Pairings Between Measures and Bounded Functions and Compensated Compactness (*).

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Summary. – *For all vectorfields $\psi \in L^\infty(\Omega, \boldsymbol{R}^n)$ whose divergence is in $L^n(\Omega)$ and for all vector measures μ in Ω whose curl is a measure we define a real valued measure (ψ, μ) in Ω, that can be considered a suitable generalization of the scalar product of ψ and μ. Several properties of the pairing (ψ, μ) are then obtained.*

Introduction.

The integral of a function f with respect to a Radon measure β is defined for instance when f is continuous, or, more generally, when f is β-measurable and summable; it is also quite clear that the integral $\langle f, \beta\rangle$ cannot be defined for a general Lebesgue-measurable (even if bounded) function f. However, we shall see that if $\mu \in M(\Omega, \boldsymbol{R}^n)$ is a $\boldsymbol{R}^n$-valued Radon measure on an open set $\Omega \subset \boldsymbol{R}^n$ and if $\psi \in L^\infty(\Omega, \boldsymbol{R}^n)$, then one can define a real valued measure (ψ, μ) on Ω, that works nicely as the scalar product of ψ and μ, provided one assumes also that

$$\text{(0.1)} \qquad \operatorname{rot} \mu = \left\{\frac{\partial \mu_i}{\partial x_j} - \frac{\partial \mu_j}{\partial x_i}\right\}_{i,j=1,\ldots,n} \quad \text{is a measure in } \Omega$$

$$\text{(0.2)} \qquad \operatorname{div} \psi \in L^n(\Omega)\,.$$

We notice that the hypothesis (0.1) is certainly satisfied in the special case that $\mu = Du$ and $u \in BV(\Omega)$. This special case is the first to be investigated, in sections 1 and 2. We remark that pairings of this type, between admissible stresses and strains $\sigma, \varepsilon(u)$ in elasto-plasticity, have been already considered in [1], [8], [2].

In section 3, we define and study the pairing (ψ, μ) in the general case. Certainly, hypotheses (0.1), (0.2) remind one of compensated compactness, and, in fact, we have also a result (theorem 4.1) that extends to our pairing (ψ, μ) the result of MURAT ([10], theorem 2). Actually, both the proof of theorem 4.1 and the definition of (ψ, μ) depend on a suitable explicit solution of the equation

$$\operatorname{rot} z = \lambda$$

(where λ is a given measure) which is obtained as in [10].

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In the appendix we have collected a few approximation and extension results that are needed in the paper.

At the beginning of each section we give an outline of its content.

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1. – The pairings $\langle \psi, \mu \rangle_{\partial\Omega}$, (ψ, Du).

It is well known that summability conditions on the divergence of a vector field ψ in Ω yield trace properties for the normal component of ψ on $\partial\Omega$, for instance compare with [13], [1], [8]. In this section (theorem 1.2) we define a function $[\psi\cdot\nu] \in L^\infty(\partial\Omega)$ which is associated to any vector field $\psi \in L^\infty(\Omega, \boldsymbol{R}^n)$ such that div ψ is a bounded measure in Ω. After that, we define the pairing (ψ, Du), when ψ and u belong to suitable spaces, and we give its first properties. Finally, the expected Green's formula relating $[\psi\cdot\nu]$ and (ψ, Du) is obtained in theorem 1.9, through lemma 1.8.

Let Ω be an open set in $\boldsymbol{R}^n$, $n \geqq 2$, and let p, q be extended real numbers such that $1 \leqq p \leqq n$, $n/(n-1) \leqq q \leqq +\infty$. We shall consider the following spaces:

$$BV(\Omega)_q = BV(\Omega) \cap L^q(\Omega)$$

$$BV(\Omega)_c = BV(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$$

$$X(\Omega)_p \;= \{\psi \in L^\infty(\Omega, \boldsymbol{R}^n) | \operatorname{div} \psi \in L^p(\Omega)\}$$

$$X(\Omega)_\mu \;= \{\psi \in L^\infty(\Omega, \boldsymbol{R}^n) | \operatorname{div} \psi \text{ is a bounded measure in } \Omega\}\,.$$

In the next theorem we define a pairing

$$\langle \psi, u \rangle_{\partial\Omega} \colon X(\Omega)_\mu \times BV(\Omega)_c \to \boldsymbol{R}$$

and in the following theorem 1.2 we show that this pairing can be represented as

$$\langle \psi, u \rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma_\psi(x)\, u(x)\, dH^{n-1}$$

where $\gamma_\psi \in L^\infty(\partial\Omega)$ is a suitable function depending on ψ.

THEOREM 1.1. – *Assume that Ω is bounded and that the boundary of Ω is locally the graph of a Lipschitz function. Denote by $\nu(x)$ the outward unit normal to $\partial\Omega$. Then* *there exists a bilinear map* $\langle\psi, u\rangle_{\partial\Omega}\colon X(\Omega)_\mu \times BV(\Omega)_c \to \boldsymbol{R}$ *such that*

$$\langle\psi, u\rangle_{\partial\Omega} = \int_{\partial\Omega} u(x)\,\psi(x)\cdot\nu(x)\, dH^{n-1} \quad \text{if } \psi \in C^1(\bar{\Omega}, \boldsymbol{R}^n) \tag{1.1}$$

$$|\langle\psi, u\rangle_{\partial\Omega}| \leqq \|\psi\|_{\infty,\Omega}\int_{\partial\Omega} |u(x)|\, dH^{n-1} \quad \text{for all } \psi, u\,. \tag{1.2}$$

PROOF. – In order for (1.1) to be satisfied, we are bound to set

$$\langle\psi, u\rangle_{\partial\Omega} = \int_\Omega u \operatorname{div}\psi\, dx + \int_\Omega \psi\cdot Du\, dx \tag{1.3}$$

for all functions $u \in BV(\Omega)_c \cap H^{1,1}(\Omega)$ and for all vectors $\psi \in X(\Omega)_\mu$. Notice that the last term on the right of (1.3) would not have a defined meaning for general ψ, if Du were just a measure. The map $\langle\psi, u\rangle_{\partial\Omega}$ is clearly bilinear, when it is defined.

Now we remark that if $u, v \in BV(\Omega)_c \cap H^{1,1}(\Omega)$ and $u = v$ on $\partial\Omega$ then one has

$$\langle\psi, u\rangle_{\partial\Omega} = \langle\psi, v\rangle_{\partial\Omega} \quad \text{for all } \psi \in X_\mu(\Omega)\,. \tag{1.4}$$

In fact, by lemma 5.4, one can find a sequence of functions $g_j \in C_0^\infty(\Omega)$ such that, for all $\psi \in X(\Omega)_\mu$, one has

$$\langle\psi, u - v\rangle_{\partial\Omega} = \int_\Omega (u - v) \operatorname{div}\psi\, dx + \int_\Omega \psi\cdot D(u - v)\, dx = \\ = \lim_{j\to\infty}\left\{\int_\Omega g_j \operatorname{div}\psi\, dx + \int_\Omega \psi\cdot Dg_j\, dx\right\} = 0\,.$$

Now we define $\langle\psi, u\rangle_{\partial\Omega}$ for all $u \in BV(\Omega)_c$ by setting

$$\langle\psi, u\rangle_{\partial\Omega} = \langle\psi, w\rangle_{\partial\Omega}$$

where w is any function in $BV(\Omega)_c \cap H^{1,1}(\Omega)$ such that $w = u$ on $\partial\Omega$. This is a valid definition, in view of the preceding remark and because of the extension lemma 5.5.

