Golab's theorem corresponds to $N = 1$. As we shall see, the general case follows rather easily. As far as the author knows, Corollary 15 is not much simpler to prove than Theorem 4.

Corollary 15 is one more illustration of the fact that for sets with finite H^1-measure, connectedness is a quite strong regularity property. Once again compare with the counterexample just after (4.4).

We start the proof of Corollary 15 by reducing to the case when $N = 1$. Note that any subsequence of our initial sequence $\{E_k\}$ satisfies the same hypotheses as $\{E_k\}$ itself. We can use this to reduce to the case when $L = \lim_{k\to+\infty} H^1(E_k)$ exists, as for (5). [Take a subsequence for which $H^1(E_k)$ converges to the right-hand side of (16).]

Let $N(k)$ denote the number of connected components of E_k. A second sequence extraction allows us to suppose that $N(k)$ is constant. Call N' this constant value. Denote by $E_k^1, E_k^2, \dots, E_k^{N'}$ the components of E_k (any choice of order will do). We can use Proposition 34.6 to extract a new subsequence for which each sequence $\{E_k^l\}_{k\geq 0}$ converges to some (closed) limit E^l. If Corollary 15 holds for $N = 1$, we get that

$$H^1(E^l) \leq \liminf_{k\to+\infty} H^1(E_k^l). \tag{17}$$

Now $E = \bigcup_{l=1}^{N'} E^l$ (because one can check from the definition of convergence that $E_k = E_k^1 \cup E_k^2 \cdots \cup E_k^{N'}$ converges to $E^1 \cup E^2 \cdots \cup E^{N'}$, and by uniqueness of limits). Then

$$\begin{aligned} H^1(E) &\leq \sum_{l=1}^{N'} H^1(E^l) \leq \sum_{l=1}^{N'} \liminf_{k\to+\infty} H^1(E_k^l) \\ &\leq \liminf_{k\to+\infty} \Big\{ \sum_{l=1}^{N'} H^1(E_k^l) \Big\} = \liminf_{k\to+\infty} \big\{ H^1(E_k) \big\}, \end{aligned} \tag{18}$$

in particular because the E_k^l, $1 \leq l \leq N'$, are disjoint.

Thus it is enough to prove Corollary 15 when $N = 1$. So let us assume that each E_k is connected.

If E is empty or reduced to one point, then $H^1(E) = 0$ and there is nothing to prove. So we may assume that $\delta = \operatorname{diam}(E) > 0$. Let us check that for each $\varepsilon > 0$, $\{E_k\}$ satisfies $\mathcal{H}(\varepsilon, C_\varepsilon)$ with $C_\varepsilon = 3$. Let $x \in E$ be given, and take $r(x) = \operatorname{Min}\{\delta/3, \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)\}$. Then let $r \leq r(x)$ be given. Choose $x_k \in E_k$, $k \geq 0$, so that $\{x_k\}$ converges to x. Also choose $z \in E$, with $|z - x| \geq 4\delta/10$, and then $z_k \in E_k$ such that $\{z_k\}$ converges to z. Thus $|z_k - x_k| \geq \delta/3$ for k large enough.

As usual, we may as well assume that $L = \lim_{k\to+\infty} H^1(E_k)$ exists. If $L = +\infty$, there is nothing to prove. Otherwise, $H^1(E_k) < +\infty$ for k large enough. This allows us to apply Proposition 30.14 and find a simple arc $\gamma_k \subset E_k$ that goes from x_k to z_k. Then let y_k be a point of $\gamma_k \cap \partial B(x_k, r/2)$. Such a point exists, because γ_k is connected and z_k lies outside of $B(x_k, r) \subset B(x_k, r(x))$. [Recall that $r(x) \le \delta/3$.]

Each of the two disjoint arcs that compose $\gamma_k \setminus \{y_k\}$ goes from y_k to a point that lies outside of $B_k = B(y_k, r/3)$. Thus $H^1(E_k \cap B_k) \ge H^1(\gamma_k \cap B_k) \ge 2r/3$. That is, B_k satisfies (3) with $\varepsilon = 0$. Also, $B_k \subset \Omega \cap B(x, r)$ for k large enough. This completes the verification of $\mathcal{H}(\varepsilon, C_\varepsilon)$, Theorem 4 applies, and we get (16). Corollary 15 follows. □

Exercise 19 (the isodiametric inequality). We want to check that

$$|A| \le \omega_d 2^{-d} (\operatorname{diam} A)^d \tag{20}$$

for all Borel sets $A \subset \mathbb{R}^d$. Here $|A|$ is the Lebesgue measure of A, and ω_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Notice that (20) is an equality when A is a ball.

We shall follow [Fe], 2.10.30, and use Steiner symmetrizations.

