Summary of $C^{0,\alpha}$ regularity results for a class of quasilinear systems

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Based on Lisa Beck's book "Elliptic Regularity Theory", Chapter 4
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Our class of problems

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This is (mostly) a summary of the results shown in this seminar, without their proofs, providing only very rough outlines of the proofs.

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(or with a nonzero right hand side) and study the full $C^{0,\alpha}$ regularity.

3. Finally, we take a few parallel approaches to the equation

$$\mathrm{div}(a(x,u)Du) = 0$$

and study the partial $C^{0,\alpha}$ regularity of the solutions - away from a small singular set.

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$$u \in W^{1,2}(\Omega, \mathbb{R}^N)$$

and

$$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$$

is a Carathéodory function (for all u, it is measurable with respect to x; for almost all x, it is continuous with respect to u).

Setting

In the equation (later referred to as (4.2) or (4.13))

$$\operatorname{div}(a(x,u)Du) = 0$$

the coefficients are uniformly elliptic and bounded, i.e. there exists $L \geq 1$ such that for almost all $x \in \Omega$, all $u \in \mathbb{R}^N$ and all $\xi, \overline{\xi} \in \mathbb{R}^{Nn}$ we have

$$a(x,u)\xi \cdot \overline{\xi} \leq L|\xi||\overline{\xi}|;$$

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$$a(x,u)\xi \cdot \overline{\xi} \leq L|\xi||\overline{\xi}|;$$

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These conditions will be referred to in the presented results as $(4.3), (4.4), (4.6)$ or (4.7).

Counterexamples to regularity

Let $n \geq 3$ and take $\Omega = B(0,1) \subset \mathbb{R}^3$. Take the simplest function in $W^{1,2}(\Omega, \mathbb{R}^n)$ with a discontinuity:

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for some $\alpha \in [1, n/2)$. Its weak derivative (classical except for $x = 0$) is

$$D_i u^\kappa(\alpha, x) = |x|^{-\alpha} \delta_{i\kappa} - \alpha |x|^{-\alpha-2} x_i x_\kappa$$

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for $1 \leq i, \kappa \leq n$.

We will see two examples which heavily use this structure of the derivative. The first one is due to de Giorgi and produces an unbounded solution (for $\alpha > 1$). The second one is due to Giusti and Miranda and produces a bounded solution with a discontinuity (for $\alpha = 1$).

De Giorgi counterexample

We introduce a family of bilinear forms $A(b_1, b_2)$ on $\mathbb{R}^{n\times n}$ via

$$A_{ij}^{\kappa\lambda}(b_1, b_2, x) = \delta_{\kappa\lambda}\delta_{ij} + \Big(b_1\delta_{i\kappa} + b_2\frac{x_i x_\kappa}{|x|^2}\Big)\Big(b_1\delta_{j\lambda} + b_2\frac{x_j x_\lambda}{|x|^2}\Big)$$

for indices $1 \le i, j, \kappa, \lambda \le n$ and $x \neq 0$.

Example 4.1 (De Giorgi) *Assume $n \geq 3$ and let $u\colon \mathbb{R}^n \supset B_1 \to \mathbb{R}^n$ be given by*

$$u(\alpha, x) = |x|^{-\alpha}x \qquad \textit{for } \alpha := \frac{n}{2}\big(1 - ((2n-2)^2 + 1)^{-1/2}\big)\,.$$

Then $u \in W^{1,2}(B_1, \mathbb{R}^n)$ is an unbounded weak solution of the elliptic system

$$\operatorname{div}\big(A(n-2, n, x)Du(\alpha)\big) = 0 \qquad \textit{in } B_1\,.$$

Giusti-Miranda counterexample

We introduce a family of coefficients $\widetilde{A}$ via

$$\tilde{A}_{ij}^{\kappa\lambda}(u) = \delta_{\kappa\lambda}\delta_{ij} + \Big(\delta_{i\kappa} + \frac{4}{n-2}\frac{u_i u_\kappa}{1+|u|^2}\Big)\Big(\delta_{j\lambda} + \frac{4}{n-2}\frac{u_j u_\lambda}{1+|u|^2}\Big)$$

for indices $1 \leq i, j, \kappa, \lambda \leq n$ and $u \in \mathbb{R}^n$.

