A-harmonic approximation

(4.13) $\operatorname{div}(a(x,u)Du) = 0$

$$\begin{cases} a(x,u)\xi\cdot\xi \geq |\xi|^2 & (4.6) \\ a(x,u)\xi\,\tilde{\xi} \leq L|\xi|\cdot|\tilde{\xi}| & (4.7) \end{cases}$$

- there exists $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ concave, nondecreasing, $\omega(t) \xrightarrow{t\searrow 0} \omega(0) = 0$ s.t. $|a(x,u) - a(\tilde{x},\tilde{u})| \leq \omega(|x-\tilde{x}| + |u-\tilde{u}|)$ (4.14)

- **Goal**: alternative proof of

Lemma 4.21: $\forall \tau \in (0,1)$ $\exists\, \varepsilon_0, R_0 > 0$ depending on n, N, L, ω, τ such that:

$u \in W^{1,2}(\Omega, \mathbb{R}^N)$ - weak sol. to (4.13)

$E(u;x_0,R) < \varepsilon_0^2$ for some $B_R(x_0) \subset \Omega$, $R \leq R_0$

$\Rightarrow E(u;x_0,\tau R) \leq$

$\leq C(n,N,L)\,\tau^2 E(u;x_0,R)$

excess decay estimate

$$\boxed{E(u;x_0,\varrho) = \fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|^2\,dx \quad \text{- excess}}$$

Def: $A \in \mathbb{R}^{Nn\times Nn}$. A function $h \in W^{1,1}(\Omega,\mathbb{R}^N)$ is called A-harmonic, if

$$\int_\Omega A\,Dh\cdot D\varphi\,dx = 0 \quad \forall \varphi \in C_0^1(\Omega,\mathbb{R}^N)$$

- if u is "close" to being A-harmonic, then u is close to an A-harmonic function in the L^2-sense

Lemma 4.27: $L \geq 1$ fixed. For every $\varepsilon > 0$ there exists $\delta = \delta(n,N,L,\varepsilon) > 0$ such that if A - constant bilinear form, elliptic and bounded by L,

$u \in W^{1,2}(B_\varrho(x_0),\mathbb{R}^N)$: $\varrho^{2\gamma-n}\int_{B_\varrho(x_0)} |Du|^2\,dx \leq 1$ for some $\gamma \in \mathbb{R}$ and u is approx A-harmonic

$$\left|\varrho^{\gamma-n}\int_{B_\varrho(x_0)} A\,Du\cdot D\varphi\,dx\right| \leq \delta \sup_{B_\varrho(x_0)}|D\varphi| \quad \forall \varphi \in C_0^1$$

then there exists A-harmonic $h \in W^{1,2}(B_\varrho(x_0),\mathbb{R}^N)$ s.t.

$$\varrho^{2\gamma-n}\int_{B_\varrho(x_0)} |Dh|^2\,dx \leq 1, \quad \varrho^{2\gamma-n-2}\int_{B_\varrho(x_0)} |u-h|^2\,dx \leq \varepsilon$$

Proof: ~~Assume $x_0=0$, $\rho=1$ (then rescaling argum.~~

Without loss of generality $x_0=0$, $\rho=1$ (+ rescaling argument)

By contradiction: take $\varepsilon>0$, $(A_j)_{j\in\mathbb{N}}$ - sequence of elliptic, bdd bilinear forms,

$(u_j)_{j\in\mathbb{N}}$ - seq. in $W^{1,2}(B_1,\mathbb{R}^N)$ s.t. $\int_{B_1}|Du_j|^2dx\le 1$

and

$$\left|\int_{B_1} A_j Du_j\cdot D\varphi\,dx\right|\le \frac{1}{k}\sup_{B_1}|D\varphi| \quad \forall\varphi\in C_0^\infty,$$

but $\int_{B_1}|u_j-h_j|^2dx > \varepsilon$ for every A_j-harmonic h_j with $\int_{B_1}|Dh_j|^2dx\le 1$

- assume $(u_j)_{B_1}=0$ (otherwise replace u_j with $u_j-(u_j)$)
- (u_j) - bounded in $W^{1,2}$ (Poincaré inequality)

$\Rightarrow$ $u_j\to v$ weakly in $W^{1,2}$

$u_j\to v$ strongly in L^2 $\leftarrow$ up to a subsequence

$A_j\to A$ in $\mathbb{R}^{Nn\times Nn}$

- $\int_{B_1}|Dv|^2dx\le 1$ (weakly lower semicontinuity of $w\mapsto\int|Dw|^2dx$)

$(v)_{B_1}=0$

- v is A-harmonic:

$$\int_{B_1}ADv\cdot D\varphi\,dx=\int_{B_1}A(Dv-Du_j)\cdot D\varphi\,dx+\int_{B_1}(A-A_j)Du_j\cdot D\varphi\,dx+\int_{B_1}A_jDu_j\cdot D\varphi\,dx$$

