B. Functions in the Sobolev Spaces $W^{1,p}$

In the next six sections we want to record elementary properties of functions in $W^{1,p}_{\mathrm{loc}}$ that will be used later. These include absolute continuity on almost every line, traces and limits on hyperplanes, Poincaré estimates, and a few others. These sections are present for self-containedness, but most readers should be able to go through them very rapidly, or skip them altogether. One may also consult [Zi] for a much more thorough treatment of these topics.

We shall also include in Sections 15–17 a few standard facts on functions f that (locally) minimize the energy $\int |\nabla f|^2$ with given boundary data. Things like harmonicity, the Neumann condition at the boundary, or conformal invariance in dimension 2. These sections also should be skippable to a large extent.

9 Absolute continuity on lines

We start our study of the Sobolev spaces $W^{1,p}$ with the absolute continuity of Sobolev functions on almost-every line.

Let $\Omega \subset \mathbb{R}^n$ be open. Recall from Definition 2.2 that $W^{1,p}(\Omega)$ is the set of functions $f \in L^1_{\mathrm{loc}}(\Omega)$ whose partial derivatives $\frac{\partial f}{\partial x_i}$ (in the sense of distributions) lie in $L^p(\Omega)$. [Note that we do not require that $f \in L^p(\Omega)$, as many authors do.] Similarly, $W^{1,p}_{\mathrm{loc}}(\Omega)$ is the set of functions $f \in L^1_{\mathrm{loc}}(\Omega)$ such that $\frac{\partial f}{\partial x_i} \in L^p_{\mathrm{loc}}(\Omega)$ for $1 \leq i \leq n$.

We start with the simple case when $n = 1$.

Proposition 1. *Let $I \subset \mathbb{R}$ be an open interval and $f \in W^{1,1}_{\mathrm{loc}}(I)$. Denote by $f' \in L^1_{\mathrm{loc}}(I)$ the derivative of f (in the sense of distributions). Then*

$$f(y) - f(x) = \int_x^y f'(t)\,dt \quad \text{for } x, y \in I. \tag{2}$$

Moreover, f is differentiable almost-everywhere on I, and its differential is equal to f' almost-everywhere.

This is what we shall mean by absolute continuity on I. To prove the proposition, fix $x \in I$ and set $F(y) = \int_x^y f'(t)\,dt$ for $y \in I$. Since F is continuous, it defines a distribution on I and we can talk about its derivative F'. We want to show that $F' = f'$, so let $\varphi \in C_c^\infty(I)$ be given, and let $[x_1, x_2]$ denote a compact interval in I that contains the support of φ. Then

$$\begin{aligned}\langle F', \varphi \rangle &= -\int_I F(y)\varphi'(y)\,dy = -\int_I [F(y) - F(x_1)]\varphi'(y)\,dy \\ &= -\int_{x_1}^{x_2} \Big\{ \int_{x_1}^{y} f'(t)\,dt \Big\} \varphi'(y)\,dy = -\int_{x_1}^{x_2}\int_{x_1}^{x_2} f'(t)\varphi'(y)\,\mathbf{1}_{\{t \leq y\}}(t,y)\,dy\,dt \\ &= \int_{x_1}^{x_2} f'(t)\varphi(t)\,dt = \langle f', \varphi \rangle,\end{aligned} \tag{3}$$

by definition of F', because $\int \varphi'(y)\,dy = 0$, and by Fubini. Thus $(F - f)' = 0$ distributionwise, and it is easy to see that it is constant (exercise!). This proves (2).

For the remaining part of Proposition 1, we apply the Lebesgue differentiation theorem (see for instance [Ru] or [Mat2]) to the function $f' \in L^1_{\mathrm{loc}}(I)$, to get that

$$\lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_{y-\varepsilon}^{y+\varepsilon} |f'(t) - f'(y)|\,dt = 0 \tag{4}$$

for almost-every $y \in I$. Proposition 1 follows, because (4) implies that f is differentiable at y, with derivative $f'(y)$. □

We may now consider larger dimensions.

