that for these functions the distributional derivative is representable by integration with respect to $\mathcal{H}^{N-1}$, and that for $\mathcal{H}^{N-1}$-almost every point the N-dimensional density of the set belongs to $\{0, 1/2, 1\}$. In order to find the proper extension to higher dimensions of the continuity and differentiability properties presented in Section 3.2, we introduce in Section 3.6 suitable notions of approximate limit, approximate jump, approximate differentiability. In Section 3.7 we study several fine properties of BV functions, proving in particular the existence of one-sided approximate limits on countably $\mathcal{H}^{N-1}$-rectifiable sets inside the domain, that $\mathcal{H}^{N-1}$-almost every approximate discontinuity point is an approximate jump point, the approximate differentiability and the characterisation of the approximate differential as the density of the absolutely continuous part of distributional derivative with respect to $\mathcal{L}^N$. In Section 3.8 we show some decomposability properties of BV spaces and prove the existence of traces on the boundary of the domain.

The next three sections contain more precise results on the structure of the distributional derivative. In Section 3.9 we split the distributional derivative into three parts, an absolutely continuous part, a jump part and a Cantor part. We prove several properties of these parts of the derivative, showing in particular that the absolutely continuous part and the jump part can be recovered, unlike the Cantor part, from a suitable blow-up analysis of the behaviour of the function. Section 3.10 is devoted to the chain rule in BV, i.e. the behaviour of the distributional derivative under Lipschitz transformations in the dependent variable. In Section 3.11 we systematically study restrictions of BV functions of N variables to one-dimensional sections, showing that the global distributional derivative can be recovered, by a disintegration method, from the distributional derivatives of the one-dimensional restrictions. The same holds for the three components of distributional derivative, the approximate jump set and the approximate one-sided limits.

Finally, in the last section we sketch the history of BV functions, from their definition up to discovery of the main fine properties and trace theorems.

3.1 The space BV

Throughout this chapter we denote by Ω a generic open set in $\mathbf{R}^N$. We begin this section with the most common definition of $BV(\Omega)$, based on the existence of a measure distributional derivative.

Definition 3.1 Let $u \in L^1(\Omega)$; we say that u is a *function of bounded variation in* Ω if the distributional derivative of u is representable by a finite Radon measure in Ω, i.e. if

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \phi \, dD_i u \qquad \forall \phi \in C_c^\infty(\Omega), \quad i = 1, \dots, N \tag{3.1}$$

for some $\mathbf{R}^N$-valued measure $Du = (D_1 u, \dots, D_N u)$ in Ω. The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.

Some remarks on this definition are in order. First, a smoothing argument shows that the integration by parts formulae (3.1) are still true for any $\phi \in C_c^1(\Omega)$, or even for

Lipschitz functions ϕ with compact support in Ω. These formulae can be summarised in a single one by writing

$$\int_\Omega u \operatorname{div}\varphi \, dx = -\sum_{i=1}^{N} \int_\Omega \varphi_i \, dD_i u \qquad \forall \varphi \in \left[C_c^1(\Omega)\right]^N. \tag{3.2}$$

We use the same notation also for functions $u \in [BV(\Omega)]^m$; in this case Du is an $m \times N$ matrix of measures $D_i u^\alpha$ in Ω satisfying

$$\int_\Omega u^\alpha \frac{\partial \phi}{\partial x_i} \, dx = -\int_\Omega \phi \, dD_i u^\alpha \qquad \forall \phi \in C_c^1(\Omega), \; i = 1, \dots, N, \; \alpha = 1, \dots, m \tag{3.3}$$

or, equivalently,

$$\sum_{\alpha=1}^{m} \int_\Omega u^\alpha \operatorname{div}\varphi^\alpha \, dx = -\sum_{\alpha=1}^{m}\sum_{i=1}^{N} \int_\Omega \varphi_i^\alpha \, dD_i u^\alpha \qquad \forall \varphi \in [C_c^1(\Omega)]^{mN}. \tag{3.4}$$

The Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$; indeed, for any $u \in W^{1,1}(\Omega)$ the distributional derivative is given by $\nabla u \mathcal{L}^N$. This inclusion is strict: there exist functions $u \in BV(\Omega)$ such that Du is singular with respect to $\mathcal{L}^N$ (for instance the Heaviside function $\chi_{(0,\infty)}$, whose distributional derivative is the Dirac measure δ_0). Our notation Du for the distributional derivative is motivated by the necessity to keep it distinct from the (approximate) pointwise differential ∇u, introduced in Section 3.6 and representing, by Theorem 3.83, only the density of Du with respect to $\mathcal{L}^N$.

