Now we show that, for general domains Ω, the approximability by smooth functions with gradients bounded in L^1 actually characterises BV functions. This theorem could be considered as the BV version of the classical Meyers–Serrin theorem for Sobolev spaces, stating the coincidence of weak and strong derivatives (see [214]).

Theorem 3.9 (Approximation by smooth functions) *Let $u \in [L^1(\Omega)]^m$. Then, $u \in [BV(\Omega)]^m$ if and only if there exists a sequence $(u_h) \subset [C^\infty(\Omega)]^m$ converging to u in $[L^1(\Omega)]^m$ and satisfying*

$$L := \lim_{h\to\infty} \int_\Omega |\nabla u_h|\, dx < \infty. \tag{3.6}$$

Moreover, the least constant L in (3.6) *is $|Du|(\Omega)$.*

Proof Assume that u can be approximated in $[L^1(\Omega)]^m$ by smooth functions satisfying (3.6). Possibly extracting a subsequence, by Theorem 1.59 we can assume that the measures $\nabla u_h \mathcal{L}^N$ weakly* converge in Ω to some $\mathbf{R}^{mN}$-valued measure μ in Ω such that $|\mu|(\Omega) \le L$. Passing to the limit as $h \to \infty$ in the classical integration by parts formula

$$\int_\Omega u_h^\alpha \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \phi \frac{\partial u_h^\alpha}{\partial x_i}\, dx \qquad \forall \phi \in C_c^1(\Omega),\ i = 1, \dots, N,\ \alpha = 1, \dots, m$$

we obtain that (3.3) is fulfilled with $Du = \mu$, i.e. $u \in [BV(\Omega)]^m$. In particular $|Du|(\Omega) = |\mu|(\Omega) \le L$.

Assuming now $u \in [BV(\Omega)]^m$ we construct for any $\delta > 0$ a function $v_\delta \in [C^\infty(\Omega)]^m$ such that

$$\int_\Omega |u - v_\delta|\, dx < \delta, \qquad \int_\Omega |\nabla v_\delta|\, dx \le |Du|(\Omega) + \delta. \tag{3.7}$$

To this aim, we notice that Ω can be written as the union of a countable family of sets $\{\Omega_h\}_{h\ge 1}$ with compact closure in Ω and such that any point of Ω belongs to at most four sets Ω_h. For instance, this family can be obtained setting

$$\Omega_{k,1} = \left\{x \in \Omega \cap B_{k+1} \setminus \overline{B}_{k-1} \colon \operatorname{dist}(x, \partial\Omega) > 1/2\right\}$$

and

$$\Omega_{k,p} := \left\{x \in \Omega \cap B_{k+1} \setminus \overline{B}_{k-1} \colon \frac{1}{p-1} > \operatorname{dist}(x, \partial\Omega) > \frac{1}{p+1}\right\}$$

for integers $k \ge 1$, $p > 1$, where $B_0 = \emptyset$. Choosing a partition of unity relative to the covering Ω_h, i.e. positive functions $\varphi_h \in C_c^\infty(\Omega_h)$ such that $\sum_h \varphi_h \equiv 1$ in Ω, for any integer $h \ge 1$ we can find $\varepsilon_h > 0$ such that $\operatorname{supp}\left((u\varphi_h) * \rho_{\varepsilon_h}\right) \subset \Omega_h$ and

$$\int_\Omega \left[|(u\varphi_h) * \rho_{\varepsilon_h} - u\varphi_h| + |(u \otimes \nabla\varphi_h) * \rho_{\varepsilon_h} - u \otimes \nabla\varphi_h|\right] dx < 2^{-h}\delta. \tag{3.8}$$

The function $v_\delta = \sum_h (u\varphi_h) * \rho_{\varepsilon_h}$ is smooth in Ω because the sum is locally finite; moreover, our choice of ε_h gives

$$\int_\Omega |v_\delta - u|\, dx \le \sum_{h=1}^{\infty} \int_\Omega \left|(u\varphi_h) * \rho_{\varepsilon_h} - u\varphi_h\right| dx < \delta.$$

