Korn’s inequality

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Note 1: Text below is written using the Einstein summation convention.

Note 2: Due to the mechanical interpretation of the problem and for simplicity we work in $\mathbb{R}^3$, but all the results and techniques translate without problems to $\mathbb{R}^d$.

We return to study of a problem stated by Miss Zaremba last week. We shall concentrate on the static case.

1 Classical formulation

$$\begin{align*}
Au &= f \quad \text{in } \Omega \\
u_i &= U_i \quad \text{on } \Gamma_U \\
\sigma_{ij}n_j &= F_i \quad \text{on } \Gamma_F
\end{align*}$$ (1)

As a reminder:

$$\begin{align*}
Au &= -(a_{ijkl}\varepsilon_{kh}(u)) \\
\sigma_{ij} &= a_{ijkl}\varepsilon_{kh}(u) \\
\varepsilon_{ij}(u) &= \frac{1}{2}(u_{i,j} + u_{j,i})
\end{align*}$$ (2-4)

2 Variational formulation

Last week’s results (or simple calculation) yields the following variational problem equivalent to the classical one:

$$a(u, v - u) = (f, v - u) + \int_{\Gamma_F} F(v - u)d\Gamma$$ (5)

$$\forall v \quad v_i = U_i \quad \text{on } \Gamma_U$$

We still need to choose suitable space of functions. The natural choice is

$$V = \{ v : v = (v_1, v_2, v_3), v_i \in H^1(\Omega) \} = (H^1(\Omega))^3$$

which is a Hilbert space for the scalar product

$$(u, v) = (u_i, v_i)_{H^1(\Omega)} = \int_{\Omega} (u_i v_i + u_{i,j} v_{i,j})dx.$$
Then, we can define the set of functions admissable as $v$ in (5).

$$U_{ad} = \{ v \in V : v_i = U_i \text{ on } \Gamma_U \}$$

In order to avoid problems with "compatibility" at the interface between $\Gamma_F$ and $\Gamma_U$ we shall assume that

$$F \in (L^2(\Gamma_F))^3.$$

To summarize, we are looking for a solution of a problem of finding a function $u \in U_{ad}$ satisfying

$$a(u, v - u) = (f, v - u) + \int_{\Gamma_F} F(v - u) \, d\Gamma \quad \forall v \in U_{ad}$$

### 3 Korn’s inequality

We would like to prove existence of a solution. In order to do that we’d like to establish whether the form $a(u, v)$ is coercive. This can be proved using Korn’s inequality.

**Twierdzenie 1** (Korn’s inequality). Let $\Omega$ be a bounded set with regular boundary$^1$. There exists a constant $C > 0$ (dependent on $\Omega$) such that

$$\int_{\Omega} \varepsilon_{ij}(v)\varepsilon_{ij}(v) \, dx + \int_{\Omega} v_i v_i \, dx \geq c \|v\|_V^2 \quad \forall v \in V. \quad (6)$$

Let us note that the converse inequality is trivial - the left hand side involves only certain combinations of first derivatives (namely $v_{ij} + v_{ji}$), while the right hand side has all of them.

So, the statement is equivalent to saying that

$$\left( \int_{\Omega} \varepsilon_{ij}(v)\varepsilon_{ij}(v) \, dx + \int_{\Omega} v_i v_i \, dx \right)^{1/2}$$

is a norm in $V$ equivalent to $\| \cdot \|_V$.

Theorem 1 is a consequence of a different theorem.