Simple but useful properties of the distributional derivative are stated in the following proposition.

Proposition 3.2 (Properties of Du) *Let $u \in [BV_{\mathrm{loc}}(\Omega)]^m$.*

(a) *If $Du = 0$, u is (equivalent to a) constant in any connected component of Ω.*

(b) *For any locally Lipschitz function $\psi : \Omega \to \mathbf{R}$ the function $u\psi$ belongs to $[BV_{\mathrm{loc}}(\Omega)]^m$ and*

$$D(u\psi) = \psi Du + (u \otimes \nabla\psi)\mathcal{L}^N.$$

(c) *If ρ is any convolution kernel and $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, then*

$$\nabla(u * \rho_\varepsilon) = Du * \rho_\varepsilon \qquad \text{in } \Omega_\varepsilon.$$

Proof (a) follows from (c) and a smoothing argument, while the verification of (b) is straightforward. To prove (c) it suffices to notice that (2.2) and the convolution identity (2.3) (applied first with $\mu = \nabla\psi\mathcal{L}^N$, then with $\mu = Du$) give

$$\begin{aligned}\int_\Omega (u * \rho_\varepsilon)\nabla\psi \, dx &= \int_\Omega u(\rho_\varepsilon * \nabla\psi) \, dx = \int_\Omega u\nabla(\psi * \rho_\varepsilon) \, dx \\ &= -\int_\Omega (\psi * \rho_\varepsilon) \, dDu = -\int_\Omega (Du * \rho_\varepsilon)\psi \, dx \qquad \forall \psi \in C_c^\infty(\Omega_\varepsilon).\end{aligned}$$

□

One of the main advantages of the BV space is that it includes, unlike Sobolev spaces, characteristic functions of sufficiently regular sets and, more generally, piecewise smooth functions. The following important example motivates the definition in Section 4.1 of the space SBV of special functions of bounded variation.

Example 3.3 Let $\Omega \subset \mathbf{R}^2$ be a bounded open set and let us assume the existence of pairwise disjoint open sets with piecewise C^1 boundary $\{\Omega_i\}_{1\le i\le p}$ such that

$$\bigcup_{i=1}^{p} \Omega_i \subset \Omega \subset \bigcup_{i=1}^{p} \overline{\Omega}_i.$$

If $u_i \in C^1(\overline{\Omega}_i)$, we can define $u : \Omega \to \mathbf{R}$ to be equal to u_i on any subdomain Ω_i, and define it arbitrarily on the remaining negligible set Σ. By applying the Gauss–Green theorem to any domain Ω_i, for $i = 1, \dots, p$, we find

$$\int_{\Omega_i} u \operatorname{div}\varphi \, dx = -\int_{\Omega_i} \langle \nabla u, \varphi \rangle \, dx - \int_{\partial\Omega_i} u_i \langle \varphi, \nu_i \rangle \, d\mathcal{H}^1 \qquad \forall \varphi \in [C^1(\overline{\Omega}_i)]^2$$

where ν_i is the inner unit normal to Ω_i. Adding with respect to i these identities we find that $u \in BV(\Omega)$, with Du given by

$$\nabla u \mathcal{L}^2 + \sum_{i=1}^{p} u_i \nu_i \mathcal{H}^1 \llcorner (\Omega \cap \partial\Omega_i).$$

Now we introduce the so-called *variation* $V(u, \Omega)$ of a function $u \in [L^1_{\mathrm{loc}}(\Omega)]^m$. The variation can be infinite, and we will see that a function $u \in [L^1(\Omega)]^m$ belongs to $[BV(\Omega)]^m$ if and only if $V(u, \Omega) < \infty$. Since $u \mapsto V(u, \Omega)$ is lower semicontinuous in the $[L^1_{\mathrm{loc}}(\Omega)]^m$ topology (cf. Remark 3.5 below), this provides a useful method of showing that some function $u \in [L^1(\Omega)]^m$ belongs to $[BV(\Omega)]^m$: one needs only to approximate u in $[L^1_{\mathrm{loc}}(\Omega)]^m$ by functions (u_h) whose variations $V(u_h, \Omega)$ are equibounded.