Proposition 5. *Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^m \times \mathbb{R}^n$ be a product of open sets, and let $f \in W^{1,1}(\Omega)$. Denote by $(x, y) \in \Omega_1 \times \Omega_2$ the points of Ω, and call $f_j = \frac{\partial f}{\partial y_j} \in L^1(\Omega)$, $1 \leq j \leq n$, the derivatives of f with respect to the last variables. For almost-every $x \in \Omega_1$, the function F_x defined by $F_x(y) = f(x,y)$ lies in $W^{1,1}(\Omega_2)$, and $\frac{\partial F_x}{\partial y_j} = f_j(x, \cdot)$ (distributionwise) for $j = 1, \ldots, n$.*

Comments. We shall mostly be interested in the case when $n = 1$ (and y is any of the coordinate functions). Our statement concerns functions in $W^{1,1}$, but it is easy to localize (to treat functions in $W^{1,1}_{\mathrm{loc}}$). Also, we get a similar statement for functions in $W^{1,p}_{\mathrm{loc}}$, $p > 1$, because these functions lie in $W^{1,1}_{\mathrm{loc}}$ and our statement says how to compute $\frac{\partial F_x}{\partial y_j}$.

In the statement of Proposition 5 (and similar ones later) we assume that a representative in the class of $f \in L^1_{\mathrm{loc}}$ has been chosen, and similarly for the functions f_j, and our statements concern these representatives. Thus the bad sets where $F_x \notin W^{1,1}(\Omega_2)$ or $\frac{\partial F_x}{\partial y_j} \neq f_j(x, \cdot)$ may depend on our choice of representatives, but this is all right.

The proof of Proposition 5 will be an exercise on distributions and Fubini. By definition,

$$\int_\Omega f(x,y)\frac{\partial \Phi}{\partial y_j}(x,y)\,dx\,dy = \Big\langle f, \frac{\partial \Phi}{\partial y_j} \Big\rangle = -\langle f_j, \Phi \rangle = -\int_\Omega f_j(x,y)\Phi(x,y)\,dx\,dy \tag{6}$$

for all test functions $\Phi \in C_c^\infty(\Omega)$. We only want to apply this to decomposed functions of the form $\varphi(x)\psi(y)$, with $\varphi \in C_c^\infty(\Omega_1)$ and $\psi \in C_c^\infty(\Omega_2)$; then (6) becomes

$$\int_\Omega f(x,y)\,\varphi(x)\,\frac{\partial\psi}{\partial y_j}(y)\,dx\,dy = -\int_\Omega f_j(x,y)\,\varphi(x)\,\psi(y)\,dx\,dy. \tag{7}$$

Since f and f_j are locally integrable on Ω, Fubini says that for each compact set $K \subset \Omega_2$,

$$\int_K \{|f(x,y)| + |f_j(x,y)|\}\,dy < +\infty \tag{8}$$

for almost every $x \in \Omega_1$. In fact, since we can cover Ω_2 by countably many compact subsets, we can find a set $Z \subset \Omega_1$, of measure zero, such that for $x \in \Omega_1 \setminus Z$, (8) holds for all compact subsets $K \subset \Omega_2$.

Fix any $\psi \in C_c^\infty(\Omega_2)$. For each $x \in \Omega_1 \setminus Z$ we can define

$$A(x) = \int_{\Omega_2} f(x,y)\,\frac{\partial\psi}{\partial y_j}(y)\,dy \tag{9}$$

and

$$B(x) = -\int_{\Omega_2} f_j(x,y)\,\psi(y)\,dy\,. \tag{10}$$

By Fubini, we even know that A and B are locally integrable on Ω_1. Moreover, (7) says that

$$\int_{\Omega_1} A(x)\,\varphi(x)\,dx = \int_{\Omega_1} B(x)\,\varphi(x)\,dx \tag{11}$$

for all $\varphi \in C_c^\infty(\Omega_1)$. Hence $A(x) = B(x)$ almost-everywhere.