To prove estimate (1.2), we take a sequence of functions $u_j \in BV(\Omega)_c \cap C^\infty(\Omega)$ that converge to u as in lemma 5.2 (actually, we do not need property 5.10) and we get

$$|\langle\psi, u\rangle_{\partial\Omega}| = |\langle\psi, u_j\rangle_{\partial\Omega}| \leqq \left|\int_\Omega u_j \operatorname{div}\psi\, dx\right| + \|\psi\|_{\infty,\Omega}\int_\Omega |Du_j|$$

for all ψ and for all j, hence, taking the limit for $j \to \infty$ we have

$$|\langle\psi, u\rangle_{\partial\Omega}| \leqq \left|\int_\Omega u \operatorname{div}\psi\, dx\right| + \|\psi\|_{\infty,\Omega}\int_\Omega |Du|\,. \tag{1.5}$$

Now, we take a fixed number $\varepsilon > 0$ and we consider a function w as in lemma 5.5. For such a function we have

$$|\langle\psi, u\rangle_{\partial\Omega}| = |\langle\psi, w\rangle_{\partial\Omega}| \leqq \|w\|_{\infty,\Omega}\int\limits_{\Omega\setminus\Omega_\varepsilon}|\operatorname{div}\psi| + \|\psi\|_{\infty,\Omega}\Big(\int\limits_{\partial\Omega}|u|\,dx + \varepsilon\Big)$$

where $\Omega_\varepsilon = \{x \in \Omega | \operatorname{dist}(w, \partial\Omega) > \varepsilon\}$ and

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega\setminus\Omega_\varepsilon}|\operatorname{div}\psi| = 0$$

because $\operatorname{div}\psi$ is a measure of bounded total variation in Ω. As $\varepsilon > 0$ is arbitrary, estimate (1.3) is proved. q.e.d.

THEOREM 1.2. – *Let Ω be as in theorem* 1.1. *Then there exists a linear operator* $\gamma\colon X(\Omega)_\mu \to L^\infty(\partial\Omega)$ *such that*

(1.6) $$\|\gamma_\psi\|_{\infty,\partial\Omega} \leqq \|\psi\|_{\infty,\Omega}$$

(1.7) $$\langle\psi, u\rangle_{\partial\Omega} = \int\limits_{\partial\Omega}\gamma_\psi(x)u(x)\,dH^{n-1} \qquad \textit{for all } u \in BV(\Omega)_c$$

(1.8) $$\gamma_\psi(x) = \psi(x)\cdot\nu(x) \qquad \textit{for all } x \in \partial\Omega \textit{ if } \psi \in C^1(\bar{\Omega}, \boldsymbol{R}^n)\,.$$

The function $\gamma_\psi(x)$ is a weakly defined trace on $\partial\Omega$ of the normal component of ψ, hence we shall denote $\gamma_\psi(x)$ by $[\psi\cdot\nu](x)$.

PROOF. – Take a fixed $\psi \in X(\Omega)_\mu$ and consider the linear functional $G\colon L^\infty(\partial\Omega) \to \boldsymbol{R}$ defined by

$$G(u) = \langle\psi, w\rangle_{\partial\Omega}$$

where $u \in L^\infty(\partial\Omega)$ and $w \in BV(\Omega)_c$ is such that $w|_{\partial\Omega} = u$. By estimate (1.3) of theorem 1.1 we have

$$|G(u)| \leqq \|\psi\|_{\infty,\Omega}\|u\|_{L^1(\partial\Omega)}$$

hence there exists a function $\gamma_\psi \in L^\infty(\partial\Omega)$ such that

$$G(u) = \int\limits_{\partial\Omega}\gamma_\psi(x)u(x)\,dH^{n-1}$$

and the theorem follows. q.e.d.

Clearly, one has $X(\Omega)_p \subset X(\Omega)_\mu$ for all $p \geqq 1$ and the trace $[\psi\cdot\nu]$ is defined for all $\psi \in X(\Omega)_p$. Our next result is quite natural.

PROPOSITION 1.3. – *Let Ω be as in theorem* 1.1 *and let p, q be extended real numbers such that*

$$\frac{1}{p}+\frac{1}{q}=1 \qquad \text{if } p>1\,,$$
$$q=+\infty \qquad \text{if } p=1\,.$$

Then, for all $\psi\in X(\Omega)_p$ and for all $u\in H^{1,1}(\Omega)\cap L^q(\Omega)$, one has

$$\int_\Omega u \operatorname{div}\psi\, dx+\int_\Omega \psi\cdot\nabla u\, dx=\int_{\partial\Omega}[\psi\cdot\nu](x)u(x)\, dH^{n-1}\,. \tag{1.9}$$

PROOF. – Take a sequence of functions $f_j\in C^\infty(\bar{\Omega})$ such that

$$f_j\to u \quad \text{in} \quad H^{1,1}(\Omega) \quad \text{and in}\begin{cases} L^q(\Omega) & \text{if } q<+\infty \\ L^\infty(\Omega) & \text{weak* if } q=+\infty\,. \end{cases} \tag{1.10}$$

Now, formula (1.9) holds for all j with f_j at the place of u and, taking the limit for $j\to\infty$, we get our result, recalling that (1.10) implies $f_j\to u$ in $L^1(\partial\Omega)$. q.e.d.

In what follows we shall consider pairs (ψ, u) such that one of the following conditions holds

$$\begin{aligned} &a)\ \ u\in BV(\Omega)_q,\ \psi\in X(\Omega)_p \quad \text{and} \quad 1<p\leqq n,\ \frac{1}{p}+\frac{1}{q}=1;\\ &b)\ \ u\in BV(\Omega)_\infty,\ \psi\in X(\Omega)_1;\\ &c)\ \ u\in BV(\Omega)_c,\ \psi\in X(\Omega)_\mu\,. \end{aligned} \tag{1.11}$$

DEFINITION 1.4. – *Let ψ, u be such that one of the conditions* (1.11) *holds for all open sets $A\subset\subset\Omega$. Then we define a linear functional (ψ, Du): $C_0^\infty(\Omega)\to\boldsymbol{R}$ as*

$$\langle(\psi, Du), \varphi\rangle=-\int_\Omega u\varphi \operatorname{div}\psi\, dx-\int_\Omega u\psi\cdot D\varphi\, dx\,.$$

Compare definition 1.4 and the rest of this section with [8].

THEOREM 1.5. – *For all open sets $A\subset\Omega$ and for all functions $\varphi\in C_0(A)$, one has*

$$|\langle(\psi, Du), \varphi\rangle|\leqq \sup|\varphi|\cdot\|\psi\|_{\infty,A}\cdot\int_A |Du| \tag{1.12}$$

hence the functional (ψ, Du) is a Radon measure in Ω.

PROOF. – Let u be fixed and take a sequence $u_j\in C^\infty(\Omega)$ that converges to u as in lemma 5.1. Take $\varphi\in C^\infty(A)$ and consider an open set V such that $A\supset\supset V\supset \operatorname{spt}\varphi$.

For all j we have then

$$|\langle(\psi, Du_j), \varphi\rangle| \leqq \sup |\varphi| \cdot \|\psi\|_{\infty,A} \cdot \int_V |Du_j|$$

and taking the limit for $j \to \infty$, we get (1.12). q.e.d.

We shall denote by $|(\psi, Du)|$ the measure total variation of (ψ, Du) and, for every Borel set $B \subset \Omega$, we shall denote by $\int_B |(\psi, Du)|$, $\int_B (\psi, Du)$ the values of these measures on B.