- Dirichlet problem:

$$\begin{cases}\operatorname{div}(A_jDv_j)=0 & \text{in } B_1\\ v_j=v & \text{on } \partial B_1\end{cases}$$

- there exists a unique weak solution $v_j\in v+W_0^{1,2}(B_1,\mathbb{R}^N)$, v_j is A_j-harmonic

- $$\int_{B_1}|Dv_j-Dv|^2dx\le\int_{B_1}A_j(Dv_j-Dv)\cdot(Dv_j-Dv)\,dx=-\int_{B_1}A_jDv\cdot(Dv_j-Dv)\,dx=$$

$$=\int_{B_1}(A-A_j)Dv\cdot(Dv_j-Dv)\,dx\le|A-A_j|\left(\int_{B_1}|Dv|^2dx\right)^{1/2}\left(\int_{B_1}|Dv_j-Dv|^2dx\right)^{1/2}$$

$\Rightarrow$ $Dv_j\to Dv$ in L^2 $\Rightarrow$ strong convergence $v_j\to v$ in $W^{1,2}$

- $v_j \to v$ in L^2
  $u_j \to v$ in L^2 $\Rightarrow$ $u_j - v_j \to 0$ in L^2

- rescale v_j: $h_j = \frac{v_j}{m_j}$, $m_j = \max\{1, (\int_{B_1} |Dv_j|^2 dx)^{1/2}\} \xrightarrow{j\to\infty} 1$

$\Rightarrow \int_{B_1} |Dh_j|^2 dx \leq 1$

$$\|u_j - h_j\|_{L^2} \leq \underbrace{\|h_j - v_j\|_{L^2}}_{\substack{\| \\ (1-\frac{1}{m_j})\|v_j\|_{L^2} \to 0}} + \underbrace{\|v_j - v\|_{L^2}}_{\searrow 0} + \underbrace{\|v - u_j\|_{L^2}}_{\searrow 0} \longrightarrow \Rightarrow \text{contradiction}$$

- rescaling argument:

$U(y) = \varrho^{\gamma-1} u(x_0 + \varrho y)$, $y \in B_1$

$\Rightarrow$ ~~there~~ there exists A-harmonic H approximating U

$\Rightarrow$ $h(x) = \varrho^{1-\gamma} H\left(\frac{x - x_0}{\varrho}\right)$ - A-harmonic function approx. u

$\boxtimes$

<u>Lemma 4.28</u>: $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ - weak solution to (4.13) with assumptions 4.6, 4.7, 4.14.

$\Rightarrow$ for every $B_\varrho(x_0) \subset \Omega$, $u_0 \in \mathbb{R}^N$:

$$\left| \varrho^{1-n} \int_{B_\varrho(x_0)} a(x_0, u_0) Du \cdot D\varphi \, dx \right| \leq c(n, L)\, \omega^{1/2}\left(\varrho + \left(\fint_{B_\varrho(x_0)} |u - u_0|^2 dx\right)^{1/2}\right)\left(\varrho^{2-n} \int_{B_\varrho(x_0)} |Du|^2 dx\right)^{1/2} \sup_{B_\varrho(x_0)} |D\varphi| \quad \forall \varphi \in C_0^1$$

<u>Proof</u>: Let $\varphi \in C_0^1$ s.t. $\sup_{B_\varrho(x_0)} |D\varphi| \leq 1$.

u - weak solution $\Rightarrow$ $\int_{B_\varrho(x_0)} a(x_0, u_0) Du \cdot D\varphi \, dx = \int_{B_\varrho(x_0)} (a(x_0, u_0) - a(x, u)) Du \, D\varphi \, dx$

$$\left| \fint_{B_\varrho(x_0)} a(x_0, u_0) Du \cdot D\varphi \, dx \right| \leq c(L) \fint_{B_\varrho(x_0)} \omega^{1/2}(|x - x_0| + |u - u_0|) |Du| \, |D\varphi| \, dx \leq$$

$$\leq c(L) \left(\fint_{B_\varrho(x_0)} \omega(|x - x_0| + |u - u_0|) dx \right)^{1/2} \left(\fint_{B_\varrho(x_0)} |Du|^2 dx \right)^{1/2} \leq$$

$$\leq c(L)\, \omega^{1/2}\left(\varrho + \left(\fint_{B_\varrho(x_0)} |u - u_0|^2 dx\right)^{1/2}\right)\left(\fint_{B_\varrho(x_0)} |Du|^2 dx\right)^{1/2}$$