**Twierdzenie 2.** Let $\Omega$ be an open set with regular boundary. Let $v$ be a distribution on $\Omega$ such that

$$v \in H^{-1}(\Omega), \quad v_i \in H^{-1}(\Omega) \quad \forall i. \quad (7)$$

Then

$$v \in L^2(\Omega).$$

$^1$The result is also valid for certain nonbounded sets.
Let $E$ be the space of $v \in (L^2(\Omega))^3$ such that $\varepsilon_{ij}(v) \in L^2(\Omega)$ $\forall i, j$;

$E$ is a Hilbert space for the norm

$$\left( \int_{\Omega} \varepsilon_{ij}(v)\varepsilon_{ij}(v)dx + \int_{\Omega} v_i v_idx \right)^{1/2}.$$

Using the definition of $\varepsilon_{ij}$ we establish that

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j} \varepsilon_{ik}(v) + \frac{\partial}{\partial x_k} \varepsilon_{ij}(v) - \frac{\partial}{\partial x_i} \varepsilon_{jk}(v)$$

If $v \in E$ then $\varepsilon_{ij} \in L^2(\Omega)$, therefore

$$\frac{\partial \varepsilon_{ij}}{\partial x_k} \in H^{-1}(\Omega)$$

so, using the established equality

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} \in H^{-1}(\Omega) \quad \forall i, j, k.$$

Applying theorem 2 to $v_{i,k}$ we see that

$v_{i,k} \in L^2(\Omega) \quad \forall i, k$.

Therefore $v \in (H^1(\Omega))^3$. We have the algebraic equality $E = (H^1(\Omega))^3$.  

Since the injection $(H^1(\Omega))^3 \to E$ is continuous, we apply the closed graph theorem (or, more precisely the bounded inverse theorem) to get an isomorphism of Banach spaces.

Now we will prove theorem 2.

Dowód. We introduce the space

$$X(\Omega) = \{ v \in H^{-1}(\Omega) : \ v_i \in H^{-1}(\Omega) \ \forall i \}$$

which is Hilbert for the norm

$$\left( \| v \|^2_{H^{-1}(\Omega)} + \sum_{i=1}^n \| v_i \|^2_{H^{-1}(\Omega)} \right)^{1/2}$$

We have to show that

$$X(\Omega) = L^2(\Omega) \quad \forall i$$

The proof consists of several steps.

\(^2\)I mean that they are isomorphic as vector spaces.
1. Relation (8) is true for $\Omega = \mathbb{R}^3$. Indeed, using the definition of $H^{-1}$ through Fourier transform the assumptions are equivalent to

$$(1 + |\xi|^2)^{-1/2} \hat{v} \in L^2(\mathbb{R}^3), \quad (1 + |\xi|^2)^{-1/2} \xi_i \hat{v} \in L^2(\mathbb{R}^3),$$

therefore

$$\int_{\Omega} (1 + |\xi|^2)^{-1} \left(1 + \sum_{i=1}^n \xi_i^2\right) |v|^2 d\xi < \infty$$

i.e.

$$v \in L^2(\Omega).$$

2. It is sufficient to prove (8) for a half-space $\Omega = \{ x : x_3 > 0 \}$.

Indeed, let $\alpha_0, \alpha_1, \ldots, \alpha_N$ be such that $\alpha_0 \in \mathcal{D}(\Omega)$, $\alpha_1 \in \mathcal{D}(\Omega)$, $\ldots$, $\alpha_N \in \mathcal{D}(\Omega)$, $\sum_{i=1}^N \alpha_i = 1$, and $\text{supp}(\alpha_i) \subset$ local chart defining $\Gamma$.

Let’s note that $\varphi \in \mathcal{D}(\Omega)$, $v \rightarrow \varphi v$ maps $X(\Omega)$ to itself. Also, we can write $v = \sum_{i=0}^N \alpha_i v$. We can consider $\alpha_0 v$ as an element of $X(\mathbb{R}^3)$ (simply extending with 0) thus, using step one $\alpha_0 v \in L^2(\Omega)$. We will have that result if we show that $\alpha_i v \in L^2(\Omega)$. If we assume that $\Gamma$ is a once continuously differentiable manifold of dimension $n - 1$, then the image of $\alpha_i v$ is in the space $X(\Omega)$ with $\Omega = \{ x : x_3 > 0 \}$, which ends the proof of this step.

