 0, 0)\}$ ↯

$$\mathcal{H}^{n-2}(\mathcal{S}_m) = 0$$

$$\Rightarrow \quad m \leq n-3 \ , \ \varphi^t(\mathcal{S}_m \cap B_1^n) = 0 \quad t > n-3.$$

$$\varphi^s(\mathcal{S}_m \cap B_1^n) > 0 \Rightarrow s \leq n-3$$

$\dim \mathcal{S} \leq n-3$. $\square$

If $m = n-3 \Rightarrow u_m \in L^2_{1,loc}(\mathbb{R}^n, N_0)$ s.t. $u_m|B_1^n \in \overline{\mathcal{H}}_{\Lambda,B}$

$$u_m(x', x'') = \tilde{u}_m(x'') \qquad x' \in \mathbb{R}^{n-3}, x'' \in \mathbb{R}^3$$

$\tilde{u}_m$ has an isolated sing. at $x'' = 0$.

by L5.2. $\quad \tilde{u}_m \in L^2_{1,loc}(\mathbb{R}^3, N_0)$ is MTM hence trivial by assumption.

$$\Rightarrow \quad m \leq n-4.$$

repeat for $\quad m = n-4, \dots, n-l$

$\Rightarrow \quad m \leq$ [redacted] $n-l-1 \Rightarrow s \leq n-l-1$

for any $s < \dim \mathcal{S} \Rightarrow \dim \mathcal{S} \leq n-l-1$ $\square$

Suppose $\underline{n = l+1}$, $\mathcal{S} \neq \emptyset$

$p_0 \in \mathcal{S}$, $u_0 \in L^2_{1,loc}(\mathbb{R}^n, N_0)$

From above argument u_0 has a $\mathcal{S}_0 = \{0\}$

If $\exists\, p_i \in \mathcal{S}$ $p_i \to p_0$ $\lambda(i) = 4\, dist(p_i, p_0)$

$u_{\lambda(i)} \in L^2_1(B^n_1, N_0)$

$\Rightarrow$ $\mathcal{S}_{\lambda(i)} \cap \partial B^n_{1/4} \neq \emptyset$ for each i. ↯

u_0 has an isolated sing at 0.

∎

$\Rightarrow$ $\mathcal{S}$ is discrete.

<u>Corollary</u> If the sectional curvature of N is nonpositive if $u(M)$ is contained in a strictly convex ball of N

$\Rightarrow$ $\mathcal{S} = \emptyset$, any $\tilde{E}$-min. map $u \in L^2_1(M, N)$ is smooth.

<u>Proof</u>

We need to show that

$TM : \mathbb{R}^j \to N$, $j \geq 3$ is trivial $\sim$

$u : S^{j-1} \to N$ is trivial

we can lift u to

$\tilde{u} : S^{j-1} \to \tilde{N}$ $\tilde{N}$ - universal cover of N

ρ^2 distance function to a point is strictly convex

$\rho^2 \circ u$ is a subharmonic func. on S^{j-1}

$\Rightarrow$ constant

Thus u is constant.

∎