Now we evaluate $|Dv_\delta|(\Omega)$; by Proposition 3.2(b), we obtain

$$\begin{aligned}
\nabla v_\delta &= \sum_{h=1}^{\infty} \nabla\left((u\varphi_h) * \rho_{\varepsilon_h}\right) = \sum_{h=1}^{\infty} (D(u\varphi_h)) * \rho_{\varepsilon_h} \\
&= \sum_{h=1}^{\infty} (\varphi_h Du) * \rho_{\varepsilon_h} + \sum_{h=1}^{\infty} (u \otimes \nabla\varphi_h) * \rho_{\varepsilon_h} \\
&= \sum_{h=1}^{\infty} (\varphi_h Du) * \rho_{\varepsilon_h} + \sum_{h=1}^{\infty} \left[(u \otimes \nabla\varphi_h) * \rho_{\varepsilon_h} - u \otimes \nabla\varphi_h\right].
\end{aligned}$$

Using (3.8) and Theorem 2.2(b), by integration we obtain

$$|Dv_\delta|(\Omega) = \int_\Omega |\nabla v_\delta|\, dx < \delta + \sum_{h=1}^{\infty} \int_\Omega \varphi_h |Du| = \delta + |Du|(\Omega).$$

This proves the existence of v_δ. Choosing $\delta_h = 2^{-h}$ and setting $u_h = v_{\delta_h}$, from (3.7) we infer

$$\lim_{h\to\infty} \int_\Omega |u_h - u|\, dx = 0, \qquad \limsup_{h\to\infty} \int_\Omega |\nabla u_h|\, dx \le |Du|(\Omega).$$

The lower semicontinuity of variation implies that $|Du_h|(\Omega)$ (i.e. $\int_\Omega |\nabla u_h|\, dx$) converge to $|Du|(\Omega)$, hence $|Du|(\Omega)$ is the least constant in (3.6). □

Remark 3.22 Theorem 3.9 says that for every open set Ω the space $[C^\infty(\Omega)]^m \cap [BV(\Omega)]^m$ is dense in $[BV(\Omega)]^m$, endowed with the topology induced by strict convergence. If Ω is an extension domain we can say something more: let $u \in [BV(\Omega)]^m$, let Tu be an extension operator and let u_ε be the mollified functions $(Tu) * \rho_\varepsilon$. We know by Proposition 3.7 and the remarks preceding it that u_ε converge to Tu in $[L^1(\mathbf{R}^N)]^m$ and $|Du_\varepsilon|(\Omega)$ converge to $|DTu|(\Omega)$ as $\varepsilon \downarrow 0$, because $|DTu|(\partial\Omega) = 0$. Since Tu coincides with u in Ω, this proves the possibility of approximating u in the strict convergence by functions in $[C^\infty(\overline{\Omega})]^m$.

The following compactness theorem for BV functions is very useful in connexion with variational problems with linear growth in the gradient (e.g. least area problems for cartesian hypersurfaces, see [175]). Since the Sobolev space $W^{1,1}$ has no similar compactness property this provides also a justification for the introduction of BV spaces in calculus of variations.

Theorem 3.23 (Compactness in BV) *Every sequence $(u_h) \subset [BV_{\rm loc}(\Omega)]^m$ satisfying*

$$\sup\left\{\int_A |u_h|\,dx + |Du_h|(A) \colon h \in \mathbf{N}\right\} < \infty \qquad \forall A \subset\subset \Omega \text{ open}$$

admits a subsequence $(u_{h(k)})$ converging in $[L^1_{\rm loc}(\Omega)]^m$ to $u \in [BV_{\rm loc}(\Omega)]^m$. If Ω is a bounded extension domain and the sequence is bounded in $[BV(\Omega)]^m$ we can say that $u \in [BV(\Omega)]^m$ and that the subsequence weakly converges to u.*

Proof Let $\Omega' \subset\subset \Omega$ be an open set. By the same diagonal argument described before Corollary 1.60 we need only to show the existence of a subsequence $(u_{h(k)})$ converging in $[L^1(\Omega')]^m$ to some function u (notice that $u \in [BV(\Omega')]^m$ by (3.11)).

