Since

$$\left|\int_D \nabla\mu\cdot G\right| = \left|\int_D \mu \,\mathrm{div}\, G\right| = \left|\lim_{k\to+\infty}\int_D u_k \,\mathrm{div}\, G\right| \\ = \left|\lim_{k\to+\infty}\int_D \nabla u_k\cdot G\right| \le \liminf_{k\to+\infty} ||\nabla u_k||_{L^2(D)}||G||_2 \tag{12}$$

for all $G \in \mathcal{C}_c^\infty(D,\mathbb{R}^2)$, (11) says that

$$||\nabla\mu||_{L^2(D)} \le \liminf_{k\to+\infty} ||\nabla u_k||_{L^2(D)}\,. \tag{13}$$

This is what we get when we consider a single relatively compact open set $D \subset\subset \Omega$. By the diagonal process, we can extract a subsequence so that the computations above work for every $D \subset\subset \Omega$. Then there is a function $u \in W^{1,2}(\Omega\setminus K)$, that coincides on each D with the corresponding function μ_D, and such that

$$\int_{\Omega\setminus K} u(x)f(x)dx = \lim_{k\to+\infty}\int_{\Omega\setminus K} u_k(x)f(x)dx \text{ for every } f\in\mathcal{C}_c(\Omega\setminus K). \tag{14}$$

Moreover,

$$\int_{\Omega\setminus K} |\nabla u|^2 \le \liminf_{k\to+\infty}\int_{\Omega\setminus K_k} |\nabla u_k|^2\,, \tag{15}$$

by (13) and because for each choice of $D \subset\subset \Omega$, the right-hand side of (13) is smaller than or equal to the right-hand side of (15).

The same reasoning, but a little simpler because we don't have to integrate by parts, yields

$$\int_{\Omega\setminus K} |u-g|^2 \le \liminf_{k\to+\infty}\int_{\Omega\setminus K_k} |u_k-g|^2\,. \tag{16}$$

Altogether we constructed a pair $(u,K)\in\mathcal{U}_N$ such that

$$\begin{aligned} J(u,K) &= \int_\Omega |u-g|^2 + \int_\Omega |\nabla u|^2 + H^1(K) \\ &\le \liminf_{k\to+\infty}\int_{\Omega\setminus K_k} |\nabla u_k|^2 + \liminf_{k\to+\infty}\int_{\Omega\setminus K_k} |u_k-g|^2 + \liminf_{k\to+\infty} H^1(K_k) \\ &\le \liminf_{k\to+\infty} J(u_k,K_k) = m_N\,, \end{aligned} \tag{17}$$

by (1), (15), (16), (6), and (3).

So we can find a minimizer (u_N, K_N) for the restriction of J to $\mathcal{U}_N$, and this for each $N \ge 1$. Now we want to repeat the argument and show that some subsequence of $\{(u_N,K_N)\}$ converges to a pair (u,K) that minimizes J on $\mathcal{U}$.

We may assume that $\inf\{J(u,K);\,(u,K)\in\mathcal{U}\} < +\infty$ because otherwise there won't be minimizers. Note that this is the case if Ω is bounded. Let us also assume that $m_N < +\infty$ for N large. One could also prove this (this is somewhat easier than (21) below), but let us not bother.

We can proceed as above with the sequence $\{(u_N, K_N)\}$, up to the point where we applied Corollary 35.15 to prove (6). Here we have no reason to believe that the sets $K_N \cup \partial\Omega$ have less than M components for some fixed M, so we need to find something else.

We want to get additional information from the fact that (u_N, K_N) minimizes J on $\mathcal{U}_N$. First, we can assume that none of the components of K_N is reduced to one point. Indeed, if x is an isolated point of K_N, u_N has a removable singularity at x, by Proposition 10.1. Then we can simply remove x from K_N, keep the same function u_N, and get an equivalent competitor with one less component.

Claim 18. The sets K_N are locally Ahlfors-regular (as in Theorem 18.1), have the property of projections (as in Theorem 24.1), and more importantly have the concentration property described in Theorem 25.1. All these properties hold with constants that do not depend on N.