Example 4.3 (Giusti and Miranda) *Assume $n \geq 3$ and let $u\colon \mathbb{R}^n \supset B_1 \to \mathbb{R}^n$ be given by $u(x) = x/|x|$. Then $u \in W^{1,2}(B_1, \mathbb{R}^n) \cap L^\infty(B_1, \mathbb{R}^n)$, and u is a discontinuous weak solution of the elliptic system*

$$\operatorname{div}(\tilde{A}(u)Du) = 0 \qquad \text{in } B_1\,. \tag{4.1}$$

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$$\operatorname{div}(\tilde{A}(u)Du) = 0 \qquad \text{in } B_1\,. \tag{4.1}$$

However, for $n = 2$ construction of examples as above is impossible: in fact, the gradient of every weak solution is as regular as its coefficients!

Caccioppoli inequality

Consider the equation (4.5):

$$\mathrm{div}(a(x,u)Du) = \mathrm{div}\ f - g$$

The following inequality valid for weak solutions of the above equation is a crucial piece of almost all approaches to $C^{0,\alpha}$ regularity.

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The following inequality valid for weak solutions of the above equation is a crucial piece of almost all approaches to $C^{0,\alpha}$ regularity.

Proposition 4.5 (Caccioppoli inequality) *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.5) with Carathéodory coefficients $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn\times Nn}$ satisfying (4.6) and (4.7), $f \in L^2(\Omega, \mathbb{R}^{Nn})$ and $g \in L^2(\Omega, \mathbb{R}^N)$. Then we have for all $\zeta \in \mathbb{R}^N$ and all balls $B_r(x_0) \Subset B_R(x_0) \subset \Omega$ the estimate*

$$\int_{B_r(x_0)} |Du|^2\, dx \le c(L)(R-r)^{-2} \int_{B_R(x_0)} |u-\zeta|^2\, dx + c \int_{B_R(x_0)} \left(|f|^2 + (R-r)^2|g|^2\right) dx\,.$$

Linear systems

Consider the equation (4.2):

$$\mathrm{div}(a(x)Du) = 0.$$

This is a linear equation - the coefficient a does not depend on the solution itself. We use the difference quotient method and the Caccioppoli inequality to obtain (first for $k = 1$, then by iteration) the following:

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Theorem 4.9 *Let $k \in \mathbb{N}$ and consider a weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ to the system (4.2) with coefficients $a \in W^{k,\infty}(\Omega, \mathbb{R}^{Nn\times Nn})$ satisfying (4.3), $f \in W^{k,2}(\Omega, \mathbb{R}^{Nn})$ and $g \in W^{k-1,2}(\Omega, \mathbb{R}^N)$. Then $u \in W^{k+1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, and for all $\Omega' \Subset \Omega$ we have*

$$\|u\|_{W^{k+1,2}(\Omega', \mathbb{R}^{Nn^{k+1}})} \leq c\big(\|u\|_{L^2(\Omega,\mathbb{R}^N)} + \|f\|_{W^{k,2}(\Omega,\mathbb{R}^{Nn})} + \|g\|_{W^{k-1,2}(\Omega,\mathbb{R}^N)}\big)$$

for a constant c depending only on n, k, $\|a\|_{W^{k,\infty}(\Omega)}$, Ω', and $\mathrm{dist}(\Omega', \partial\Omega)$.

Decay estimates

A particular feature of linear systems we will use later are decay estimates. Consider the equation (4.11) with constant coefficients:

$$\operatorname{div}(a\, Du) = 0.$$

Lemma 4.11 (Decay estimates I; Campanato) *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.11) with constant coefficients $a \in \mathbb{R}^{Nn \times Nn}$ satisfying (4.3) and (4.4). Then for all balls $B_r(x_0) \subset B_R(x_0) \subset \Omega$ we have*

$$\int_{B_r(x_0)} |u|^2 \, dx \leq c\Big(\frac{r}{R}\Big)^n \int_{B_R(x_0)} |u|^2 \, dx$$

and

$$\int_{B_r(x_0)} |u - (u)_{B_r(x_0)}|^2 \, dx \leq c\Big(\frac{r}{R}\Big)^{n+2} \int_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^2 \, dx \,,$$

with constants c depending only on n, N, and L. Moreover, the same estimates are true if u is replaced by any derivative $D^k u$ for $k \in \mathbb{N}$.

