Measure Theory mid-term exam, 15.11.2019

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The rules. The exam runs from 17:00 on Friday, 15.11.2019 until midnight. No paper is accepted after 0:00. Paper may be submitted only as pdf file. A link to cloud disk is acceptable, but it may not be password protected. A student submits solution of only up to five problems. This is an open-book-exam, students may communicate only with their peers attending this class. However, solutions are to be written independently.

Problem 1 Let $\mathbb{X}$ be any infinite set and $x_0 \in A \subset \mathbb{X}$, $\{x_0\} \neq A$. We define set functions $\mu, \nu : P(\mathbb{X}) \to \mathbb{R}_+ \cup \{+\infty\}$ by the following formulas,

$$\mu(E) = \begin{cases} 0, & \text{if } \mathbb{X} \setminus E \text{ is infinite}, \\ 1, & \text{if } \mathbb{X} \setminus E \text{ is finite}, \end{cases} \qquad \nu(E) = \begin{cases} 0, & \text{if } x_0 \notin E, \\ 1, & \text{if } x_0 \in E,\ A \nsubseteq E, \\ +\infty, & \text{if } A \subseteq E. \end{cases}$$

(a) Check if μ or ν are outer measures.
(b) If μ or ν is an outer measure, then characterize its σ-field of measurable sets.

Problem 2 Let us suppose that μ is an outer measure defined on a set $\mathbb{X}$. Prove that μ is σ-finite if and only if there exists a positive μ-summable function f.

Definition 1 An outer measure μ on $\mathbb{R}^n$ is said to be *translation invariant* if for all $x \in \mathbb{R}^n$ and all $A \subset \mathbb{R}^n$ we have $\mu(A) = \mu(x + A)$. Here $x + A := \{x + a : a \in A\}$.

Problem 3 Let μ be a translation invariant outer measure on $\mathbb{R}$, such that

$$\int_{\mathbb{R}^2} (x+y)^2 \, \mathrm{d}(\mu \times \mu) < \infty.$$

Prove that

$$\int_{\mathbb{R}} x^2 \, \mathrm{d}\mu < \infty.$$

Problem 4 Let μ and ν be a pair of finite Radon measures on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu = \int_{\mathbb{R}^n} \phi \, \mathrm{d}\nu$$

for all continuous and bounded functions ϕ. Prove that $\mu(A) = \nu(A)$ for all $A \subset \mathbb{R}^n$.

Problem 5 (a) Let μ be a Radon measure on $\mathbb{R}^n$ and let f be a continuous function with compact support. Prove that

$$\lim_{y \to 0} \int_{\mathbb{R}^n} \big(f(x+y) - f(x) \big) \, \mathrm{d}_x\mu = 0. \tag{1}$$

(b) Prove (1) for a μ-summable function f.

Problem 6 Let μ be the counting measure on $\mathbb{N}$, i.e. for all $n \in \mathbb{N}$ we have $\mu(\{n\}) = 1$. Let us suppose that $f_n : \mathbb{N} \to \mathbb{R}$ and the sequence $\{f_n\}_{n=1}^\infty$ converges in measure to f. Show that $\{f_n\}_{n=1}^\infty$ converges to f uniformly.