Definition 3.4 (Variation) Let $u \in [L^1_{\mathrm{loc}}(\Omega)]^m$. The *variation* $V(u, \Omega)$ of u in Ω is defined by

$$V(u, \Omega) := \sup \left\{ \sum_{\alpha=1}^{m} \int_{\Omega} u^\alpha \operatorname{div}\varphi^\alpha \, dx : \varphi \in [C^1_c(\Omega)]^{mN}, \ \|\varphi\|_\infty \le 1 \right\}.$$

A simple integration by parts proves that $V(u, \Omega) = \int_\Omega |\nabla u| \, dx$ if u is continuously differentiable in Ω. Other useful properties of the variation are listed in the following remark.

Remark 3.5 (Properties of the variation)
(Lower semicontinuity) The mapping $u \mapsto V(u, \Omega) \in [0, \infty]$ is lower semicontinuous in the $[L^1_{\mathrm{loc}}(\Omega)]^m$ topology. To check this, we need only to notice that

$$u \mapsto \sum_{\alpha=1}^{m} \int_{\Omega} u^\alpha \operatorname{div}\varphi^\alpha \, dx$$

is continuous in the $[L^1_{\mathrm{loc}}(\Omega)]^m$ topology for any choice of $\varphi \in [C^1_c(\Omega)]^{mN}$.

(Additivity) Notice also that $V(u, A)$ is defined for any open set $A \subset \Omega$ (in this case the test vector fields φ must be supported in A); it can be proved (see Exercise 3.1) that

$$\tilde{V}(u, B) = \inf\{V(u, A) \colon A \supset B, \quad A \text{ open}\} \qquad B \in \mathcal{B}(\Omega)$$

extends $V(u, \cdot)$ to a Borel measure in Ω.
(Locality) The mapping $u \mapsto V(u, A)$ is also local, i.e. $V(u, A) = V(v, A)$ if u coincides with v $\mathcal{L}^N$-a.e. in $A \subset \Omega$.

Proposition 3.6 (Variation of BV functions) *Let $u \in [L^1(\Omega)]^m$. Then, u belongs to $[BV(\Omega)]^m$ if and only if $V(u, \Omega) < \infty$. In addition, $V(u, \Omega)$ coincides with $|Du|(\Omega)$ for any $u \in [BV(\Omega)]^m$ and $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $[BV(\Omega)]^m$ with respect to the $[L^1_{\text{loc}}(\Omega)]^m$ topology.*

Proof If $u \in [BV(\Omega)]^m$ we can estimate the supremum defining $V(u, \Omega)$ observing that

$$\sum_{\alpha=1}^{m} \int_{\Omega} u^{\alpha} \text{div}\varphi^{\alpha}\, dx = -\sum_{i=1}^{N}\sum_{\alpha=1}^{m} \int_{\Omega} \varphi_i^{\alpha}\, dD_i u^{\alpha}$$

for any $\varphi \in [C_c^1(\Omega)]^{mN}$. Since in the computation of $V(u, \Omega)$ we require that $\|\varphi\|_{\infty} \le 1$, from Proposition 1.47 we infer that $V(u, \Omega) \le |Du|(\Omega) < \infty$.

Conversely, if $V(u, \Omega) < \infty$ a homogeneity argument shows that

$$\left|\sum_{\alpha=1}^{m} \int_{\Omega} u^{\alpha} \text{div}\varphi^{\alpha}\, dx\right| \le V(u, \Omega)\|\varphi\|_{\infty} \qquad \forall \varphi \in [C_c^1(\Omega)]^{mN}.$$

Since $[C_c^1(\Omega)]^{mN}$ is dense in $[C_0(\Omega)]^{mN}$, we can find a continuous linear functional L on $[C_0(\Omega)]^{mN}$ coinciding with

$$\varphi \mapsto \sum_{\alpha=1}^{m} \int_{\Omega} u^{\alpha} \text{div}\varphi^{\alpha}\, dx$$

on $[C_c^1(\Omega)]^{mN}$ and satisfying $\|L\| \le V(u, \Omega)$. By the Riesz theorem, there exists a $\mathbf{R}^{mN}$-valued finite Radon measure $\mu = (\mu_i^{\alpha})$ such that $\|L\| = |\mu|(\Omega)$ and

$$L(\varphi) = \sum_{i=1}^{N}\sum_{\alpha=1}^{m} \int_{\Omega} \varphi_i^{\alpha}\, d\mu_i^{\alpha} \qquad \forall \varphi \in [C_0(\Omega)]^{mN}.$$