Decay estimates

A similar result holds for a nonzero right hand side. When a is uniformly continuous, since our result are local, we may "freeze the coefficients": if

$$\mathrm{div}(a(x)Du) = \mathrm{div}\, f - g,$$

then

$$\mathrm{div}\,\big(a(x_0)Du\big) = \mathrm{div}\,\big((a(x_0) - a(x))Du + f\big) - g$$

so u satisfies a different equation with constant coefficients and nonzero right hand side. Hence, we may use an inhomogenous version of the previous result to conclude that a similar decay estimate holds.

Decay estimates

Lemma 4.16 (Decay estimates III; Campanato) *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.2) with continuous coefficients $a \in C^0(\Omega, \mathbb{R}^{Nn\times Nn})$ satisfying (4.3), (4.4), and (4.12), let $f \in L^2(\Omega, \mathbb{R}^{Nn})$ and $g \in L^2(\Omega, \mathbb{R}^N)$. Then for all balls $B_r(x_0) \subset B_R(x_0) \subset \Omega$ we have*

$$\int_{B_r(x_0)} |Du|^2\,dx \le c\Big[\Big(\Big(\frac{r}{R}\Big)^n + \omega(R)^2\Big)\int_{B_R(x_0)} |Du|^2\,dx + \int_{B_R(x_0)} \big(|f|^2 + R^2|g|^2\big)\,dx\Big]$$

and

$$\int_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|^2\,dx \le c\Big[\Big(\frac{r}{R}\Big)^{n+2}\int_{B_R(x_0)} |Du-(Du)_{B_R(x_0)}|^2\,dx + \omega(R)^2\int_{B_R(x_0)}|Du|^2\,dx + \int_{B_R(x_0)} \big(|f-(f)_{B_R(x_0)}|^2 + R^2|g|^2\big)\,dx\Big],$$

Decay estimates

Finally, we use the new decay estimate to prove

Theorem 4.18 *Let $k \in \mathbb{N}$ and consider a weak solution $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ to the system (4.2) with coefficients $a \in C^k(\Omega, \mathbb{R}^{Nn \times Nn})$ satisfying (4.3), $f \in W^{k,2}(\Omega, \mathbb{R}^{Nn})$ and $g \in W^{k-1,2}(\Omega, \mathbb{R}^N)$. Then we have the implications (with corresponding estimates):*

(i) *If $D^k f \in L^{2,\lambda}(\Omega, \mathbb{R}^{Nn^{k+1}})$ and $D^{k-1} g \in L^{2,\lambda}(\Omega, \mathbb{R}^{Nn^{k-1}})$ for some $\lambda \in (0, n)$, then we have $D^{k+1} u \in L^{2,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn^{k+1}})$;*

(ii) *If $a \in C^{k,(\lambda-n)/2}(\Omega, \mathbb{R}^{Nn \times Nn})$, $D^k f \in \mathcal{L}^{2,\lambda}(\Omega, \mathbb{R}^{Nn^{k+1}})$ and $D^{k-1} g \in \mathcal{L}^{2,\lambda}(\Omega, \mathbb{R}^{Nn^{k-1}})$ for some $\lambda \in (n, n+2)$, then we have $D^{k+1} u \in \mathcal{L}^{2,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn^{k+1}}) \simeq C^{0,(\lambda-n)/2}(\Omega, \mathbb{R}^{Nn^{k+1}})$.*

Partial $C^{0,\alpha}$ regularity

Consider the equation (4.13):

$$\operatorname{div}(a(x,u)Du) = 0.$$

In this generality, due to the counterexamples by de Giorgi, Giusti and Miranda, we cannot expect Hölder continuity of solutions inside the whole Ω. Therefore, we aim for partial regularity results:

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We will discuss three approaches:

- A blow-up technique;
- $\mathcal{A}-$harmonic approximations;
- Using Gehring's lemma.

