Exercise 21. We want to use Exercise 19 to show that $2^d c_d = \omega_d$. We keep the same notation.

1. Let $\{A_i\}_{i\in I}$ be a covering of $B(0,1)$ in $\mathbb{R}^d$ by countably many sets. Use (20) to show that $\sum(\operatorname{diam} B_i)^d \geq 2^d$.
2. Show that $\omega_d \geq 2^d c_d$. [Recall that c_d was chosen so that $H^d(B(0,1)) = \omega_d$.]
3. Let $\tau > 0$ be given. Show that we can find disjoint balls $B_i \subset B$, of diameters less than τ, and such that $|B(0,1) \setminus \bigcup_i B_i| = 0$. You may use Theorem 33.12 and Remark 33.37.
4. Show that for each $\varepsilon > 0$ we can cover $B(0,1) \setminus \bigcup_i B_i$ by (countably many) sets D_j of diameters less than τ and for which $\sum_j(\operatorname{diam} D_j)^d \leq \varepsilon$.
5. Show that $H^d(B(0,1)) \leq c_d \sum(\operatorname{diam} B_i)^d + c_d\varepsilon \leq c_d 2^d \omega_d^{-1} H^d(B(0,1)) + c_d\varepsilon$. Conclude.

36 A little more on the existence of minimizers

In this section we return briefly to the issue of existence of minimizers for the Mumford-Shah functional on a domain Ω. We want to say how it is possible to get these minimizers as limits of suitable sequences of nearly minimizing pairs, using in particular the lower semicontinuity result of the previous section. Our description here will be a little more precise than what we did in Section 4, but we shall not give a complete proof. Also, we shall restrict to dimension 2, as often in this book. The technique that we describe in this section also works in higher dimensions; see [MaSo3].

We should remind the reader that the standard, and somewhat simpler way to prove existence for this sort of functional is to use its weak description in terms of SBV functions, and then the compactness theorem from [Am]. This is well described in [AFP3]; here we take the slightly silly attitude of trying to see what we can get without using SBV.

The idea of trying to use the uniform concentration property from Section 25 to get existence results for minimizers comes from [DMS]. It is also described in [MoSo2], but in both cases the authors do not go all the way to a complete proof without SBV, due to technical complications. See the comments near the end of this section.

Let us start our description. Let Ω be a domain in the plane, $g \in L^\infty(\Omega)$, and set

$$J(u,K) = \int_{\Omega\setminus K} |u-g|^2 + \int_{\Omega\setminus K} |\nabla u|^2 + H^1(K), \tag{1}$$

where the competitors lie in the class $\mathcal{U}$ of pairs (u,K) for which K is a relatively closed set in Ω such that $H^1(K) < \infty$, and $u \in W^{1,2}(\Omega \setminus K)$ is defined on $\Omega \setminus K$ and has one derivative in L^2 there. See Section 2. It is customary to assume that

Ω is bounded, but the main point is to find competitors for which $J(u, K) < \infty$. However, to make the proof below work more easily, we shall assume that $\partial\Omega$ has finitely many connected components, and $H^1(\partial\Omega) < \infty$.

Our first goal will be to minimize J in the smaller class

$$\mathcal{U}_N = \{(u, K) \in \mathcal{U}\,; K \cup \partial\Omega \text{ has at most } N \text{ connected components}\}, \tag{2}$$

where $N \geq 1$ is a given integer. So let $(u_k, K_k) \in \mathcal{U}_N$ be given, with

$$\lim_{k \to +\infty} J(u_k, K_k) = m_N, \tag{3}$$

where we set

$$m_N = \inf\{J(u, K); (u, K) \in \mathcal{U}_N\}. \tag{4}$$

We may assume that $m_N < +\infty$ and that u_k minimizes $J(u_k, K_k)$ with the given K_k, i.e., that

$$J(u_k, K_k) = \inf\{J(v, K_k)\,; v \in W^{1,2}(\Omega \setminus K_k)\}. \tag{5}$$

This is possible, because Proposition 3.3 says that for any given K_k, the infimum in (5) is reached.

We want to extract from $\{(u_k, K_k)\}$ a convergent subsequence, and then show that the limit minimizes J on $\mathcal{U}_N$. We extract a first subsequence (which we shall still denote $\{(u_k, K_k)\}$ to keep the notation reasonably pleasant), so that the sets $K_k \cup \partial\Omega$ converge (in $\mathbb{R}^n$) to some closed set K^*. Here we can use convergence for the Hausdorff metric on compact sets, or equivalently the notion of convergence from Section 34. The existence of the desired subsequence follows from Proposition 34.6, say.

