5.5 Coarea Formula for BV functions

Next we relate the variation measure of f and the perimeters of its level sets.

NOTATION For $f : U \to \mathbb{R}$ and $t \in \mathbb{R}$, define

$$E_t \equiv \{x \in U \mid f(x) > t\}.$$

LEMMA 1
If $f \in BV(U)$, the mapping

$$t \mapsto \|\partial E_t\|(U) \qquad (t \in \mathbb{R})$$

is $\mathcal{L}^1$-measurable.

PROOF The mapping

$$(x,t) \mapsto \chi_{E_t}(x)$$

is $(\mathcal{L}^n \times \mathcal{L}^1)$-measurable, and thus, for each $\varphi \in C^1_c(U;\mathbb{R}^n)$, the function

$$t \mapsto \int_{E_t} \operatorname{div}\varphi \, dx = \int_U \chi_{E_t} \operatorname{div}\varphi \, dx$$

is $\mathcal{L}^1$-measurable. Let D denote any countable dense subset of $C^1_c(U;\mathbb{R}^n)$. Then

$$t \mapsto \|\partial E_t\|(U) = \sup_{\substack{\varphi \in D \\ |\varphi| \le 1}} \int_{E_t} \operatorname{div}\varphi \, dx$$

is $\mathcal{L}^1$-measurable. ∎

THEOREM 1 COAREA FORMULA FOR BV FUNCTIONS
Let $f \in BV(U)$. Then

(i) E_t has finite perimeter for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$ and

(ii) $\|Df\|(U) = \int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt$.

(iii) Conversely, if $f \in L^1(U)$ and

$$\int_{-\infty}^{\infty} \|\partial E_t\|(U)\, dt < \infty,$$

then $f \in BV(U)$.

REMARK Compare this with Proposition 2 in Section 3.4.4. ∎

PROOF Let $\varphi \in C_c^1(U;\mathbb{R}^n)$, $|\varphi| \leq 1$.

1. Claim #1: $\int_U f \operatorname{div} \varphi \, dx = \int_{-\infty}^{\infty} \left(\int_{E_t} \operatorname{div} \varphi \, dx \right) dt$.

Proof of Claim #1: First suppose $f \geq 0$, so that

$$f(x) = \int_0^\infty \chi_{E_t}(x) \, dt \qquad (\text{a.e. } x \in U).$$

Thus

$$\begin{aligned} \int_U f \operatorname{div} \varphi \, dx &= \int_U \left(\int_0^\infty \chi_{E_t}(x) \, dt \right) \operatorname{div} \varphi(x) \, dx \\ &= \int_0^\infty \left(\int_U \chi_{E_t}(x) \operatorname{div} \varphi(x) \, dx \right) dt \\ &= \int_0^\infty \left(\int_{E_t} \operatorname{div} \varphi \, dx \right) dt. \end{aligned}$$

Similarly, if $f \leq 0$,

$$f(x) = \int_{-\infty}^0 (\chi_{E_t}(x) - 1) \, dt,$$

whence

$$\begin{aligned} \int_U f \operatorname{div} \varphi \, dx &= \int_U \left(\int_{-\infty}^0 (\chi_{E_t}(x) - 1) \, dt \right) \operatorname{div} \varphi(x) \, dx \\ &= \int_{-\infty}^0 \left(\int_U (\chi_{E_t}(x) - 1) \operatorname{div} \varphi(x) \, dx \right) dt \\ &= \int_{-\infty}^0 \left(\int_{E_t} \operatorname{div} \varphi \, dx \right) dt. \end{aligned}$$

For the general case, write $f = f^+ + (-f^-)$.

2. From Claim #1 we see that for all φ as above,

$$\int_U f \operatorname{div} \varphi \, dx \leq \int_{-\infty}^\infty \|\partial E_t\|(U) \, dt.$$

Hence

$$\|Df\|(U) \leq \int_{-\infty}^\infty \|\partial E_t\|(U) \, dt. \qquad (\star)$$