Notation for the rest of the talk

From now on, we assume a stronger version of Carathéodory condition on a: we assume that there exists a modulus of continuity ω (a concave, continuous increasing function with $\omega(0) = 0$) such that

$$|a(x, u) - a(\widetilde{x}, \widetilde{u})| \leq \omega(|x - \widetilde{x}| + |u - \widetilde{u}|)$$

for all $x, \widetilde{x} \in \Omega$ and all $u, \widetilde{u} \in \mathbb{R}^N$. This will be denoted as (4.14).

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We also introduce the α-regular and singular set of $f : \Omega \to \mathbb{R}^N$ as follows:

$$\mathrm{Reg}_\alpha(f) := \{x_0 \in \Omega \colon f \text{ is locally continuous near } x_0 \text{ with Hölder exponent } \alpha\}$$

for $\alpha \in [0,1]$, and the singular set of f as its complement in Ω, i.e.

$$\mathrm{Sing}_\alpha(f) := \Omega \setminus \mathrm{Reg}_\alpha(f).$$

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- Excess decay estimate; we show a bound on the function

$$E(u; x_0, \varrho) := \fint_{B_\varrho(x_0)} |u - (u)_{B_\varrho(x_0)}|^2 \, dx$$

of the (Campanato-style) type: $E(u; x_0, r) \leq c(r/R)^2 E(u; x_0, R)$.

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- A regularity improvement of $\mathrm{Reg}_0(u)$ to $\mathrm{Reg}_\alpha(u)$;
- A characterisation of the set of singular points.

Giusti-Miranda technique (via blow-up)

The main idea is as follows. First, we want to show the excess decay estimate. Assume that it does not hold for a sequence u_j of weak solutions on balls B_j. We translate and rescale these balls and functions to obtain a sequence v_j on the unit ball, which fail to satisfy the excess decay estimate. The v_j's are solutions to some related system of equations.

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Since the decay estimate is given in terms of L^2 norm of the solution, then also the sequence v_j from some point satisfies the excess decay estimate, contradiction.

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$\rightarrow$ The excess decay estimate holds if the initial excess is small enough.

Giusti-Miranda technique (via blow-up)

In this way, we obtain (a.e. in x_0) an excess decay estimate of the form

$$E(u; x_0, r) \leq c(r/R)^2 E(u; x_0, R).$$

This leads to the following characterisation of the singular set:

Theorem 4.23 (Giusti and Miranda, Morrey) *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.13) with continuous coefficients $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ satisfying (4.6), (4.7) and (4.14). Then we have the characterization of the singular set via*

$$\operatorname{Sing}_0(u) = \left\{ x_0 \in \Omega\colon \liminf_{\varrho \searrow 0} \fint_{\Omega(x_0,\varrho)} |u - (u)_{\Omega(x_0,\varrho)}|^2 \, dx > 0 \right\}$$

and in particular $\mathcal{L}^n(\operatorname{Sing}_0(u)) = 0$. Moreover, for every $\alpha \in (0,1)$ there holds $\operatorname{Reg}_0(u) = \operatorname{Reg}_\alpha(u)$, i.e. $u \in C^{0,\alpha}(\operatorname{Reg}_0(u), \mathbb{R}^N)$.

Giusti-Miranda technique (via blow-up)

The fact that the singular set has Lebesgue measure zero follows from the Lebesgue differentiation theorem (under this characterisation). Away from the singular set, the excess decay estimate gives a bound in the Campanato space $\mathcal{L}^{2,n+2\alpha}$. Since

$$\mathcal{L}^{2,n+2\alpha}(B_\delta(x_0), \mathbb{R}^N) \simeq C^{0,\alpha}(\overline{B_\delta(x_0)}; \mathbb{R}^N)$$

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If the coefficients $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ are k-times differentiable with respect to both variables and if the k-th order derivatives are uniformly Hölder continuous with exponent $\alpha \in (0,1)$, then we obtain $\mathrm{Reg}_0(u) = \mathrm{Reg}_\alpha(D^{k+1}u)$ with similar arguments.

