some M that does not depend on k), and equicontinuous on H. This last means that for each $\varepsilon > 0$ we can find $\eta > 0$ (independent of k) such that $|f_k(x) - f_k(y)| \leq \varepsilon$ as soon as $|x - y| \leq \eta$.

1. Show that there is a subsequence of $\{f_k\}$ that converges uniformly on H. [You may use a dense sequence $\{x_m\}$ in H, and reduce to the case when each $\{f_k(x_m)\}_{k \geq 0}$ converges.]

2. Show that if the f_k are uniformly bounded and equicontinuous on H, and if $\{f_k\}$ converges to some function, then the convergence is uniform.

35 Uniform concentration and lower semicontinuity of Hausdorff measure

We are now ready to see why Dal Maso, Morel, and Solimini cared so much about the uniform concentration property of Section 25.

Let us set notation for the next theorem. We are given an open set $\Omega \subset \mathbb{R}^n$, and a sequence $\{E_k\}_{k \geq 0}$ of relatively closed subset of Ω. We assume that

$$\lim_{k \to +\infty} E_k = E \tag{1}$$

for some set $E \subset \Omega$, which we naturally assume to be closed in Ω. Here we use the notion of convergence introduced in the last section. That is, we take an exhaustion of Ω by compact subsets H_m and define the corresponding pseudodistances d_m as in (34.1); then (1) means that $d_m(E_k, E)$ tends to 0 for each $m \geq 0$. See Definition 34.4.

We want to give a sufficient condition under which

$$H^d(E) \leq \liminf_{k \to +\infty} H^d(E_k), \tag{2}$$

where $0 < d < n$ is a given integer. Note that something is needed, (2) is not true in general. See the simple counterexample just after (4.4) (i.e., dotted lines). Our main hypothesis is the following. For each $\varepsilon > 0$, we assume that we can find a constant $C_\varepsilon \geq 1$ such that the following property $\mathcal{H}(\varepsilon, C_\varepsilon)$ holds.

$\mathcal{H}(\varepsilon, C_\varepsilon)$: for each $x \in E$, we can find $r(x) > 0$ with the following property. Let $0 < r \leq r(x)$ be given. Then for k large enough we can find a ball $B(y_k, \rho_k) \subset \Omega \cap B(x, r)$ such that $\rho_k \geq C_\varepsilon^{-1} r$ and

$$H^d(E_k \cap B(y_k, \rho_k)) \geq (1 - \varepsilon)\, \omega_d\, \rho_k^d. \tag{3}$$

Here ω_d denotes the Lebesgue (or H^d-) measure of the unit ball in $\mathbb{R}^d$.

Theorem 4 [DMS], [MoSo2]. *If $\{E_k\}_{k \geq 0}$ is a sequence of (relatively) closed subsets of Ω, E is a closed subset of Ω, and if for every $\varepsilon > 0$ we can find C_ε such that $\mathcal{H}(\varepsilon, C_\varepsilon)$ holds, then we have the semicontinuity property (2).*

The reader may be surprised that we did not include (1) in the hypotheses. It is not really needed; the main point is that because of $\mathcal{H}(\varepsilon, C_\varepsilon)$, every $x \in E$ is a limit of some sequence of points $x_k \in E_k \cap B(y_k, \rho_k)$. The fact that points of the E_k lie close to E is not needed for (2). But (1) will always hold when we apply Theorem 4 anyway.

Our proof will essentially follow [MoSo2]. Before we start for good, let us notice that we can reduce to the case when

$$L = \lim_{k \to +\infty} H^d(E_k) \quad \text{exists.} \tag{5}$$

Indeed we can always find a subsequence $\{E_{k_j}\}$ for which $H^d(E_{k_j})$ converges to the right-hand side of (2). This subsequence still satisfies the hypotheses of Theorem 4, and if we prove (2) for it, we also get it for $\{E_k\}$ itself.

So we assume (5) and try to prove that $H^d(E) \leq L$. By the same argument as above, we shall always be able to replace $\{E_k\}$ with any subsequence of our choice (because the hypotheses are still satisfied and the conclusion is the same). We shall use this possibility a few times.

Let ε be given, and let C_ε be such that $\mathcal{H}(\varepsilon, C_\varepsilon)$ holds. For each $x \in E$, $\mathcal{H}(\varepsilon, C_\varepsilon)$ gives us a radius $r(x)$. For each integer N, set $r_N(x) = \text{Min}\{2^{-N}, r(x)\}$ and $B_N(x) = B(x, r_N(x))$. We can cover E by countably many balls $B_N(x)$, because each $E \cap H_m$, H_m compact, can be covered by finitely many such balls. Now take all the balls that we get this way (when N varies). This gives a quite large, but still countable, family $\{B_i\}_{i \in I}$ of balls centered on E. By construction,

$$\{B_i\}_{i \in I} \quad \text{is a Vitali covering of } E \tag{6}$$

(see Definition 33.9).

