3.6 Approximate continuity and differentiability

We have seen in Section 3.5 an example of a BV function u of $N > 1$ variables (the characteristic function of an open set) discontinuous in a set with strictly positive Lebesgue measure, together with any function in its equivalence class. By a similar strategy examples can also be produced with functions in Sobolev spaces $W^{1,p}$, with $p \le N$. So, apparently, there is no hope to get "good representatives" as in dimension 1 (cf. Theorem 3.28). These remarks show the necessity of weak continuity and differentiability notions suitable to be satisfied by functions in Sobolev spaces or in BV. These notions can be introduced following the basic idea that not only sets with zero measure, as in dimension 1, but also sets with zero density can be disregarded. We start with approximate limits.

Definition 3.63 (Approximate limit) Let $u \in [L^1_{\mathrm{loc}}(\Omega)]^m$; we say that u has an *approximate limit at* $x \in \Omega$ if there exists $z \in \mathbf{R}^m$ such that

$$\lim_{\varrho \downarrow 0} ⨍_{B_\varrho(x)} |u(y) - z| \, dy = 0. \tag{3.65}$$

The set S_u of points where this property does not hold is called the *approximate discontinuity set*. For any $x \in \Omega \setminus S_u$ the vector z, uniquely determined by (3.65), is called approximate limit of u at x and denoted by $\tilde{u}(x)$.

In the following we say that u is *approximately continuous* at x if $x \notin S_u$ and $\tilde{u}(x) = u(x)$, i.e. x is a Lebesgue point of u. Notice that the set of points where the approximate limit exists does not depend on the representative in the equivalence class of u, i.e. if $v = u$ $\mathcal{L}^N$-a.e. in Ω then $x \notin S_u$ if and only if $x \notin S_v$ and $\tilde{u}(x) = \tilde{v}(x)$. On the other hand, the property of being approximately continuous at x depends on the value of u at the point, and this value could be different for functions in the same equivalence class.

Proposition 3.64 (Properties of approximate limits) *Let u be a function in $[L^1_{\mathrm{loc}}(\Omega)]^m$.*

(a) S_u is a $\mathcal{L}^N$-negligible Borel set and $\tilde{u} : \Omega \setminus S_u \to \mathbf{R}^m$ is a Borel function, coinciding $\mathcal{L}^N$-a.e. in $\Omega \setminus S_u$ with u;

(b) if $x \in \Omega \setminus S_u$ the functions $u * \rho_\varepsilon(x)$ converge to $\tilde{u}(x)$ as $\varepsilon \downarrow 0$;

(c) if $f : \mathbf{R}^m \to \mathbf{R}^p$ is a Lipschitz map and $v = f \circ u$, then $S_v \subset S_u$ and $\tilde{v}(x) = f(\tilde{u}(x))$ for any $x \in \Omega \setminus S_u$.

Proof Since the complement of the set of Lebesgue points of u is $\mathcal{L}^N$-negligible, we infer that S_u is $\mathcal{L}^N$-negligible and $\tilde{u}$ coincides $\mathcal{L}^N$-a.e. with u. One can prove that S_u is a Borel set noticing that

$$\Omega \setminus S_u = \bigcap_{n=1}^{\infty} \bigcup_{q \in \mathbf{Q}^m} \left\{ x \in \Omega : \limsup_{\varrho \downarrow 0} ⨍_{B_\varrho(x)} |u(y) - q| \, dy < \frac{1}{n} \right\}.$$

Indeed, the inclusion $\subset$ is trivial, by the density of $\mathbf{Q}^m$. If x belongs to the set on the right side, then for any integer $n \geq 1$ we can find $q_n \in \mathbf{Q}^m$ such that

$$\limsup_{\varrho \downarrow 0} \fint_{B_\varrho(x)} |u(y) - q_n| \, dy < \frac{1}{n}.$$

It is easily seen that (q_n) is a Cauchy sequence and that its limit z satisfies (3.65), hence $x \notin S_u$. As a consequence of (3.65), for any $x \in \Omega \setminus S_u$ the mean values $u_{B_\varrho(x)}$ of u on $B_\varrho(x)$ converge to $z = \tilde{u}(x)$ as $\varrho \downarrow 0$. Hence, the Borel property of $\tilde{u}$ in $\Omega \setminus S_u$ simply follows from its representation as the pointwise limit as $\varrho \downarrow 0$ of the continuous functions $x \mapsto u_{B_\varrho(x)}$.