By theorem 1.5 we get immediately the following corollary.

COROLLARY 1.6. – *The measures (ψ, Du), $|(\psi, Du)|$ are absolutely continuous with respect to the measure $|Du|$ in Ω and one has*

$$\left|\int_B (\psi, Du)\right| \leqq \int_B |(\psi, Du)| \leqq \|\psi\|_{\infty,A} \int_B |Du|$$

for all Borel sets B and for all open sets A such that $B \subset A \subset \Omega$.

Moreover, by the Radon-Nicodym theorem, for fixed ψ, u, there exists a $|Du|$-measurable function

$$\theta(\psi, Du, x) \colon \Omega \to \boldsymbol{R}$$

such that

$$\int_B (\psi, Du) = \int_B \theta(\psi, Du, x)|Du| \qquad \textit{for all Borel sets } B \subset \Omega$$

$$\|\theta(\psi, Du, x\|_{L^\infty(\Omega, |Du|)} \leqq \|\psi\|_{\infty,\Omega} .$$

REMARK 1.7. – If E is an open set with lipschitz boundary in $\boldsymbol{R}^n$, then the characteristic function u of E

$$u(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

belongs to the space $BV_{\text{loc}}(\boldsymbol{R}^n)$ and the measure (ψ, Du) in $\boldsymbol{R}^n$ coincides with the measure $[\psi \cdot \nu] H^{n-1}|_{\partial E}$.

We shall need the following continuity lemma in the proof of theorem 1.9.

LEMMA 1.8. – *Assume that u, ψ satisfy to one of the conditions* (1.11) *and let $u_j \in C^\infty(\Omega) \cap BV(\Omega)$ converge to u as in lemma* 5.2 (*actually, here we do not need*

(5.10)). *Then we have*

$$\int_\Omega (\psi, Du_j) \to \int_\Omega (\psi, Du) .$$

PROOF. – Take a number $\varepsilon > 0$, then take an open set $A \subset\subset \Omega$ such that

$$\int_\Omega |Du| < \varepsilon$$

and let $g \in C_0^\infty(\Omega)$ be such that $0 \leqq g(x) \leqq 1$ in Ω and $g(x) \equiv 1$ in A. We have then

$$\left|\int_\Omega (\psi, Du_j) - \int_\Omega (\psi, Du)\right| \leqq$$
$$\leqq |\langle(\psi, Du_j), g\rangle - \langle(\psi, Du), g\rangle| + \int_\Omega |(\psi, Du_j)|(1-g) + \int_\Omega |(\psi, Du)|(1-g)$$

where

$$\lim_{j\to\infty} \langle(\psi, Du_j), g\rangle = \langle(\psi, Du), g\rangle$$
$$\max\lim_{j\to\infty} \int_\Omega |(\psi, Du_j)|(1-g) \leqq \|\psi\|_{\infty,\Omega} \max\lim_{j\to\infty} \int_{\Omega\setminus A} |Du_j| < \varepsilon \|\psi\|_{\infty,\Omega}$$
$$\int_\Omega |(\psi, Du)|(1-g) \leqq \varepsilon \|\psi\|_{\infty,\Omega}$$

and the lemma is proved, as ε is arbitrary. q.e.d.

We conclude this section by the expected Green's formula, compare with theorem 3.2 in [8], relating the function $[\psi\cdot\nu]$ and the measure (ψ, Du).

THEOREM 1.9. – *Let Ω be a bounded open set with Lipschitz boundary and let ψ, u be such that one of the conditions* (1.11) *holds, then one has*

$$\int_\Omega u \operatorname{div} \psi \, dx + \int_\Omega (\psi, Du) = \int_{\partial\Omega} [\psi\cdot\nu] u \, dH^{n-1} .$$

PROOF. – Take a sequence of functions $u_j \in C^\infty(\Omega) \cap BV(\Omega)$ that converge to u as in lemma 5.2. Then, by lemma 1.8 and proposition 1.3, one has

$$\int_\Omega u \operatorname{div} \psi \, dx + \int_\Omega (\psi, Du) = \lim \left\{ \int_\Omega u_j \operatorname{div} \psi \, dx + \int (\psi, Du_j) \right\} =$$
$$= \lim_{j\to\infty} \int_{\partial\Omega} [\psi\cdot\nu] u_j \, dH^{n-1} = \int_{\partial\Omega} [\psi\cdot\nu] u \, dH^{n-1}$$

because

$$\left.\begin{aligned}&\int_\Omega (\psi, Du_j) = \int_\Omega \psi\cdot Du_j\,dx\\ &u_j|_{\partial\Omega} = u|_{\partial\Omega}\end{aligned}\right\}\quad \text{for all } j\,. \qquad \text{q.e.d.}$$

2. – Representation of $\theta(\psi, Du, x)$.

In this section we shall be concerned with the problem of whether or not one can write

$$\theta(\psi, Du, x) = \psi(x)\cdot\frac{Du}{|Du|}(x) \tag{2.1}$$

where $(Du/|Du|)(x)$ is the density function of the measure Du with respect to the measure $|Du|$. First, we shall see that the answer is affirmative if $Du\in L^1_{\text{loc}}(\Omega)$ or if $\psi\in C^0(\Omega)$; then we shall see that, in any case, (2.1) holds $|Du|^a$-almost everywhere, where $|Du|^a$ denotes the absolutely continuous part of the measure $|Du|$ with respect to the Lebesgue measure $\mathcal{L}^n$ in Ω. An example shows that, in general, (2.1) does not hold $|Du|^s$-almost everywhere (where $|Du|^s$ is the singular part of $|Du|$), as one is not able to define $\psi(x)$ $|Du|^s$-a.e. in Ω. However, even if one does not have a representation formula for $\theta(\psi, Du, x)$ in the singular zone of $|Du|$, the function $\theta(\psi, Du, x)$ still enjoys a few properties (proposition 2.6, 2.7, 2.8) that can be useful. In particular, the results in this section will be used in [3] (compare also with [2]) to get some regularity properties of the vector field $(Du/|Du|)(x)$ when u is a solution to a problem $\int_\Omega f(x, Du)\to\min$ and $f(x,p)$ is asimptotically of linear growth in p for large $|p|$.

For the sake of simplicity, we shall assume throughout this section that $\psi\in X(\Omega)_n$ and that $u\in BV(\Omega)$, but it is clear that analogous results can be obtained for pairs (ψ, u) satisfying any one of the conditions (1.11). No assumption is needed in this section on the open set $\Omega\subset\mathbf{R}^n$.

Here is a continuity result.

PROPOSITION 2.1. – *Assume that*

$$\psi_j \rightharpoonup \psi \qquad \textit{in } L^\infty(A)\textit{-weak*} \tag{2.2}$$

$$\operatorname{div}\psi_j \rightharpoonup \operatorname{div}\psi \qquad \textit{in } L^n(A)\textit{-weak} \tag{2.3}$$

for all open sets $A\subset\subset\Omega$; *then, for all* $u\in BV_{\text{loc}}(\Omega)$, *one has*

$$(\psi_j, Du) \rightharpoonup (\psi, Du) \tag{2.4}$$

as measures in Ω, and

$$\theta(\psi_j, Du, x) \rightharpoonup \theta(\psi, Du, x) \tag{2.5}$$

in $L^\infty(A)$-weak for all $A \subset\subset \Omega$.*

PROOF. – For all $A \subset\subset \Omega$ and for all j we have $\int_A |(\psi_j, Du)| \leqq \|\psi_j\|_{\infty,A} \cdot \int_A |Du|$ where

$$\sup_{j\in N} \|\psi_j\|_{\infty,A} = c(A) < +\infty$$

because of (2.2), hence it is sufficient to check the weak convergence (2.4) on $C_0^1(\Omega)$ functions. On the other hand, if $\varphi \in C_0^1(\Omega)$ one has

$$\langle(\psi_j, Du), \varphi\rangle = -\int_\Omega u\varphi \operatorname{div} \psi_j \, dx - \int_\Omega u\psi_j D\varphi \, dx \to \langle(\psi, Du), \varphi\rangle$$

and (2.4) is proved.