Let $i \in I$ be given. Write $B_i = B(x_i, r_i)$, with $x_i \in E$ and $0 < r_i \leq r(x_i)$. We know from $\mathcal{H}(\varepsilon, C_\varepsilon)$ that for k large enough we can find a ball $D_{i,k} = B(y_{i,k}, \rho_{i,k})$ such that

$$C_\varepsilon^{-1} r_i \leq \rho_{i,k}\,, \qquad D_{i,k} \subset \Omega \cap B_i\,, \tag{7}$$

and

$$H^d(E_k \cap D_{i,k}) \geq (1 - \varepsilon)\,\omega_d\, \rho_{i,k}^d\,. \tag{8}$$

Since we can replace $\{E_k\}$ with any subsequence and I is at most countable, we can assume that for each $i \in I$, the sequence $\{y_{i,k}\}$ converges to some limit y_i, and similarly $\{\rho_{i,k}\}$ converges to some $\rho_i \in [C_\varepsilon^{-1} r_i, r_i]$.

Set $D_i = B(y_i, (1+\varepsilon)\rho_i)$. Note that $D_{i,k} \subset D_i$ for k large enough. Hence

$$H^d(E_k \cap D_i) \geq H^d(E_k \cap D_{i,k}) \geq (1-\varepsilon)\,\omega_d\, \rho_{i,k}^d \geq (1-\varepsilon)^2\, \omega_d\, \rho_i^d \tag{9}$$

for k large enough.

Note that $B_i \subset 3C_\varepsilon D_i$ by (7), hence (6) says that $\{D_i\}_{i\in I}$ is a Vitali family for E (see Definition 33.10). Now we want to apply Theorem 33.12. Pick any $\lambda < H^d(E)$ and $\tau > 0$. We get a set $I_0 \subset I$ such that

$$\text{the } D_i, i \in I_0, \text{ are disjoint,} \tag{10}$$

$$\operatorname{diam} D_i < \tau \text{ for } i \in I_0, \tag{11}$$

and

$$c_d \sum_{i\in I_0} (\operatorname{diam} D_i)^d \geq \lambda. \tag{12}$$

Let us choose a finite set $I_1 \subset I_0$ such that

$$\lambda \leq (1+\varepsilon)\, c_d \sum_{i\in I_1} (\operatorname{diam} D_i)^d = (1+\varepsilon)^{d+1}\, 2^d c_d \sum_{i\in I_1} \rho_i^d\,. \tag{13}$$

Recall that c_d is the constant in the definition of H^d (see Definition 2.6). We need to know that $2^d c_d \leq \omega_d$, where as before $\omega_d = H^d(B)$ and B is the unit ball in $\mathbb{R}^d$. This is a nasty little verification, which amounts to checking that in the definition of $H^d(B)$, covering B with little balls is asymptotically optimal. See [Fe], 2.10.35 (and compare with 2.10.2 and 2.7.16 (1)) or look at Exercises 19 and 21.

For k large enough, (9) holds for all $i \in I_1$, and then

$$\begin{aligned}\lambda &\leq (1+\varepsilon)^{d+1}\, \omega_d \sum_{i\in I_1} \rho_i^d \leq (1-\varepsilon)^{-2}(1+\varepsilon)^{d+1} \sum_{i\in I_1} H^d(E_k \cap D_i) \\ &\leq (1-\varepsilon)^{-2}(1+\varepsilon)^{d+1} H^d(E_k),\end{aligned} \tag{14}$$

by (13), (9), and (10). We can let k tend to $+\infty$; we get that $\lambda \leq (1-\varepsilon)^{-2}(1+\varepsilon)^{d+1}L$, where $L = \lim_{k\to+\infty} H^d(E_k)$ as in (5). Now λ was any number strictly smaller than $H^d(E)$, so $H^d(E) \leq (1-\varepsilon)^{-2}(1+\varepsilon)^{d+1}L$. Finally, this holds for every choice of $\varepsilon > 0$, and (2) follows. This completes our proof of Theorem 4. □

Theorem 4 has the following simple consequence (essentially, Golab's theorem).

Corollary 15. *Let $\{E_k\}$ be a sequence of closed sets in $\Omega \subset \mathbb{R}^n$. Suppose that $\{E_k\}$ converges to some closed subset E of Ω, and that there is an integer N such that each E_k has at most N connected components. Then*

$$H^1(E) \leq \liminf_{k\to+\infty} H^1(E_k). \tag{16}$$