From (3.4) and the identity

$$\sum_{\alpha=1}^{m} \int_{\Omega} u^{\alpha}\, \text{div}\,\varphi^{\alpha}\, dx = \sum_{i=1}^{N}\sum_{\alpha=1}^{m} \int_{\Omega} \varphi_i^{\alpha}\, d\mu_i^{\alpha} \qquad \forall \varphi \in [C_c^1(\Omega)]^{mN}$$

we obtain that $u \in [BV(\Omega)]^m$, $Du = -\mu$ and

$$|Du|(\Omega) = |\mu|(\Omega) = \|L\| \le V(u, \Omega)\ .$$

Finally, the lower semicontinuity of $u \mapsto |Du|(\Omega)$ follows directly from Remark 3.5.

□

Motivated by Proposition 3.6, also $|Du|(\Omega)$ will be sometimes called the variation of u in Ω; however, unlike $V(u, \Omega)$, the expression $|Du|(\Omega)$ will be used for BV functions only. We notice that $[BV(\Omega)]^m$, endowed with the norm

$$\|u\|_{BV} := \int_\Omega |u|\,dx + |Du|(\Omega)$$

is a Banach space, but the norm-topology is too strong for many applications. Indeed, one can notice that, even for $m = 1$, continuously differentiable functions are not dense in $BV(\Omega)$; one can consider any $u \in BV(\Omega)$ such that Du is not zero and singular with respect to $\mathcal{L}^N$ and notice that

$$|D(u - v)|(\Omega) = |Du|(\Omega) + |Dv|(\Omega) \geq |Du|(\Omega) > 0$$

for any $v \in C^1(\Omega) \cap BV(\Omega)$. This is true because, as it can be easily checked, $|\lambda - \mu| = |\lambda| + |\mu|$ for mutually singular measures λ, μ.

However, $[BV(\Omega)]^m$ functions can be approximated, in the $[L^1(\Omega)]^m$ topology, by smooth functions whose gradients are bounded in $[L^1(\Omega)]^m$. To see this, assume first that $\Omega = \mathbf{R}^N$. Let ρ be a convolution kernel and let $u_\varepsilon = u * \rho_\varepsilon$ be the mollified functions. Recalling Proposition 3.2(c), Theorem 2.2(b) gives

$$|Du_\varepsilon|(\mathbf{R}^N) = \int_{\mathbf{R}^N} |\nabla u_\varepsilon|\,dx = \int_{\mathbf{R}^N} |Du * \rho_\varepsilon|\,dx \leq |Du|(\mathbf{R}^N).$$

In particular, by the lower semicontinuity of the variation, $|Du_\varepsilon|(\mathbf{R}^N)$ converges to $|Du|(\mathbf{R}^N)$ as $\varepsilon \downarrow 0$. A local version of this result is the following.

Proposition 3.7 *Let $u \in [BV(\Omega)]^m$ and let $U \subset\subset \Omega$ such that $|Du|(\partial U) = 0$. Then,*

$$\lim_{\varepsilon \downarrow 0} |Du_\varepsilon|(U) = |Du|(U).$$

Proof By the lower semicontinuity of the variation, we have $\liminf_\varepsilon |Du_\varepsilon|(U) \geq |Du|(U)$. On the other hand, denoting by U_ε the open ε-neighbourhood of U, from Theorem 2.2(b) we infer

$$\limsup_{\varepsilon \downarrow 0} |Du_\varepsilon|(U) \leq \limsup_{\varepsilon \downarrow 0} |Du|(U_\varepsilon) = |Du|(\overline{U}) = |Du|(U).$$

□

Remark 3.8 In particular

$$\lim_{\varepsilon \downarrow 0} |Du_\varepsilon|(B_\varrho(x)) = |Du|(B_\varrho(x)) \tag{3.5}$$

for any ball $B_\varrho(x) \subset\subset \Omega$ such that $|Du|(\partial B_\varrho(x)) = 0$. This continuity property is quite useful because, given x, the set of all $\varrho > 0$ such that $B_\varrho(x) \subset\subset \Omega$ and $|Du|(\partial B_\varrho(x)) > 0$ is at most countable (cf. Example 1.63).