3. We introduce

$$H^1_0(0, \infty; L^2(\mathbb{R}^2)) = \{ \varphi | \varphi, \frac{d\varphi}{dx_3} \in L^2(0, \infty; L^2(\mathbb{R}^2)), \varphi(x', 0) = 0 \},$$

where $x' = (x_1, x_2)$.

$$H^{-1}(0, \infty; L^2(\mathbb{R}^2)) = \text{dual of } H^1_0(0, \infty; L^2(\mathbb{R}^2))$$

where $L^2(\mathbb{R}^2)$ is identified with its dual,

$$Y(\Omega) = \{ v \mid v, dv/dx_3 \in H^{-1}(0, \infty; L^2(\mathbb{R}^2)) \}$$

We will now show that $Y(\Omega)$ is dense in $X(\Omega)$. Indeed, let $\rho_m$ be a regularizing sequence for $\mathcal{D}(\mathbb{R}^2)^3$.

For $v \in X(\Omega)$ we define $v_m = v(x') * \rho_m$, or more precisely

$$v_m(x) = \int_{\mathbb{R}^2} v(x' - y', x_3) \rho_m(y') dy'.$$

Then, as $m \rightarrow \infty$, we have $v_m \rightarrow v$ in $X(\Omega)$. Note that $v_m$ is in particular contained in $Y(\Omega)$.

$^3\rho_m$ is a sequence of functions $\geq 0$ of $\mathcal{D}(\mathbb{R}^2)$, $\int \rho_m = 1$, $\text{supp} \rho_m \subset B(0, \epsilon_m)$, $\epsilon_m \rightarrow 0$.
4. \( \mathcal{D}(\overline{\Omega}) \) is dense in \( X(\Omega) \). It suffices to show density in \( Y(\Omega) \). As a simple consequence of the Hahn-Banach theorem we only need to show, that for every functional \( M \) which is zero on \( \mathcal{D}(\Omega) \) is zero on a whole space \( Y(\Omega) \).

Let \( v \to M(v) \) be a linear form continuous on \( Y(\Omega) \) and therefore of the form

\[
M(v) = \int_0^\infty [(f, v) + (g, dv/dx_3)]dx_3, \quad f, g \in H_0^1(0, \infty; L^2(\mathbb{R}_x^2))
\]

Assume that \( M = 0 \) on \( \mathcal{D}(\Omega) \). If \( \tilde{f}, \tilde{g} \) denote the continuations of \( f \) and \( g \) as 0 for \( x_3 < 0 \), the assumptions is equivalent to

\[
\tilde{f} - \frac{dg}{dx_3} = 0
\]

therefore \( d\tilde{g}/dx_3 \in H^1(-\infty, +\infty, L^2(\mathbb{R}_x^2)) \), but then

\[
\int_0^\infty (g, dv/dx_3)dx_3 = -\int_0^\infty (dg/dx_3, v)dx_3 \quad \forall v \in Y(\Omega)
\]

so \( M = 0 \) on \( Y(\Omega) \).

5. For \( v \in \mathcal{D}(\Omega) \) we set

\[
Pv(x) = \begin{cases} v(x) & \text{if } x_3 > 0 \\ a_1 v(x', -x_3) + a_2 v(x', -2x_3) & \text{if } x_3 < 0 \end{cases}
\]

where \( a_1 + a_2 = 1, a_1 + a_2/2 = -1 \). We verify that \( v \to Pv \) is continuous in \( \mathcal{D}(\Omega) \), provided with the topology induced by \( X(\Omega) \to X(\mathbb{R}^3) \).

Using that, we see from the previous point that we can extend \( P \) to a linear, continuous mapping of \( X(\Omega) \to X(\mathbb{R}^3) \) and such that \( P \) restricted to \( \Omega \) equals to \( v \). Then, for \( v \in X(\Omega) \) \( Pv \in X(\mathbb{R}^3) \), therefore as a consequence of step one \( Pv \in L^2(\mathbb{R}^3) \) therefore \( v \in L^2(\Omega) \).