To show (2.5) we notice that for all j, by corollary 1.6, one has

$$\|\theta(\psi_j, Du, x)\|_{L^\infty(A, |Du|)} \leqq \|\psi_j\|_{\infty,A} \leqq c(A)$$

hence the convergence (2.5) has to be checked only on $C_0^0(\Omega)$ functions, where it reduces to (2.4). q.e.d.

We shall need the following simple fact.

LEMMA 2.2. – *For every function $\psi \in X(\Omega)_n$, there exists a sequence of functions $\psi_j \in C^\infty(\Omega) \cap L^\infty(\Omega)$ such that*

$\|\psi_j\|_{\infty,\Omega} \leqq \|\psi\|_{\infty,\Omega}$	*for all j*
$\psi_j \rightharpoonup \psi$	in $L^\infty(\Omega)$-weak* and in $L^p_{loc}(\Omega)$ for $1 \leqq p < +\infty$
$\psi_j(x) \to \psi(x)$	at every Lebesgue point x of ψ, and uniformly in any set of uniform continuity for ψ.
$\operatorname{div} \psi_j \to \operatorname{div} \psi$	in $L^n_{loc}(\Omega)$.

PROOF. – Just take a sequence $\{\eta_j\}$ of mollifiers and set $\psi_j = \eta_j * \tilde{\psi}$, where $\tilde{\psi}$ is defined by

$$\tilde{\psi}(x) = \begin{cases} \psi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega . \end{cases} \quad \text{q.e.d.}$$

Now we give the representation results for $\theta(\psi, Du, x)$.

PROPOSITION 2.3. – *If* $\psi \in X(\Omega)_n \cap C^0(\Omega)$ *and* $u \in BV(\Omega)$ *then one has*

$$\theta(\psi, Du, x) = \psi(x)\cdot \frac{Du}{|Du|}(x)\,, \qquad |Du| - \text{a.e. } in\ \Omega\,. \tag{2.6}$$

PROOF. – Formula (2.6) is equivalent to

$$\langle(\psi, Du), \varphi\rangle = \int_\Omega \varphi\psi\, Du\,, \qquad \forall \varphi \in C_0^1(\Omega) \tag{2.7}$$

and (2.7) is true by definition if $\psi \in C^1(\Omega)$. If ψ is general, we take a sequence ψ_j as in lemma 2.2 and, by lemma 2.1, for all $\varphi \in C_0^1(\Omega)$, we have

$$\langle(\psi, Du), \varphi\rangle = \lim_{j\to\infty} \langle(\psi_j, Du), \varphi\rangle = \lim_{j\to\infty}\int_\Omega \varphi\psi_j\, Du = \int_\Omega \varphi\psi\, Du$$

where, in the last step, we have used the fact that ψ_j converges uniformly to ψ on $\operatorname{spt}\varphi$. q.e.d.

If $u \in H^{1,1}(\Omega)$, then, for all $\psi \in X(\Omega)_n$ and for all $\varphi \in C_0^1(\Omega)$ one has

$$\int_\Omega \varphi\psi\, Du\, dx = -\int_\Omega u \operatorname{div}(\varphi\psi)\, dx = \langle(\psi, Du), \varphi\rangle$$

and this implies that

$$\theta(\psi, Du, x) = \psi(x)\cdot \frac{Du}{|Du|}(x)\,, \qquad |Du| - \text{a.e. in } \Omega\,.$$

For a general $u \in BV(\Omega)$ one has the following result.

THEOREM 2.4. ⊂ If $\psi \in X(\Omega)_n$ and $u \in BV(\mathbf{R})$, one has

$$\theta(\psi, Du, x) = \psi(x)\cdot \frac{Du}{|Du|}(x)\,, \qquad |Du|^a - \text{a.e. in } \Omega\,. \tag{2.8}$$

PROOF. – Formula (2.8) is equivalent to

$$\int_B \theta(\psi, Du, x)|Du|^a(x)\, dx = \int_B \psi(x)\cdot (Du)^a(x)\, dx \tag{2.9}$$

for all Borel $B \subset \Omega$. Let E^a and E^s be two Borel sets such that $E^a \cup E^s = \Omega$, $E^a \cap E^s = \emptyset$, $\int_{E^s}|Du|^a = \int_{E^a}|Du|^s = 0$ and let $\varepsilon > 0$ be fixed. Then let K be a compact set, with $K \subset E^s$, such that

$$\int_{E^s\setminus K} |Du|^s < \varepsilon \tag{2.10}$$

and take any compact set $B_0 \subset E^a$. We can find an open set L with regular boundary, such that

$$B_0 \subset L \subset \Omega \setminus K , \qquad \int_{L \setminus B_0} |Du| < \varepsilon$$

and, by (2.10) it follows that one has also

$$\int_L |Du|^s < \varepsilon .$$

Now, take a sequence $u_j \in C^\infty(L) \cap BV(L)$ approximating u as in lemma 5.2. By lemma 1.8 and corollary 5.3 we have

$$\left|\int_L \theta(\psi, Du, x)\, Du - \int_L \psi(x)\cdot(Du)^a(x)\, dx\right| = \lim_{j\to\infty}\left|\int_L \psi(x)\cdot Du_j(x)\, dx - \int_L \psi(x)\cdot (Du)^a(x)\, dx\right| \leqq$$
$$\leqq \|\psi\|_{\infty,L} \lim_{j\to\infty}\int_L |Du_j - (Du)^a| \leqq \|\psi\|_{\infty,\Omega}\int_L |Du|^s \leqq \|\psi\|_{\infty,\Omega} .$$

On the other hand, we have

$$\left|\int_L \psi\cdot(Du)^a\, dx - \int_{B_0} \psi\cdot(Du)^a\, dx\right| \leqq \|\psi\|_{\infty,\Omega}\int_{L\setminus B_0} |Du| \leqq \varepsilon\|\psi\|_{\infty,\Omega}$$

and, by corollary 1.6, we have also

$$\left|\int_L \theta(\psi, Du, x)|Du| - \int_{B_0} \theta(\psi, Du, x)|Du|\right| \leqq \|\psi\|_{\infty,\Omega}\int_{L\setminus B_0} |Du| \leqq \varepsilon\|\psi\|_{\infty,\Omega} .$$

In conclusion we get

$$\left|\int_{B_0} \theta(\psi, Du, x)|Du| - \int_{B_0} \psi\cdot(Du)^a\, dx\right| \leqq 3\varepsilon\|\psi\|_{\infty,\Omega} .$$

Hence (2.9) is proved for all compact sets $B \subset E^a$. By the regularity properties of Radon measures we have then that (2.9) holds for all Borel sets in Ω. q.e.d.