Let $A \subset \mathbb{R}^d$ be a Borel set and H a hyperplane through the origin. The Steiner symmetrization of A with respect to H is the set $S_H(A)$ such that, for each line L perpendicular to H, $S_H(A) \cap L$ is empty if $A \cap L$ is empty, and is the closed interval of L centered on $H \cap L$ and with length $H^1(A \cap L)$ otherwise.

1. Check that S_H is measurable and $|S_H(A)| = |A|$.
2. Let E, F be Borel sets in $\mathbb{R}$ and I, J intervals centered at 0 and such that $H^1(I) = H^1(E)$ and $H^1(J) = H^1(F)$. Show that
$$\sup\{|x-y|\,;\, x \in I \text{ and } y \in J\} \le \sup\{|x-y|\,;\, x \in E \text{ and } y \in F\}.$$
3. Show that $\operatorname{diam} S_H(A) \le \operatorname{diam} A$.
4. Set $\lambda = \inf\{\operatorname{diam} A\,;\, A \in \mathbb{R}^d \text{ is a Borel set and } |A| = \omega_d\}$, and
$$\Sigma = \{A \subset \mathbb{R}^d\,;\, A \text{ is closed, } |A| = \omega_d, \text{ and } \operatorname{diam} A = \lambda\}.$$
Show that Σ is not empty. [Hint: use Fatou.]
5. Now set $\alpha = \inf\{r > 0\,;$ we can find $A \in \Sigma$ such that $A \subset \overline{B}(0, r)\}$. Show that $\Sigma_1 = \{A \in \Sigma\,;\, A \subset \overline{B}(0, \alpha)\}$ is not empty.
6. Show that $S_H(A) \in \Sigma_1$ when $S \in \Sigma_1$ and H is a hyperplane through the origin.
7. Let $A \in \Sigma_1$ be given, and suppose that $z \in \partial B(0, \alpha)$ and $\varepsilon > 0$ are such that $\partial B(0, \alpha) \cap B(z, \varepsilon)$ does not meet A. Let H be a a hyperplane through the origin and call w the symmetric image of z with respect to H. Show that $S_H(A)$ does not meet $\partial B(0, \alpha) \cap [B(z, \varepsilon) \cup B(w, \varepsilon)]$.
8. Show that $\partial B(0, \alpha) \subset A$ for $A \in \Sigma_1$. Then show that $A = \overline{B}(0, \alpha)$, hence $\lambda = 2$ and (21) holds.

Corollary 2.

Proposition 14. *Let $\Gamma \subset \mathbb{R}^n$ be a compact connected set such that $H^1(\Gamma) < +\infty$. Then for each choice of $x_0, y_0 \in \Gamma$, with $y_0 \neq x_0$, we can find an injective Lipschitz mapping $f : [0,1] \to \Gamma$ such that $f(0) = x_0$ and $f(1) = y_0$.*

A reasonable option would be to take an arc in Γ from x_0 to y_0 and then remove the loops in it until it becomes simple (see for instance [Fa]). Since we like to minimize things here, let us try to find f directly, essentially by minimizing the length of the corresponding arc. Let Γ, x_0, y_0 be given, as in the statement, and set

$$M = \inf \Big\{ m\,;\ \text{there is an } m\text{-Lipschitz function } f : [0,1] \to \Gamma \\ \text{such that } f(0) = x_0 \text{ and } f(1) = y_0 \Big\}. \tag{15}$$

We know from Theorem 1 that $M < +\infty$: we can even find Lipschitz mappings from $[0,1]$ onto Γ. Let $\{f_k\}$ be a minimizing sequence. That is, $f_k : [0,1] \to \Gamma$ is m_k-Lipschitz, $f_k(0) = x_0$, $f_k(1) = y_0$, and $\{m_k\}$ tends to M. As before, we can extract a subsequence, which we shall still denote by $\{f_k\}$, that converges uniformly on $[0,1]$ to some limit f. Then f is M-Lipschitz, $f(0) = x_0$, $f(1) = y_0$, and $f([0,1]) \subset \Gamma$ (because Γ is compact).

Suppose f is not injective. We can find $0 \leq t_1 < t_2 \leq 1$ such that $f(t_1) = f(t_2)$. Then we can remove the needless loop between t_1 and t_2, reparameterize our arc $[0,1]$, and get an $(1-t_2+t_1)M$-Lipschitz mapping $\widetilde{f}$ with the usual properties. Namely, we can take

$$\begin{cases} \widetilde{f}(t) = f((1-t_2+t_1)\,t) & \text{for} \quad 0 \leq t \leq \dfrac{t_1}{1-t_2+t_1}, \\ \widetilde{f}(t) = f\Big((1-t_2+t_1)\,t + (t_2-t_1)\Big) & \text{for} \quad \dfrac{t_1}{1-t_2+t_1} \leq t \leq 1. \end{cases} \tag{16}$$

The existence of $\widetilde{f}$ contradicts (15), so f is injective and satisfies all the required properties. Proposition 14 follows. □