It remains to prove the continuity of \( P \).

\[
\frac{\partial}{\partial x_3} Pv = \begin{cases} \frac{\partial w}{\partial x_3} & x_3 > 0 \\ -a_1 \frac{\partial w}{\partial x_3}(x', -x_3) - 2a_2 \frac{\partial w}{\partial x_3}(x', -2x_3), & x_3 < 0 \end{cases}
\]

Setting \( \partial v/\partial x_3 = w \), we introduce

\[
Q w = \begin{cases} w(x) & x_3 > 0 \\ -a_1 w(x', -x_3) - 2a_2 w(x', -2x_3) & x_3 < 0 \end{cases}
\]

We need to prove that \( P \) (resp. \( Q \)) is continuous on \( \mathcal{D}(\overline{\Omega}) \), with the topology induced by \( H^{-1}(\Omega) \to H^{-1}(\mathbb{R}^3) \), thus by transposition, that \( \ell P \) (resp.
\(^tQ\) is continuous as a functional \(H^1(\mathbb{R}^3) \rightarrow H^1_0(\Omega)\). Now, for \(H^1(\mathbb{R}^3)\), we have:

\[
\begin{align*}
^tP\varphi(x) &= \varphi(x) + a_1\varphi(x', -x_3) + \frac{1}{2} a_2\varphi(x', -x/2), \\
^tQ\varphi(x) &= \varphi(x) - a_1\varphi(x', -x_3) - a_2\varphi(x', -x_3/2)
\end{align*}
\]

Then \(^tP\varphi(x', 0) = ^tQ\varphi(x', 0) = 0\), from which the result follows.

\[\square\]

4 Proving coercivity

Assumptions as in theorem 1. Let \(\Gamma_U \subset \Gamma\) and \(\Gamma_U\) has positive measure. Let

\[V_0 = \{ v | v \in (H^1(\Omega))^3, \quad v = 0 \text{ on } \Gamma_U \}\]

Then there exists \(\alpha_0 > 0\) such that

\[a(v, v) \geq \alpha_0 \|v\|^2_{V_0} \quad \forall \in V_0\]

Dowód. In general

\[a(v, v) = 0 \iff v \in \mathcal{R}\]

where

\[\mathcal{R} = \{ v | v(x) = a + b \times xa, b \in \mathbb{R}^3 \}\]

Because \(\Gamma_U\) has positive measure

\[v \in \mathcal{R} \cap V_0 \Rightarrow v = 0\]

and consequently \(a(v, v) = 0, \quad v \in V_0 \iff v = 0\).

Thus we see that \(a(v, v)\) is a norm on \(V_0\) and we have to show that it is equivalent to \(\| \cdot \|_V\). To simplify notation, we set

\[\varepsilon(v) = \int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)dx.\]

We now from the previous lecture that

\[a(v, v) \geq \alpha \varepsilon(v),\]

so, we only need to find a \(c_0 > 0\) such that

\[\varepsilon(v) \geq c_0 |v|^2, \quad |v|^2 = \int_\Omega v_i v_i dx, \quad \forall v \in V_0.\]

If we replace \(v\) by \(v|v|^{-1}\) we can assume that \(|v| = 1\); then we have to prove that \(\varepsilon(v) \geq c_0\). We argue by contradiction. There exists a sequence \(v_n \in V_0\) with:

\[|v_n| = 1, \quad \varepsilon(v_n) \to 0.\]
From Korn’s inequality we then have $\|v_n\|_V \leq const$; then we can select a subsequence such that

$$v_n \rightharpoonup v \quad \text{weakly in } V.$$  

But then

$$\liminf \varepsilon(v_n) \geq \varepsilon(v),$$

therefore $\varepsilon(v) = 0$, therefore $v = 0$. Using Sobolev embedding theorem we choose a subsequence $v_n \to 0$ strongly in $L^2(\Omega)^3$ which is a contradiction. $\Box$