REMARK 2.5. – If $\psi_\varrho(x) = \fint_{B_\varrho(x)} \psi(y)\, dy$ is the mean value of ψ in the ball of radius ϱ and center x, then we have shown that

$$(2.11) \qquad \psi_\varrho(x)\cdot \frac{Du}{|Du|}(x) \rightharpoonup \theta(\psi, Du, x) \quad \text{in} \quad L^\infty_{\text{loc}}(\psi, |Du|)\text{-weak*}$$

where

$$\left.\begin{array}{l}\psi_\varrho(x) \rightharpoonup \psi(x)\\ \theta(\psi, Du, x) = \psi(x)\cdot \dfrac{Du}{|Du|}(x)\end{array}\right\} |Du|^a\text{— a.e. in } \Omega\,.$$

On the other hand, in general, one need not have $\psi_\varrho(x) \to \psi(x)$ in any sense in the zone where $|Du|^s$ is concentrated, and the convergence (2.11) only makes sense. As an example of this situation one can take

$$\Omega = \boldsymbol{R}^2\,, \qquad E = \{x \in \boldsymbol{R}^2 | x_2 < 0\}\,, \qquad u(x) = \begin{cases} 1 & \text{if } x \in E\\ 0 & \text{if } x \in \boldsymbol{R}^2 \setminus E\end{cases}$$

$$\psi = (\psi_1, \psi_2)\,, \qquad \psi_1(x_1, x_2) = \operatorname{sen}\frac{1}{x_2}\,, \qquad \psi_2 = 0$$

and it is easily seen that $\psi \in L^\infty(\Omega, \boldsymbol{R}^2)$, $\operatorname{div}\psi = 0$, $\theta(\psi, Du, x) = [\psi\cdot\nu](x)$ on ∂E (where ν is the normal to ∂E), while the mean values $\psi_\varrho(x)$ do not converge on ∂E.

Even though the function $\theta(\psi, Du, x)$ cannot be represented in terms of a well defined value of $\psi(x)$ $|Du|^s$-a.e., it enjoys a few nice properties that are studied in the rest of this section.

PROPOSITION 2.6. – *If* $\psi \in X(\Omega)_n$ *and* $u \in BV(\Omega)$, *then one has*

(i) $\theta(\psi, D(u+g), x) = \theta(\psi, Du, x)$ $|Du|^s$-a.e. *in* Ω *for all* $g \in H^{1,1}(\Omega)$;

(ii) $\theta(\psi, D(gu), x) = \operatorname{segn} g(x), \theta(\psi, Du, x)$, $|g||Du|^s$-a.e. *in* Ω *for all* $g \in C^1(\Omega)$.

PROOF. – (i) Recall that if $Dg \in L^1(\Omega)$ then one has $(D(u+g))^s = (Du)^s$, then notice that

$$(\psi, D(u+g)) = \theta(\psi, D(u+g), x)|D(u+g)|^s + \theta(\psi, D(u+g), x)|D(u+g)|^a$$

while, on the other hand

$$(\psi, D(u+g)) = (\psi, Du) + (\psi, Dg) =$$
$$= \theta(\psi, Du, x)|Du|^s + \theta(\psi, Du, x)|Du|^a + \psi(x)\cdot Dg(x)\,.$$

Equating the two expressions for the singular part of $(\psi, D(u+g))$ we get

$$\theta(\psi, D(u+g), x)|Du|^s = \theta(\psi, Du, x)|Du|^s$$

and (i) follows.

(ii) For all test functions $\varphi \in C^1_0(\Omega)$ we have

$$\langle(\psi, D(gu)), \varphi\rangle = \langle(\psi, Du), g\varphi\rangle + \int_\Omega (\psi\cdot Dg)u\varphi\, dx$$

hence we have, for all Borel sets $B \subset \Omega$,

$$\int_B \theta(\psi, D(gu), x)|D(gu)| = \int_B \theta(\psi, Du, x) g |Du|^s + \int_B \theta(\psi, Du, x) g |Du|^a + \int_B \psi \cdot Dgu\, dx . \tag{2.12}$$

Recalling that $|D(gu)|^s = |g||Du|^s$ and equating the singular parts on the two sides of (2.12) we get (ii). q.e.d.

For all functions $u: \Omega \to \mathbf{R}$ let us consider the sets

$$E_{u,t} = \{x \in \Omega | u(x) > t\} .$$

If $u \in BV(\Omega)$, it is well known [9], [5] that the characteristic functions

$$\chi_{u,t}(x) = \begin{cases} 1 & \text{if } x \in E_{u,t} \\ 0 & \text{if } x \notin E_{u,t} \end{cases}$$

of the sets $E_{u,t}$ are in $BV(\Omega)$ for $\mathfrak{L}^1$-almost all $t \in \mathbf{R}$; moreover, the function $t \mapsto \int_\Omega |D\chi_{u,t}|$ is $\mathfrak{L}^1$-measurable and the coarea formula

$$\int_\Omega f(x)|Du| = \int_{-\infty}^{+\infty} dt \int_\Omega f(x)|D\chi_{u,t}| \tag{2.13}$$

holds for every $|Du|$-summable function $f: \Omega \to \mathbf{R}$. It follows that a set $B \subset \Omega$ has $|Du|$-measure zero if and only if for $\mathfrak{L}^1$-almost all $t \in \mathbf{R}$ one has $\int_B |D\chi_{u,t}| = 0$. For later use we recall also that one has

$$\frac{Du}{|Du|}(x) = \frac{D\chi_{u,t}}{|D\chi_{u,t}|}(x) , \qquad |D\chi_{u,t}| - \text{a.e. in } \Omega$$

for $\mathfrak{L}^1$-almost all $t \in \mathbf{R}$.

Now we shall give a «slicing» result that links the measure (ψ, Du) with the measures $(\psi, D\chi_{u,t})$.

PROPOSITION 2.7. – *If $\psi \in X(\Omega)_n$ and $u \in BV(\Omega)$, then we have:*

(i) *for all functions $\varphi \in C_0^0(\Omega)$, the function $t \mapsto \langle(\psi, D\chi_{u,t}), \varphi\rangle$ is $\mathfrak{L}^1$-measurable and*

$$\langle(\psi, Du), \varphi\rangle = \int_{-\infty}^{+\infty} \langle(\psi, D\chi_{u,t}), \varphi\rangle\, dt;$$

(ii) *for all Borel sets* $B \subset \Omega$, *the function* $t \to \int_B (\psi, D\chi_{u,t})$ *is* $\mathfrak{L}^1$*-measurable and*

$$\int_B (\psi, Du) = \int_{-\infty}^{+\infty} dt \int_B (\psi, D\chi_{u,t});$$

(iii) $\theta(\psi, Du, x) = \theta(\psi, D\chi_{u,t}, x)$ $|D\chi_{u,t}|$*-a.e. in* Ω *for* $\mathfrak{L}^1$*-almost all* $t \in \mathbf{R}$.

PROOF. – (i) Take a sequence of functions $\psi_j \in C^\infty(\Omega) \cap L^\infty(\Omega)$ that converge to ψ as in lemma 2.2. Then, for all j, we have, by the coarea formula,

$$\text{(2.14)} \quad \langle(\psi_j, Du), \varphi\rangle = \int_\Omega \psi_j(x) \cdot \frac{Du}{|Du|}(x)\varphi(x)|Du| = \\ = \int_{-\infty}^{+\infty} dt \int_\Omega \psi_j(x) \cdot \frac{D\chi_{u,t}}{|D\chi_{u,t}|}(x)\varphi(x)|D\chi_{u,t}| = \int_{-\infty}^{+\infty} \langle(\psi_j, D\chi_{u,t}), \varphi\rangle\, dt$$

where

$$|\langle(\psi_j, D\chi_{u,t}), \varphi\rangle| \leq \|\psi\|_{\infty,\Omega}\|\varphi\|_{\infty,\Omega}\int_\Omega |D\chi_{u,t}|\,.$$

Recalling proposition 2.1, taking the limit in (2.14) for $j \to \infty$, by the dominated convergence theorem we get the proof of (i).