Finally, (b) can be proved noticing that

$$|u * \rho_\varepsilon(x) - \tilde{u}(x)| \leq \int_{\mathbf{R}^N} |u(x - \varepsilon z) - \tilde{u}(x)| \rho(z) \, dz \leq \frac{\|\rho\|_\infty}{\varepsilon^N} \int_{B_\varepsilon(x)} |u(y) - \tilde{u}(x)| \, dy$$

(with the change of variables $x - \varepsilon z = y$) and (c) follows at once from the estimate $|v(y) - f(\tilde{u}(x))| \leq \mathrm{Lip}(f) |u(y) - \tilde{u}(x)|$. $\square$

values a and b along a direction ν. To this aim we introduce the convenient notation

$$\begin{cases} B_\varrho^+(x,\nu) := \{y \in B_\varrho(x) \colon \langle y-x, \nu\rangle > 0\} \\ B_\varrho^-(x,\nu) := \{y \in B_\varrho(x) \colon \langle y-x, \nu\rangle < 0\} \end{cases} \tag{3.67}$$

$$u_{a,b,\nu}(y) := \begin{cases} a & \text{if } \langle y, \nu\rangle > 0 \\ b & \text{if } \langle y, \nu\rangle < 0 \end{cases} \tag{3.68}$$

for the two half balls contained in $B_\varrho(x)$ determined by ν and for the function jumping between a and b along the hyperplane orthogonal to ν.

Definition 3.67 (Approximate jump points) Let $u \in [L^1_{\mathrm{loc}}(\Omega)]^m$ and $x \in \Omega$. We say that x is an *approximate jump point of u* if there exist $a,\ b \in \mathbf{R}^m$ and $\nu \in \mathbf{S}^{N-1}$ such that $a \neq b$ and

$$\lim_{\varrho\downarrow 0} \fint_{B_\varrho^+(x,\nu)} |u(y)-a|\,dy = 0, \qquad \lim_{\varrho\downarrow 0} \fint_{B_\varrho^-(x,\nu)} |u(y)-b|\,dy = 0. \tag{3.69}$$

The triplet (a, b, ν), uniquely determined by (3.69) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$.
The set of approximate jump points is denoted by J_u.

To simplify several statements it is also convenient to say that two triples (a, b, ν), (a', b', ν') are *equivalent* if

$$\text{either} \quad (a,b,\nu) = (a',b',\nu') \qquad \text{or} \quad (a,b,\nu) = (b',a',-\nu'). \tag{3.70}$$

Example 3.68 (Characteristic functions) If u is a characteristic function, say $u = \chi_E$, then S_u is the essential boundary $\partial^* E$ of E introduced in Definition 3.60. In fact, if u has approximate limit z at x, then either $z = 0$ or $z = 1$ because the range of u is $\{0, 1\}$. For similar reasons J_u is a subset of $E^{1/2}$ and $\{u^+(x), u^-(x)\} = \{0, 1\}$ for any $x \in J_u$. The inclusion is strict, as shown by the set $\{xy > 0\} \subset \mathbf{R}^2$ at the origin.

For sets of finite perimeter, De Giorgi theorem implies that the reduced boundary $\mathcal{F}E$ is contained in J_u. Also this inclusion is strict: setting

$$E := \left\{(x,y) \in \mathbf{R}^2 \colon y < \phi(x)\right\} \qquad \text{with} \qquad \phi(x) := x^2 \sin\frac{1}{x}$$

we find that $0 \in J\chi_E$ (because $\phi'(0) = 0$) but condition (3.57) fails to be satisfied at $x_0 = 0$. Hence, the origin does not belong to $\mathcal{F}E$. However, by Theorem 3.61 the sets

$$\mathcal{F}E \cap \Omega, \quad J_{\chi_E} \cap \Omega, \quad E^{1/2} \cap \Omega, \quad \partial^* E \cap \Omega$$

have the same $\mathcal{H}^{N-1}$ measure if E has finite perimeter in Ω.

Now we state the main properties of the approximate jump set J_u and of the triplets $\big(u^+(x), u^-(x), \nu_u(x)\big)$.

Proposition 3.69 *Let* $u \in [L^1_{\mathrm{loc}}(\Omega)]^m$.

(a) *The set* J_u *is a Borel subset of* S_u *and there exist Borel functions*

$$\big(u^+(x), u^-(x), \nu_u(x)\big) : J_u \to \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{S}^{N-1}$$

such that (3.69) *is fulfilled at any* $x \in J_u$;

(b) *if* $x \in J_u$ *the functions* $u * \rho_\varepsilon(x)$ *converge to* $[u^+(x) + u^-(x)]/2$ *as* $\varepsilon \downarrow 0$;

(c) *if* $f : \mathbf{R}^m \to \mathbf{R}^p$ *is a Lipschitz map,* $v = f \circ u$ *and* $x \in J_u$, *then* $x \in J_v$ *if and only if* $f(u^+(x)) \neq f(u^-(x))$, *and in this case*

$$\big(v^+(x), v^-(x), \nu_v(x)\big) = \big(f(u^+(x)), f(u^-(x)), \nu_u(x)\big).$$

Otherwise, $x \notin S_v$ *and* $\tilde{v}(x) = f(u^+(x)) = f(u^-(x))$.