$\boxtimes$

Def. ~~$E(u;x_0,\varrho) = \frac{1}{|B_\varrho(x_0)|}\int_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|$~~

$$E(u;x_0,\varrho) = \fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|^2\,dx \quad \text{- excess}$$

$$\tilde{E}(u;x_0,\varrho) = \varrho^{2-n}\int_{B_\varrho(x_0)} |Du|^2\,dx$$

- $E(u;x_0,\varrho) \le c(n,N)\tilde{E}(u;x_0,\varrho) \le c(n,N,L)E(u;x_0,\varrho)$ (Poincaré + Caccioppoli ineq)

<u>Proof of Lemma 4.21:</u> $B_R(x_0) \Subset \Omega$

- suppose $E(u;x_0,\frac{R}{2}) \neq 0$ (otherwise nothing to prove)

$$\Rightarrow \tilde{E}(u;x_0,\tfrac{R}{2}) \neq 0$$

- $w(x) = u(x)\left(\tilde{E}(u;x_0,\tfrac{R}{2})\right)^{-1/2}$

$$\Rightarrow \left(\frac{R}{2}\right)^{2-n}\int_{B_{R/2}(x_0)} |Dw|^2\,dx \le 1$$

- $u_0 = (u)_{B_{R/2}(x_0)} \Rightarrow \left|\left(\frac{R}{2}\right)^{1-n}\int_{B_{R/2}(x_0)} a(x_0,u_0)Dw\cdot D\varphi\,dx\right| \le c(n,L)\omega^{1/2}\left(R+E(u;x_0,\tfrac{R}{2})^{1/2}\right)$ $\sup_{B_{R/2}(x_0)}|D\varphi|$

  (Lemma 4.28)

- $\varepsilon > 0$, ~~$\delta(n,L$~~, $\delta = \delta(n,N,L,\varepsilon)$ from Lemma 4.27. ~~Ass~~

  Assume

  $$c(n,L)\omega^{1/2}\left(R+E(u;x_0,\tfrac{R}{2})^{1/2}\right) \le \delta \quad (4.25)$$

  $\Rightarrow$ there exists A-harmonic $h \in W^{1,2}$ ~~s.t.~~ ($A = a(x_0,u_0)$) s.t.

  Lemma 4.27 $\left(\frac{R}{2}\right)^{-n}\int_{B_{R/2}(x_0)} |w-h|^2\,dx \le \varepsilon$, $\left(\frac{R}{2}\right)^{2-n}\int_{B_{R/2}(x_0)} |Dh|^2\,dx \le 1$

For every $\tau \in (0,\frac{1}{2})$:

$$E(u;x_0,\tau R) = \fint_{B_{\tau R}(x_0)} |u-(u)_{B_{\tau R}(x_0)}|^2\,dx \le \fint_{B_{\tau R}(x_0)} |u-\tilde{E}(u;x_0,\tfrac{R}{2})^{1/2}(h)_{B_{\tau R}(x_0)}|^2\,dx =$$

$$= \tilde{E}(u;x_0,\tfrac{R}{2})\fint_{B_{\tau R}(x_0)} |w-(h)_{B_{\tau R}(x_0)}|^2\,dx \le \tilde{E}(u;x_0,\tfrac{R}{2})\left(\fint_{B_{\tau R}(x_0)} |w-h|^2\,dx + \fint_{B_{\tau R}(x_0)} |h-(h)_{B_{\tau R}}|^2\,dx\right) \le$$

$$\le 2\tilde{E}(u;x_0,\tfrac{R}{2})\left((2\tau)^{-n}\int_{B_{R/2}} |w-h|^2\,dx + c(n,N,L)\tau^2\fint_{B_{R/2}(x_0)} |h-(h)_{B_{R/2}(x_0)}|^2\,dx\right)$$

[decay estimate for h

$$\int_{B_r} |u-(u)_{B_r}|^2\,dx \le c\left(\frac{r}{R}\right)^{n+2}\int_{B_R} |u-(u)_{B_R}|^2\,dx \Big]$$

$$\Rightarrow E(u;x_0,\tau R) \le c(n,N,L)\,\tilde{E}(u;x_0,R/2)\left(\tau^{-n}\varepsilon + \tau^2\left(\frac{R}{2}\right)^{2-n}\int_{B_{R/2}(x_0)} |Dh|^2\,dx\right) \le$$

$$\le c(n,N,L)\,\tilde{E}(u;x_0,R/2)\left(\tau^{-n}\varepsilon+\tau^2\right)$$

- choose $\varepsilon = \tau^{n+2}$, then

$$E(u;x_0,\tau R) \le c(n,N,L)\,\tau^2\,\tilde{E}(u;x_0,\tfrac{R}{2}) \le c(n,N,L)\,\tau^2\,E(u;x_0,R)$$

← Caccioppoli ineq.

- ~~Why can we assume (4.28)?~~

~~choose ε, R_0 such that $E(u,x_0$~~