We shall prove (ii) after (iii). Let's prove (iii). Take $a, b \in \mathbf{R}$ and consider the function $v \in BV(\Omega)$ defined by

$$v(x) = \begin{cases} b & \text{if } b \leq u(x) \\ u(x) & \text{if } a \leq u(x) \leq b \\ a & \text{if } u(x) \leq a \end{cases}$$

then we have $E_{u,t} = E_{v,t}$ for all t such that $a \leq t < b$, hence

$$\left.\begin{array}{l} D\chi_{u,t} = D\chi_{v,t} \\ \dfrac{D\chi_{u,t}}{|D\chi_{u,t}|}(x) = \dfrac{D\chi_{v,t}}{|D\chi_{v,t}|}(x) \end{array}\right\} \text{ if } a \leq t < b$$

$$D\chi_{v,t} = 0 \qquad \text{if } t \geq b, \text{ or } t < a$$

and it follows that

$$\frac{Du}{|Du|}(x) = \frac{D\chi_{u,t}}{|D\chi_{u,t}|}(x) = \frac{D\chi_{v,t}}{|D\chi_{v,t}|}(x) = \frac{Dv}{|Dv|}(x)\,,$$

$$|D\chi_{v,t}| - \text{a.e. in } \Omega \text{ for } \mathfrak{L}^1\text{-almost all } t \in \mathbf{R}$$

that is

$$\frac{Du}{|Du|}(x) = \frac{Dv}{|Dv|}(x), \qquad |Dv| - \text{a.e. in } \Omega .$$

Now it follows that, for every $\psi \in X(\Omega)$ we have

$$\theta(\psi, Du, x) = \theta(\psi, Dv, x), \qquad |Dv|\text{-a.e. in } \Omega . \tag{2.15}$$

In fact, if $\psi_j \to \psi$ as in lemma 2.2, we have, for all j

$$\theta(\psi_j, Du, x) = \psi_j(x)\cdot \frac{Du}{|Du|}(x) = \theta(\psi_j, Dv, x), \qquad |Dv| - \text{a.e. in } \Omega$$

and taking the limit for $j \to \infty$, by the uniqueness of the limit in the $L^\infty(\Omega, |Dv|)$-weak* topology, we get (2.15). Finally, using statement (i) for $v(x)$, we have, for all $a < b$ and for a fixed $\varphi \in C_0^\infty(\Omega)$,

$$\langle(\psi, Dv), \varphi\rangle = \int_{-\infty}^{+\infty} \langle(\psi, D\chi_{v,t}), \varphi\rangle\, dt$$

i.e., by the coarea formula and (2.15):

$$\int_a^b dt \int_\Omega \theta(\psi, Du, x)\varphi(x)|D\chi_{v,t}| = \int_a^b dt \int_\Omega \theta(\psi, D\chi_{v,t}, x)\varphi(x)|D\chi_{v,t}|$$

and this implies that

$$\int_\Omega \theta(\psi, Du, x)\varphi(x)|D\chi_{u,t}| = \int_\Omega \theta(\psi, D\chi_{u,t}, x)\varphi(x)|D\chi_{u,t}| \tag{2.16}$$

for $\mathfrak{L}^1$-almost all $t \in \boldsymbol{R}$. If S is a countable dense set in $C_0^\infty(\Omega)$ with respect to the uniform convergence, it is possible to find a set $N \subset \boldsymbol{R}$ such that $\mathfrak{L}^1(N) = 0$ and that (2.16) holds for all $t \in \boldsymbol{R} \setminus N$ and for all $\varphi \in S$. It follows that for all $t \in \boldsymbol{R}\setminus N$ one has

$$\theta(\psi, Du, x) = \theta(\psi, D\chi_{u,t}, x)$$

as wanted.

To prove (ii) we notice that, by (iii), we have

$$\int_B (\psi, Du) = \int_B \theta(\psi, Du, x)|Du| = \int_{-\infty}^{+\infty} dt \int_B \theta(\psi, Du, x)|D\chi_{u,t}| =$$

$$= \int_{-\infty}^{+\infty} dt \int_B \theta(\psi, D\chi_{u,t}, x)|D\chi_{u,t}| = \int_{-\infty}^{+\infty} dt \int_B (\psi, D\chi_{u,t}) . \qquad \text{q.e.d.}$$

Our next result is a consequence of proposition 2.7.

PROPOSITION 2.8. – *If* $\alpha: \boldsymbol{R} \to \boldsymbol{R}$ *is an increasing function of class* C^1, *then one has*

$$\theta(\psi, D(\alpha \circ u), x) = \theta(\psi, Du, x), \quad |Du|\text{-a.e. } in\ \Omega \tag{2.17}$$

where $(\alpha \circ u)(x) = \alpha(u(x))$.

PROOF. – First, notice that

$$E_{u,t} = \{x \in \Omega | u(x) > t\} = \{x \in \Omega | (\alpha \circ u)(x) > \alpha(t)\} = E_{\alpha \circ u, \alpha(t)}$$

so that, for almost all $t \in \boldsymbol{R}$, one has

$$D\chi_{u,t} = D\chi_{\alpha \circ u, \alpha(t)}$$

hence, for almost all $t \in \boldsymbol{R}$ one has also

$$\theta(\psi, Du, x) = \theta(\psi, D\chi_{u,t}, x) = \theta(\psi, D\chi_{\alpha \circ u, \alpha(t)}, x) = \theta(\psi, D(\alpha \circ u), x)$$

$|D\chi_{u,t}|$-a.e. in Ω, and (2.17) follows. q.e.d.

3. – The pairing (ψ, μ).

In this section we define a pairing (ψ, μ) when $\psi \in X(\Omega)_n$ and μ is a measure whose curl is also a measure. The key lemma is lemma 3.4; the idea for solving the equation $\operatorname{rot} z = \lambda$ is the same as in [10], but we cannot use Rellich theorem to show the compactness of the operator $Z: \lambda \to z$, as we do not have sufficient information on the derivatives of $Z(\lambda)$, and we use instead the information on the translations of $Z(\lambda)$.

The pairing (ψ, μ) is then defined, when μ has a compact support, noticing that one can write $\mu = f + Du$, where $f \in L^1_{\text{loc}}(\boldsymbol{R}^n)$, $\operatorname{rot} f = \operatorname{rot} \mu$, $u \in BV_{\text{loc}}(\boldsymbol{R}^n)$, and using the results of section 1. When μ does not have a compact support we localize and then we glue together the pieces. The results of section 1 and 2 are then used to derive a few properties of the pairing (ψ, μ), that are collected in theorem 3.8.

We shall denote by $M(\Omega, \boldsymbol{R}^N)$ the space of the $\boldsymbol{R}^N$ valued Radon measures in Ω. We shall set $M_0(\Omega, \boldsymbol{R}^N) = \{\mu \in M(\Omega, \boldsymbol{R}^N)$ such that $\operatorname{spt} \mu$ is compact$\}$ and we shall write simply $M(\Omega)$ instead of $M(\Omega, \boldsymbol{R})$.

We shall use the following well known facts, that we recall for convenience.