Let $\delta = \operatorname{dist}(\Omega', \partial\Omega) > 0$, $U \subset \Omega$ the open $\delta/2$ neighbourhood of Ω' and let $u_{h,\varepsilon} = u_h * \rho_\varepsilon$. If $\varepsilon \in (0, \delta/2)$ the functions $u_{h,\varepsilon}$ are smooth in $\overline{\Omega'}$ and satisfy

$$\|u_{h,\varepsilon}\|_{C(\overline{\Omega'})} \le \|u_h\|_{L^1(U)}\|\rho_\varepsilon\|_\infty, \qquad \|\nabla u_{h,\varepsilon}\|_{C(\overline{\Omega'})} \le \|u_h\|_{L^1(U)}\|\nabla\rho_\varepsilon\|_\infty.$$

By our assumption on (u_h), the sequence $(u_{h,\varepsilon})$ is equibounded and equicontinuous for ε fixed. This means that, with ε fixed, we can find converging subsequences of $(u_{h,\varepsilon})$ in $C(\overline{\Omega'})$. By a diagonal argument we can find a subsequence $(h(k))$ such that $(u_{h(k),\varepsilon})$ converges in $C(\overline{\Omega'})$ for any $\varepsilon = 1/p$, with $p > 2/\delta$ integer. Applying Lemma 3.24 below we find

$$\begin{aligned}
\limsup_{k,\,k'\to\infty} \int_{\Omega'} |u_{h(k)} - u_{h(k')}|\,dx &\le \limsup_{k,\,k'\to\infty} \int_{\Omega'} |u_{h(k),1/p} - u_{h(k'),1/p}|\,dx \\
&+ \limsup_{k,\,k'\to\infty} \int_{\Omega'} \left[|u_{h(k)} - u_{h(k),1/p}| + |u_{h(k'),1/p} - u_{h(k')}|\right] dx \\
&\le \frac{2}{p} \sup_{h\in\mathbf{N}} |Du_h|(U).
\end{aligned}$$

Since arbitrarily large p can be chosen and $L^1(\Omega')$ is a Banach space, this proves that $(u_{h(k)})$ converges in $L^1(\Omega')$.

Finally, we prove the last part of the statement. If we assume that Ω is a bounded extension domain, we can apply the first part of the statement to the extensions $Tu_h \in [BV(\mathbf{R}^N)]^m$ to obtain $[L^1_{\mathrm{loc}}(\mathbf{R}^N)]^m$ convergence of a subsequence $(Tu_{h(k)})$ to some function $u \in [BV(\mathbf{R}^N)]^m$. In particular $(u_{h(k)})$ converges in $[L^1(\Omega)]^m$ to u and $u \in [BV(\Omega)]^m$ by (3.11). The weak* convergence follows at once from (3.11). □

Lemma 3.24 *Let $u \in [BV(\Omega)]^m$ and $K \subset \Omega$ a compact set. Then*

$$\int_K |u * \rho_\varepsilon - u|\,dx \le \varepsilon |Du|(\Omega) \qquad \forall \varepsilon \in (0, \mathrm{dist}(K, \partial\Omega)).$$

Proof By Theorem 3.9 we can assume without loss of generality that $u \in [C^1(\Omega)]^m$. Starting from the identity

$$u(x - \varepsilon y) - u(x) = -\varepsilon \int_0^1 \langle \nabla u(x - \varepsilon t y), y\rangle\,dt \qquad x \in K,\ \ y \in B_1$$

we can take norms in both sides, integrate with respect to x and use Fubini's theorem to obtain

$$\int_K |u(x - \varepsilon y) - u(x)|\,dx \le \varepsilon \int_0^1 \int_K |\nabla u(x - \varepsilon t y)|\,dxdt \le \varepsilon |Du|(\Omega).$$

Multiplying both sides by $\rho(y)$ and integrating we obtain

$$\int_K \left(\int_{\mathbf{R}^N} |u(x - \varepsilon y) - u(x)| \rho(y)\,dy \right) dx \le \varepsilon |Du|(\Omega). \tag{3.21}$$

Since $u * \rho_\varepsilon(x) - u(x)$ is equal to $\int_{\mathbf{R}^N} [u(x - \varepsilon y) - u(x)]\rho(y)\,dy$, which can be estimated with the integral among parentheses in (3.21), the statement follows. □