Let us apply this argument for $\psi \in \mathcal{D}$, where $\mathcal{D} \subset C_c^\infty(\Omega_2)$ is a countable set of test functions. For each $\psi \in \mathcal{D}$ we get functions A and B and a set $Z(\psi)$ of measure zero such that $A(x) = B(x)$ on $\Omega_1 \setminus Z(\psi)$. Set $Z_1 = Z \cup \big(\bigcup_{\psi\in\mathcal{D}} Z(\psi)\big)$. Thus Z_1 is still negligible.

Let $x \in \Omega_1 \setminus Z_1$ be given. We know that

$$\int_{\Omega_2} f(x,y)\,\frac{\partial\psi}{\partial y_j}(y)\,dy = -\int_{\Omega_2} f_j(x,y)\,\psi(y)\,dy \tag{12}$$

for all $\psi \in \mathcal{D}$, because $A(x) = B(x)$ with the notation above. Let us assume that we chose $\mathcal{D}$ so large that, for each $\psi \in C_c^1(\Omega_2)$, we can find a sequence $\{\psi_j\}$ in $\mathcal{D}$ such that the ψ_j have a common compact support in Ω_2 and $\{\psi_j\}$ converges to ψ for the norm $||\psi||_\infty + ||\nabla\psi||_\infty$. (This is easy to arrange.) Then (12) extends to all $\psi \in C_c^1(\Omega_2)$, because since $x \notin Z$, (8) holds for all compact sets K$\subset \Omega_2$.

Continue with $x \in \Omega_1 \setminus Z_1$ fixed. Set $F_x(y) = f(x,y)$, as in the statement of the proposition. Note that $f_j(x,\cdot) \in L^1_{\text{loc}}(\Omega_2)$ because (8) holds for all compact sets K$\subset \Omega_2$. Since (12) holds for all $\psi \in C_c^\infty(\Omega_2)$ we get that $\frac{\partial F_x}{\partial y_j} = f_j(x,\cdot)$. To complete the proof of Proposition 5 we just need to observe that $f_j(x,\cdot) \in L^1(\Omega_2)$ (and not merely $L^1_{\text{loc}}(\Omega_2)$). This is the case, by Fubini. □

Corollary 13. *If $\Omega = \Omega_1 \times I$ for some open interval I, and $f \in W^{1,p}_{\rm loc}(\Omega)$ for some $p \in [1, +\infty)$, then for almost every $x \in \Omega_1$, $f(x, \cdot)$ is absolutely continuous on I (i.e., satisfies the hypotheses and conclusions of Proposition 1). Furthermore, its derivative equals $\frac{\partial f}{\partial y}(x, y)$ almost-everywhere.*

This is easy to prove. First note that we can restrict to $p = 1$, because $W^{1,p}_{\rm loc} \subset W^{1,1}_{\rm loc}$ anyway. Since we only assumed f to be locally in $W^{1,p}$, we need a small localization argument. Cover Ω_1 by a countable collection of relatively compact open subsets $\Omega_{1,k} \subset\subset \Omega_1$, and similarly cover I by an increasing sequence of open intervals $I_k \subset\subset I$. Then $f \in W^{1,1}(\Omega_{1,k} \times I_k)$ for each k, and Proposition 5 says that for almost every $x \in \Omega_{1,k}$, $f(x, .)$ is absolutely continuous on I_k, with a derivative that coincides with $\frac{\partial f}{\partial y}(x, .)$ almost-everywhere. The conclusion follows: when $x \in \Omega_1$ does not lie in any of the exceptional sets in the $\Omega_{1,k}$, $f(x, .)$ is absolutely continuous on I, still with the same derivative. □

10 Some removable sets for $W^{1,p}$

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $K \subset \Omega$ be a relatively closed subset such that*

$$H^{n-1}(K) = 0. \tag{2}$$

If $1 \leq p \leq +\infty$ and $f \in W^{1,p}(\Omega \setminus K)$, then $f \in W^{1,p}(\Omega)$, with the same derivative.