FACT 3.1. – If $f \in L^1_{loc}(\boldsymbol{R}^n)$ and $\mu \in M_0(\boldsymbol{R}^n)$, then the convolution $f * \mu$ is in $L^1_{loc}(\boldsymbol{R}^n)$ and

$$\int_A |f * \mu| \leqq \int_{A-\operatorname{spt}\mu} |f| \cdot \|\mu\|$$

where $\|\mu\| = \int_{\boldsymbol{R}^n} |\mu|$ and $A - \operatorname{spt}\mu = \{x - y | x \in A,\ y \in \operatorname{spt}\mu\}$.

FACT 3.2. (*Compactness criterion.*) – Let A be a bounded set in $\boldsymbol{R}^n$ and let $E \subset L^1(\boldsymbol{R}^n)$ be such that

$$\sup_{f \in E} \int_{\boldsymbol{R}^n} |f| < +\infty$$

$$\operatorname{spt} f \subset A \qquad \text{for all } f \in E$$

$$\int_{\boldsymbol{R}^n} |T_a f - f|\, dx \leqq \omega(|a|) \qquad \text{for all } f \in E$$

$$\lim_{\delta \to 0} \omega(\delta) = 0$$

where $(T_a f)(x) = f(x - a)$, then E is a relatively compact set in $L^1(\boldsymbol{R}^n)$.

By using Facts 3.1 and 3.2, it is easy to prove the following lemma.

LEMMA 3.3. – *Suppose that* $f \in L^1_{loc}(\boldsymbol{R}^n)$, *let* A *be a bounded set in* $\boldsymbol{R}^n$ *and let* $L \subset M(\boldsymbol{R}^n)$ *be such that*

$$\sup_{\lambda \in L} \|\lambda\| < +\infty$$

$$\operatorname{spt} \lambda \subset A \qquad \textit{for all } \lambda \in L$$

then the set $E = \{(f * \lambda)|_V;\ \lambda \in L\}$ *is relatively compact in* $L^1(V)$ *for all bounded rectangles* $V \subset \boldsymbol{R}^n$.

Here is the key lemma for what follows.

LEMMA 3.4. – *Let* A *be an open bounded subset of* $\boldsymbol{R}^n$ *and let us consider the space* $M_R(A) = \{\lambda \in M(\boldsymbol{R}^n, \boldsymbol{R}^{n^2})$ *such that* $\lambda = \operatorname{rot} T$ *for some distribution* $T \in \mathcal{D}'(\boldsymbol{R}^n)^n$ *with* $\operatorname{supt} T \subset\subset A\}$. *Then there exists a linear operator*

$$Z\colon M_R(A) \to L^1_{loc}(\boldsymbol{R}^n, \boldsymbol{R}^n)$$

such that

(i) $\operatorname{rot} Z(\lambda) = \lambda$ *in* $\boldsymbol{R}^n$, *for all* $\lambda \in M_R(A)$;

(ii) *the map* $\lambda \to Z(\lambda)|_V$ *is a completely continuous operator* $M_R(A) \to L^1(V, \boldsymbol{R}^n)$, *for any bounded rectangle* $V \subset \boldsymbol{R}^n$.

PROOF. – Let us consider the kernels

$$E_j(x) = \frac{1}{\alpha_n}\frac{x_j}{|x|^n} = \frac{\partial}{\partial x_j} G(x)\,, \qquad 1 \leqq j \leqq n\,,$$

where α_n is the $(n-1)$-dimensional measure of the unit sphere in $\boldsymbol{R}^n$ and

$$G(x) = \begin{cases} -\dfrac{1}{n-2}\dfrac{1}{\alpha_n}\dfrac{1}{|x|^{n-2}} & \text{if } n \geqq 3 \\ -\dfrac{1}{\alpha_2}\ln\left(\dfrac{1}{|x|}\right) & \text{if } n = 2 \end{cases}$$

is the fundamental solution of the Laplace equation, i.e.

$$\sum_{j=1}^{n} \frac{\partial E_j}{\partial x_j} = \Delta G = \delta\,.$$

For every $\lambda \in M_R(A)$ we consider the function $z = Z(\lambda) \in L^1_{\text{loc}}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ defined by

$$z_i = \sum_{j=1}^{n} \lambda_{ij} * E_j\,.$$

If $T \in \mathcal{E}'(A)^n$ is such that $\operatorname{rot} T = \lambda$ we have, in the sense of distributions,

$$(\operatorname{rot} z)_{ij} = \sum_{k=1}^{n} \frac{\partial}{\partial x_j}\left\{\left[\frac{\partial T_i}{\partial x_k} - \frac{\partial T_k}{\partial x_i}\right] * E_k\right\} - \frac{\partial}{\partial x_i}\left\{\left[\frac{\partial T_j}{\partial x_k} - \frac{\partial T_k}{\partial x_j}\right] * E_k\right\} = $$

$$= \left[\frac{\partial T_i}{\partial x_j} - \frac{\partial T_j}{\partial x_i}\right] * \sum_{k=1}^{n} \frac{\partial E_k}{\partial x_k} = \lambda_{ij}$$

and (i) is proved.

Using Lemma 3.3 one gets immediately (ii). q.e.d.

LEMMA 3.5. – *For every measure* $\mu \in M_0(\boldsymbol{R}^n, \boldsymbol{R}^n)$ *such that* $\operatorname{rot} \mu \in M(\boldsymbol{R}^n, \boldsymbol{R}^{n^2})$ *there exist a function* $f \in L^1_{\text{loc}}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ *and a function* $u \in BV_{\text{loc}}(\boldsymbol{R}^n)$ *such that*

$$\mu = Du + f \quad \text{in } \boldsymbol{R}^n\,.$$

PROOF. – If $\mu \in M_0(\boldsymbol{R}^n, \boldsymbol{R}^n)$, then, by lemma 3.4, we can consider the function $f = Z(\operatorname{rot} \mu) \in L^1_{\text{loc}}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ and we have

$$\operatorname{rot}(\mu - f) = 0 \quad \text{in } \boldsymbol{R}^n$$

hence there exists [11] a distribution $u \in \mathcal{D}'(\boldsymbol{R}^n)$ such that $\mu - f = Du$. On the other hand $\mu - f$ is a measure and it follows [11] that $u \in BV_{\text{loc}}(\boldsymbol{R}^n)$. q.e.d.

DEFINITION 3.6. – *For every measure* $\mu \in M_0(\boldsymbol{R}^n, \boldsymbol{R}^n)$ *such that* rot $\mu \in M(\boldsymbol{R}^n, \boldsymbol{R}^{n^2})$ *and for every vector field* $\psi \in X(\boldsymbol{R}^n)_n$, *we define the measure* $(\psi, \mu) \in M(\boldsymbol{R}^n)$ *as*

$$\langle(\psi, \mu), \varphi\rangle = \langle(\psi, Du), \varphi\rangle + \int_\Omega f\varphi \, dx\,, \qquad \varphi \in C_0^\infty(\boldsymbol{R}^n)$$

where

$$\mu = f + Du\,, \qquad f \in L^1_{\text{loc}}(\boldsymbol{R}^n, \boldsymbol{R}^n)\,, \qquad u \in BV_{\text{loc}}(\boldsymbol{R}^n)\,. \tag{3.1}$$

We remark that definition 3.6 is valid, because for every measure μ whose curl is a measure there exists (lemma 3.5) at least a pair f, u that satisfies (3.1); moreover, the definition is easily seen to be independent of the choice of f, u.

Now we shall define the pairing (ψ, μ) without the assumption on the support of μ.