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The fact that the singular set has Lebesgue measure zero follows from the Lebesgue differentiation theorem (under this characterisation). Away from the singular set, the excess decay estimate gives a bound in the Campanato space $\mathcal{L}^{2,n+2\alpha}$. Since

$$\mathcal{L}^{2,n+2\alpha}(B_\delta(x_0), \mathbb{R}^N) \simeq C^{0,\alpha}(\overline{B_\delta(x_0)}; \mathbb{R}^N)$$

this gives Hölder regularity of u. Moreover

> If the coefficients $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ are k-times differentiable with respect to both variables and if the k-th order derivatives are uniformly Hölder continuous with exponent $\alpha \in (0,1)$, then we obtain $\mathrm{Reg}_0(u) = \mathrm{Reg}_\alpha(D^{k+1}u)$ with similar arguments.

Finally, estimates on the set of non-Lebesgue points of a Sobolev function yield $\dim_{\mathcal{H}}(\mathrm{Sing}_0(u)) \leq n-2$.

Duzaar-Grotowski technique (via $\mathcal{A}$-harmonic approximations)

Definition 4.26 Let $\mathcal{A} \in \mathbb{R}^{Nn \times Nn}$. A function $h \in W^{1,1}(\Omega, \mathbb{R}^N)$ is called $\mathcal{A}$-harmonic if it satisfies

$$\int_\Omega \mathcal{A}Dh \cdot D\varphi \, dx = 0 \qquad \text{for all } \varphi \in C_0^1(\Omega, \mathbb{R}^N)\,.$$

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• Prove that if the excess of a solution u is small (locally), then u is close to an $\mathcal{A}$-harmonic function in the L^2 norm;

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• The $\mathcal{A}$-harmonic function satisfies the excess decay estimate (it is a solution to a system with constant coefficients).

• Since the excess decay estimate is expressed in terms of the L^2 norm, u also satisfies the excess decay estimate $\rightarrow$ $C^{0,\alpha}$ regularity as before.

Giaquinta-Giusti technique (the direct method, via Gehring's lemma)

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A key feature is the increased integrability of u following from the Gehring's lemma.

Proposition 4.29 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (4.13) with Carathéodory coefficients $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^{Nn \times Nn}$ which satisfy (4.6) and (4.7). Then there exists a number $p > 2$ depending only on n, N, and L such that we have $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, and for every ball $B_R(x_0) \subset \Omega$ there holds*

$$\left(\fint_{B_{R/2}(x_0)} (1 + |Du|)^p \, dx \right)^{\frac{2}{p}} \leq c(n, N, L) \fint_{B_R(x_0)} (1 + |Du|^2) \, dx \, .$$

Giaquinta-Giusti technique (the direct method, via Gehring's lemma)

Then, we (locally) compare u to v, a solution of the following system

$$\begin{cases} \operatorname{div}\left(a(x_0, (u)_{B_{R/4}(x_0)})Dv\right) = 0 & \text{in } B_{R/4}(x_0)\,, \\ v = u & \text{on } \partial B_{R/4}(x_0)\,. \end{cases}$$

Since this system is linear, it satisfies an excess decay estimate. The only thing that is left is to prove that u and v are close in the L^2 norm - this can be done using the increased integrability of u. $\rightarrow C^{0,\alpha}$ regularity

Minimizers to quadratic variational integrals

Finally, we note that the above methods are also suitable for the study of $C^{0,\alpha}$ regularity of minimizers to functionals of the form

$$\mathcal{Q}[w;\Omega] := \int_{\Omega} a(x,w) Dw \cdot Dw \, dx$$

This is not a simple consequence of the above results, because these methods are suitable even if the underlying functional lacks the sufficient regularity to have an Euler-Lagrange equation!