Proof (a) Let $D = \{(a_n, b_n, \nu_n)\}$ be a countable dense set in $\mathbf{R}^m \times \mathbf{R}^m \times \mathbf{S}^{N-1}$ and let $w_n(y) = u_{a_n, b_n, \nu_n}$, according to (3.68). Then, the same argument used in the proof of Proposition 3.64(a) shows that

$$(\Omega \setminus S_u) \cup J_u = \bigcap_{p=1}^{\infty} \bigcup_{n=0}^{\infty} \left\{ x \in \Omega :\ \limsup_{\varrho \downarrow 0} \fint_{B_\varrho} |u(x+y) - w_n(y)|\, dy < \frac{1}{p} \right\}.$$

Since the right side is a Borel set and $J_u \subset S_u$ we infer that J_u is a Borel set. Let us select for any $x \in J_u$ a triplet $\big(\bar{u}^+(x), \bar{u}^-(x), \bar{\nu}_u(x)\big)$ satisfying the conditions of Definition 3.67 (notice that $\bar{\nu}_u$ need not to be a Borel function, because the sign of $\bar{\nu}_u$ is not uniquely determined) and let us prove that $x \mapsto \phi(x) = \big(\bar{u}^+(x) - \bar{u}^-(x)\big) \otimes \bar{\nu}_u(x)$ is a Borel map in J_u; to this aim we define

$$w_x(y) := \begin{cases} \bar{u}^+(x) & \text{if } \langle y, \bar{\nu}_u(x) \rangle > 0 \\ \bar{u}^-(x) & \text{if } \langle y, \bar{\nu}_u(x) \rangle < 0 \end{cases}$$

and notice (cf. Remark 3.72) that the rescaled functions $u^{x,\varrho}(y) = u(x + \varrho y)$ converge in $[L^1_{\mathrm{loc}}(\mathbf{R}^N)]^m$ to w_x as $\varrho \downarrow 0$. In particular

$$x \mapsto \int_{B_1} w_x(y) \otimes \nabla\psi(y)\, dy = \lim_{\varrho \downarrow 0} \varrho^{-N} \int_{B_\varrho(x)} u(y) \otimes \nabla\psi\left(\frac{y-x}{\varrho}\right) dy$$

is a Borel map in J_u for any $\psi \in C^\infty_c(B_1)$. Taking a sequence $(\psi_h) \subset C^\infty_c(B_1)$ monotonically converging to χ_{B_1} we get

$$\omega_{N-1}\phi(x) = Dw_x(B_1) = \lim_{h \to \infty} \int_{B_1} \psi_h(y)\, dDw_x(y)$$
$$= -\lim_{h \to \infty} \int_{B_1} w_x(y) \otimes \nabla\psi_h(y)\, dy$$

and this proves that ϕ is a Borel map. For any $\alpha \in \{1, \ldots, m\}$, let E_α be the set of all $x \in J_u$ such that α is the least index such that the α-th row of $\phi(x)$ is nonzero. Since ϕ is a Borel map it can be easily seen that $\{E_\alpha\}$ is a Borel partition of J_u. On any set E_α we can define ν_u as $\phi^\alpha/|\phi^\alpha|$; this defines a Borel map on J_u. Accordingly, since ν_u and $\bar{\nu}_u$ are either equal or opposite, we can define $\big(u^+(x), u^-(x)\big)$ to be equal to $\big(\bar{u}^+(x), \bar{u}^-(x)\big)$ if $\nu_u(x) = \bar{\nu}_u(x)$ and to be equal to $\big(\bar{u}^-(x), \bar{u}^+(x)\big)$ if $\nu_u(x) = -\bar{\nu}_u(x)$. An argument similar to the one used in the proof of Proposition 3.64(a) with $B^\pm_\varrho(x, \nu_u(x))$ instead of $B_\varrho(x, \nu_u(x))$ shows that $u^\pm$ are Borel maps in J_u.

(b)–(c) The proof is analogous to the one of Proposition 3.64, splitting the region of integration in $B^+_\varrho(x, \nu)$ and $B^-_\varrho(x, \nu)$. □