Note that we do not need to extend f to Ω, because K has vanishing Lebesgue measure by (2) (see Exercise 2.20). For the same reason, the derivatives $\frac{\partial f}{\partial x_i}$ are initially defined almost-everywhere on $\Omega \setminus K$, but we can easily (and uniquely) extend them to Ω.

Proposition 1 says that closed sets of H^{n-1}-measure zero are removable for $W^{1,p}$ functions. There are lots of other removable sets, but Proposition 1 will be enough here.

It is enough to prove the proposition with $p = 1$, because the derivative of f on Ω is the same as on $\Omega \setminus K$. The proof will be somewhat easier in the special case when we know that

$$f \in L^1_{\text{loc}}(\Omega), \tag{3}$$

which we shall treat first. (Note that (3) is not automatic: we only know that f lies in $L^1_{\text{loc}}(\Omega \setminus K)$, and a priori it may have monstrous singularities near K.) In our applications to Mumford-Shah minimizers, this special case would often be enough because f is bounded.

So let us assume that $f \in W^{1,1}(\Omega \setminus K)$ and that (3) holds. All we have to do is show that the distributional derivatives $\frac{\partial f}{\partial x_j}$ on $\Omega \setminus K$ also work on Ω. Let us only do this for $j = 1$. We want to show that for all $\varphi \in C^\infty_c(\Omega)$,

$$\int_\Omega f(x) \frac{\partial \varphi}{\partial x_1}\, dx = -\int_\Omega \frac{\partial f}{\partial x_1}(x)\,\varphi(x)\, dx. \tag{4}$$

Clearly the problem is local: it is enough to check (4) when φ is supported in a small rectangle $R \subset\subset \Omega$ (because we can cut the original φ into finitely many small pieces).

So let us assume that φ is supported in $R = I \times J$, where I is a (bounded) open interval and J is a (bounded) product of open intervals. Denote by $(x, y) \in I \times J$ the points of R, and call $\pi : (x, y) \to y$ the projection on the hyperplane that contains J. Set

$$Z = \pi(K \cap \overline{R}). \tag{5}$$

Then $H^{n-1}(Z) = 0$, by (2) and because π is Lipschitz (see Remark 2.11). Also, Z is closed if we took R relatively compact in Ω. Set $F_y(x) = f(x, y)$ for $y \in J \setminus Z$ and $x \in I$. Since $f \in W^{1,1}(I \times (J \setminus Z))$, Corollary 9.13 says that for almost-every $y \in J \setminus Z$, F_y is absolutely continuous on I, with derivative

$$F'_y(x) = \frac{\partial f}{\partial x}(x, y) \text{ for almost every } x \in I. \tag{6}$$

Call $J' \subset J \setminus Z$ the set of full measure for which this holds. Then

$$\begin{aligned}\int_R f(x,y)\frac{\partial \varphi}{\partial x}(x,y)\,dx\,dy &= \int_{J'} \Big\{ \int_I f(x,y)\frac{\partial \varphi}{\partial x}(x,y)\,dx \Big\}\,dy \\ &= \int_{J'} \Big\{ -\int_I F_y'(x)\varphi(x,y)\,dx \Big\}\,dy \\ &= -\int_{J'}\int_I \frac{\partial f}{\partial x}(x,y)\varphi(x,y)\,dx\,dy = -\int_\Omega \frac{\partial f}{\partial x}\varphi, \qquad (7)\end{aligned}$$

by Proposition 9.1 and (6). This is just (4) with different notations; our verification is now complete in the special case when (3) holds.