3. Claim #2: Assertion (ii) holds for all $f \in BV(U) \cap C^\infty(U)$.

Proof of Claim #2: Let

$$m(t) \equiv \int_{U-E_t} |Df|\,dx = \int_{\{f\leq t\}} |Df|\,dx.$$

Then the function m is nondecreasing, and thus m' exists $\mathcal{L}^1$ a.e., with

$$\int_{-\infty}^{\infty} m'(t)\,dt \leq \int_U |Df|\,dx. \qquad (\star\star)$$

Now fix any $-\infty < t < \infty$, $r > 0$, and define $\eta : \mathbb{R} \to \mathbb{R}$ this way:

$$\eta(s) \equiv \begin{cases} 0 & \text{if } s \leq t \\ \dfrac{s-t}{r} & \text{if } t \leq s \leq t+r \\ 1 & \text{if } s \geq t+r. \end{cases}$$

Then

$$\eta'(s) = \begin{cases} \dfrac{1}{r} & \text{if } t < s < t+r \\ 0 & \text{if } s < t \ \text{ or } \ s > t+r. \end{cases}$$

Hence, for all $\varphi \in C_c^1(U;\mathbb{R}^n)$,

$$\begin{aligned} -\int_U \eta(f(x)) \operatorname{div}\varphi\,dx &= \int_U \eta'(f(x)) Df\cdot\varphi\,dx \\ &= \frac{1}{r}\int_{E_t - E_{t+r}} Df\cdot\varphi\,dx. \qquad (\star\star\star) \end{aligned}$$

Now

$$\begin{aligned} \frac{m(t+r)-m(t)}{r} &= \frac{1}{r}\left[\int_{U-E_{t+r}} |Df|\,dx - \int_{U-E_t} |Df|\,dx\right] \\ &= \frac{1}{r}\int_{E_t - E_{t+r}} |Df|\,dx \\ &\geq \frac{1}{r}\int_{E_t - E_{t+r}} Df\cdot\varphi\,dx \\ &= -\int_U \eta(f(x)) \operatorname{div}\varphi\,dx \qquad \text{by } (\star\star\star). \end{aligned}$$

For those t such that $m'(t)$ exists, we then let $r \to 0$:

$$m'(t) \geq -\int_{E_t} \operatorname{div}\varphi\,dx \qquad \mathcal{L}^n \text{ a.e. } t.$$

Take the supremum over all φ as above:

$$||\partial E_t||(U) \leq m'(t),$$

and recall $(\star\star)$ to find

$$\int_{-\infty}^{\infty} ||\partial E_t||(U)\, dt \leq \int_U |Df|\, dx = ||Df||(U).$$

This estimate and $(\star)$ complete the proof.

4. Claim #3: Assertion (ii) holds for each function $f \in BV(U)$.

Proof of Claim #3: Fix $f \in BV(U)$ and choose $\{f_k\}_{k=1}^{\infty}$ as in Theorem 2 in Section 5.2.2. Then

$$f_k \to f \qquad \text{in } L^1(U) \text{ as } k \to \infty.$$

Define

$$E_t^k \equiv \{x \in U \mid f_k(x) > t\}.$$

Now

$$\int_{-\infty}^{\infty} |\chi_{E_t^k}(x) - \chi_{E_t}(x)|\, dt = \int_{\min\{f(x), f_k(x)\}}^{\max\{f(x), f_k(x)\}} dt = |f_k(x) - f(x)|;$$

consequently,

$$\int_U |f_k(x) - f(x)|\, dx = \int_{-\infty}^{\infty} \left(\int_U \left| \chi_{E_t^k}(x) - \chi_{E_t}(x) \right| dx \right) dt.$$

Since $f_k \to f$ in $L^1(U)$, there exists a subsequence which, upon reindexing by k if needs be, satisfies

$$\chi_{E_t^k} \to \chi_{E_t} \text{ in } L^1(U), \qquad \text{for } \mathcal{L}^1 \text{ a.e. } t.$$

Then, by the Lower Semicontinuity Theorem,

$$||\partial E_t||(U) \leq \liminf_{k \to \infty} ||\partial E_t^k||(U).$$

Thus Fatou's Lemma implies

$$\begin{aligned} \int_{-\infty}^{\infty} ||\partial E_t||(U)\, dt &\leq \liminf_{k \to \infty} \int_{-\infty}^{\infty} ||\partial E_t^k||(U)\, dt \\ &= \lim_{k \to \infty} ||Df_k||(U) \\ &= ||Df||(U). \end{aligned}$$

This calculation and $(\star)$ complete the proof. ∎