Set $K = K^* \setminus \partial\Omega \subset \Omega$. Because of Corollary 35.15,

$$H^1(K) = H^1(K^*) - H^1(\partial\Omega) \leq \liminf_{k \to +\infty} H^1(K_k \cup \partial\Omega) - H^1(\partial\Omega) = \liminf_{k \to +\infty} H^1(K_k). \tag{6}$$

We shall need to know that $K^* = K \cup \partial\Omega$ has at most N connected components. First note that we can find a subsequence for which each $K_k \cup \partial\Omega$ has the same number of components, and moreover each of these converges to some limit (see the first lines of the proof of Corollary 35.15). For the separate existence of limits for the components, we can also proceed as follows. Each component K_k^l of $K_k \cup \partial\Omega$ has a finite length $L_k^l \leq H^1(K_k \cup \partial\Omega) \leq m_N + H^1(\partial\Omega) + 1 < +\infty$, by (1) and (3), and if k is large enough. Then Proposition 30.1 says that we can find a CL_k^l-Lipschitz mapping $f_{k,l}$ from $[0,1]$ onto K_k^l. Since $L_k^l \leq m_N + H^1(\partial\Omega) + 1$ for k large, we can extract a subsequence so that each $\{f_{k,l}\}_k$ converges. Then the sets $\{E_{k,l}\}_k$ also converge. Also, each limit set is connected, since it is a Lipschitz image of $[0,1]$. Thus K^* has at most N components.

The reader may wonder why we considered the number of components of $K_k \cup \partial\Omega$, rather than K_k itself, especially since this is what cost us the unnatural assumption that $H^1(\partial\Omega) < +\infty$. The point is that we may have a sequence of connected sets K_k, with some parts that tend to the boundary, and for which the intersection of the limit with Ω is no longer connected. There are ways to change the definition of $\mathcal{U}_N$ so that this issue does not arise, and yet we don't need to assume that $H^1(\partial\Omega) < +\infty$. But we do not wish to elaborate here.

Next we want to make a subsequence of $\{u_k\}$ converge. For this we need some little, but uniform amount of regularity for the functions u_k. Here we shall try to use fairly little information (namely, the L^∞-bound (7) and later the fact that $u_k \in W^{1,2}(\Omega \setminus K_k)$), but we could go a little faster by using stronger properties of the u_k. Observe that

$$||u||_\infty \leq ||g||_\infty, \tag{7}$$

by uniqueness of the minimizing function in (5) and a comparison with a truncation of u_k. See the proof of (3.20) for details.

Let $D \subset\subset \Omega \setminus K$ be any relatively compact subset of $\Omega \setminus K$. By (7), the functions u_k define bounded linear forms on $\mathcal{C}_c(D)$, the set of continuous functions with compact support in D (with the sup norm), with uniform bounds. Hence, modulo extraction of a new subsequence, we can assume that these linear forms converge weakly. That is, there is a measure $\mu = \mu_D$ on D such that $\lim_{k\to+\infty} \int_D f(x)u_k(x)dx = \int_D f(x)d\mu(x)$ for every f in (a dense class of) $\mathcal{C}_c(D)$. The measure μ defines a distribution on D, and its derivative is given by

$$\langle \partial_i \mu, f\rangle = -\int_D \partial_i f(x) d\mu(x) = -\lim_{k\to+\infty} \int_D \partial_i f(x) u_k(x)dx \tag{8}$$

for $f \in \mathcal{C}_c^\infty(D)$ and $i = 1, 2$.

Recall that D is relatively compact in $\Omega \setminus K$, and hence lies at positive distance from $K^* = K \cup \partial\Omega$. Since K^* is the limit of the $K_k \cup \partial\Omega$, D does not meet $K_k \cup \partial\Omega$ for k large enough. Then $\int_D \partial_i f(x)u_k(x)dx = -\int_D f(x)\partial_i u_k(x)dx$, and

$$\langle \partial_i \mu, f\rangle = \lim_{k\to+\infty} \int_D f(x)\partial_i u_k(x)dx. \tag{9}$$

In particular,

$$\begin{aligned} |\langle \partial_i \mu, f\rangle| &\leq \liminf_{k\to+\infty} ||f||_2 \, ||\partial_i u_k||_{L^2(D)} \\ &\leq ||f||_2 \liminf_{k\to+\infty} J(u_k, K_k)^{1/2} \leq ||f||_2 \, m_N^{1/2} < +\infty. \end{aligned} \tag{10}$$

The Riesz representation theorem says that $\partial_i \mu \in L^2(D)$, and so $\mu \in W^{1,2}(D)$. [Note that $\mu \in L^1_{\rm loc}(D)$; it is even bounded, because

$$\int_D f(x)d\mu(x) \leq ||g||_\infty \int_D |f(x)|dx \text{ for } f \in \mathcal{C}_c(D), \text{ by (7).]}$$

Next

$$||\nabla\mu||_{L^2(D)} = \sup\Big\{\Big|\int_D \nabla\mu \cdot G\Big| ; G \in \mathcal{C}_c^\infty(D, \mathbb{R}^2), ||G||_2 \leq 1\Big\}. \tag{11}$$