The proof is the same as in Sections 18–25. Let us try to say why rapidly. All the estimates that we proved in these sections were obtained by comparing our initial minimizer (or quasiminimizer) (u, K) with other competitors $(\widetilde{u}, \widetilde{K})$. Our point now is that for all the competitors $(\widetilde{u}, \widetilde{K})$ that we used, $\widetilde{K} \cup \partial\Omega$ never has more connected components than $K \cup \partial\Omega$ itself.

For instance, to get the trivial estimates (18.20), we replaced $K \cap \overline{B}(x,r)$ with $\partial B(x,r)$. This does not increase the number of components as soon as $B(x,r) \subset \Omega$ and $K \cap B(x,r)$ is not empty, which is always the case when we use (18.20). For the local Ahlfors-regularity, we removed $K \cap B(x,r)$ from K for some ball $B(x,r) \subset \Omega$ such that $\partial B(x,r)$ does not meet K, and this is also all right. The stories for Theorems 24.1 and 25.1 are similar.

We are now ready to continue the argument above. We have a subsequence of $\{(u_N, K_N)\}$ that converges to some limit (u, K). Because of Claim 18, this subsequence satisfies the hypotheses of Theorem 35.4. [Take $r(x) = \mathrm{Min}(r_0, \mathrm{dist}(x, \mathbb{R}^2 \setminus \Omega))$ and compare $\mathcal{H}(\varepsilon, C_\varepsilon)$ in Section 35 with the conclusion of Theorem 25.1]. Hence

$$H^1(K) \leq \liminf_{N\to+\infty} H^1(K_N). \tag{19}$$

The analogues of (15) and (16) hold with the same proofs, and finally

$$J(u,K) \leq \lim_{N\to+\infty} J(u_N, K_N) = \lim_{N\to+\infty} m_N\,, \tag{20}$$

as in (17), because (u_N, K_N) minimizes J on $\mathcal{U}_N$.

Recall also that $\{m_N\}$ is nonincreasing. Since we want to show that (u, K) minimizes J on $\mathcal{U}$, it is enough to check that

$$\inf\{J(u,K); (u,K) \in \mathcal{U}\} = \lim_{N\to+\infty} m_N. \tag{21}$$

In other words, for each $\varepsilon > 0$, we have to find some pair $(v, G) \in \mathcal{U}$ such that $J(v, G) \leq \inf \{J(u, K); (u, K) \in \mathcal{U}\} + \varepsilon$, and for which $G \cup \partial\Omega$ has a finite number of connected components.

To do this, the idea is to start from a pair $(v_0, G_0) \in \mathcal{U}$ such that $J(v_0, G_0) - \inf \{J(u, K); (u, K) \in \mathcal{U}\} << \varepsilon$, and modify it slightly. As far as the author knows, (21) is not proved like this in [DMS], where the authors prefer to use the existence result from the SBV approach, nor in [MoSo2], where they avoid the issue. There is a (hopefully) complete argument in [BoDa], Sections 6–8, but which is somewhat more painful than needed because a slightly more complicated version of the Mumford-Shah functional is studied there. Recall that a full proof of existence (in any dimension) is given in [MaSo3]; Probably it contains a proof of (21), but I did not check.

Let us also mention that F. Dibos and E. Séré [DiSé] showed that in dimension 2, the minimum is approached by competitors for which K is composed of arcs of circles.

No matter how, the argument cannot be too simple. The difficulty with it is that we need to show that many of the good properties of minimizers (like local Ahlfors-regularity, or rectifiability) hold on very large parts of the set G_0. This needs the same sort of arguments as in Sections 20–24, but one needs to be more careful because the little piece of G_0 where the Ahlfors-regularity fails, say, could play nasty tricks on you when you want to deal with the property of projections, for instance. In the case of minimizers, this issue simply did not arise because the little piece was empty. In particular, it would seem that the use of some covering lemma is really unavoidable here. We leave the details and refer the reader to [BoDa] or [MaSo3] for a full proof.

Once we have (21), our proof of existence of minimizers for the Mumford-Shah functional J in (1) is complete. □