DEFINITION 3.7. – *Let* Ω *be an open set in* $\boldsymbol{R}^n$ *and suppose that* $\psi \in X(\Omega)_n$, $\mu \in M(\Omega, \boldsymbol{R}^n)$, rot $\mu \in M(\Omega, \boldsymbol{R}^{n^2})$. *For all open sets* $A \subset\subset \Omega$ *choose a function* $g \in C_0^\infty(\Omega)$ *such that* $g \equiv 1$ *on* A *and consider the distribution*

$$T_A = (\psi, g\mu)|_A$$

where $(\psi, g\mu)$ *is defined in definition* 3.6. *It is easy to see that if* A_1, A_2 *are such that* $A_1 \cap A_2 \neq \emptyset$ *one has*

$$T_{A_1}|_{A_1 \cup A_2} = T_{A_1 \cup A_2} = T_{A_2}|_{A_1 \cup A_2}$$

and by a well known glueing principle [11], *there exists one and only one measure in* Ω, *that we shall denote* (ψ, μ), *such that* $(\psi, \mu)|_A = T_A$ *for all* $A \subset\subset \Omega$.

Now we collect a few properties of (ψ, μ).

THEOREM 3.8. – (i) *The map that takes* ψ, μ *to* (ψ, μ) *is bilinear.* (ii) *The measure* (ψ, μ) *is absolutely continuous with respect to the measure* $|\mu|$ *and one has precisely, for all Borel sets* $B \subset \Omega$,

$$\int_B |(\psi, \mu)| \leqq \|\psi\|_{\infty,\Omega} \int_B |\mu|\,. \tag{3.2}$$

(iii), *For all functions* $g \in C^1(\Omega)$ *with* $\sup_\Omega (|g| + |Dg|) < +\infty$, *one has*

$$(\psi, g\mu) = (g\psi, \mu) = (\psi, \mu)g\,.$$

Moreover, if we consider the function $\theta(\psi, \mu, x)\colon \Omega \to \mathbf{R}$ *such that*

$$\int_B (\psi, \mu) = \int_B \theta(\psi, \mu, x)|\mu| \quad \textit{for all Borel sets } B \subset \Omega$$

we have also

(iv) $\theta(\psi, \mu, x) = \psi(x)\cdot \dfrac{d\mu}{d|\mu|}(x)\,, \quad |\mu|^a$— a.e. *in* Ω;

(v) $\theta(\psi, \mu, x) = \theta(\psi, \mu_1, x)$ $|\mu|^s$-a.e., *if* $\mu^s = \mu_1^s$;

(vi) $\theta(\psi, g\mu, x) = \theta(\psi, \mu, x)\,\mathrm{segn}\, g(x) \quad |g||\mu|$-a.e. *in* Ω

$\theta(g\psi, \mu, x) = g(x)\theta(\psi, \mu, x) \quad |\mu|$-a.e. *in* Ω

if $g \in C^1(\Omega)$ *and* $\sup_\Omega (|g| + |Dg|) < +\infty$.

PROOF. – (i) is obvious. To prove (ii) it is sufficient to show that (3.2) holds for all Borel sets $B \subset A \subset\subset \Omega$. To do that, we can write $\mu = f + Du$ in A, for suitable f and u, and we have $\mu^a = f + (Du)^a$, $\mu^s = (Du)^s$ and

$$(\psi, \mu) = \theta(\psi, Du, x)|Du|^s + \big((Du)^a(x) + f(x)\big)\cdot\psi(x)\, dx \quad \text{in } A \tag{3.3}$$

hence we get

$$\int_B |(\psi, \mu)| \leq \|\psi\|_{\infty,A}\Big\{\int_B |Du|^s + \int_B |(Du)|^a(x) + f(x)\, dx\Big\} = \|\psi\|_{\infty,A}\int_B |\mu|$$

and (ii) is proved. To show (iii), we take a function $\varphi \in C_0(\Omega)$, then we write $\mu = Du + f$ on the support of φ and we have

$$\langle(\psi, gu), \varphi\rangle = -\int ug \operatorname{div} \psi\varphi\, dx - \int ug\psi\nabla\varphi\, dx + \int fg\psi\varphi\, dx - \int u\, Dg\psi\varphi\, dx =$$
$$= \langle(g\psi, \mu), \varphi\rangle = \langle(\psi, \mu), g\varphi\rangle$$

which proves (iii). To show (iv), again we take $A \subset\subset \Omega$ and write $\mu = f + Du$ in A so that (3.3) holds. On the other hand, we have by definition

$$(\psi, \mu) = \theta(\psi, \mu, x)|\mu| = \theta(\psi, \mu, x)|Du|^s + \theta(\psi, \mu, x)|(Du)^a(x) + f(x)|\, dx \tag{3.4}$$

and, equating the regular parts of the measures on the right sides of (3.3), (3.4), we obtain that (iv) holds $|\mu|^a$-a.e. in A. Varying the set A, (iv) follows. Finally, (v) and (vi) are proved by similar methods; we omit the details. q.e.d.

4. – Compensated compactness for the pairing (ψ, μ).

As a general reference for compensated compactness, we give [12].

We have the following compensated compactness result.

THEOREM 4.1. – *Let* ψ_j, ψ, μ_j, μ *be such that* $\psi, \psi_j \in X(\Omega)_n$; $\mu, \mu_j \in M(\Omega, \boldsymbol{R}^n)$; $\operatorname{rot}\mu$, $\operatorname{rot}\mu_j \in M(\Omega, \boldsymbol{R}^n)$ *and assume that*

$$\psi_j \rightharpoonup \psi \quad \text{in } L^\infty(\Omega)\text{-weak*}$$

$$\|\psi_j\|_{\infty,\Omega} + \|\operatorname{div}\psi_j\|_{L^{n+\delta}(\Omega)} \leqq C_1 \quad \text{for all } j \text{ for some fixed } \delta > 0$$

$$\mu_j \rightharpoonup \mu \quad \text{weakly in } M(\Omega, \boldsymbol{R}^n)$$

$$\|\mu_j\| + \|\operatorname{rot}\mu_j\| \leqq c_2 \quad \text{for all } j$$

then one has also

$$(\psi_j, \mu_j) \rightharpoonup (\psi, \mu) \quad \text{weakly in } M(\Omega)\,.$$

PROOF. – It is sufficient to show that for all $\varphi \in C_0^\infty(\Omega)$ one has

$$\langle(\psi_j, \mu_j), \varphi\rangle \rightharpoonup \langle(\psi, \mu), \varphi\rangle \tag{4.1}$$

in fact, as, for all j, we have

$$\int_\Omega |(\psi_j, \mu_j)| \leqq \|\psi_j\|_{\infty,\Omega} \int_\Omega |\mu_j| \leqq c_1 c_2$$

the convergence (4.1) holds then also for all $\varphi \in C_0^0(\Omega)$.

Let $\varphi \in C_0^\infty(\boldsymbol{R})$ be fixed and let $g \in C_0^1(\Omega)$ be such that $g \equiv 1$ on the support of φ, then consider the measures $\tilde{\mu}$, $\tilde{\mu}_j \in M_0(\boldsymbol{R}^n, \boldsymbol{R}^n)$ defined by

$$\tilde{\mu} = g\mu\,, \qquad \tilde{\mu}_j = g\mu_j\,.$$

We still have $\operatorname{rot}\tilde{\mu}$, $\operatorname{rot}\tilde{\mu}_j \in M(\boldsymbol{R}^n, \boldsymbol{R}^{n^2})$ and (4.1) is equivalent to

$$\langle(\psi_j, \tilde{\mu}_j), \varphi\rangle \to \langle(\psi, \mu), \varphi\rangle\,. \tag{4.2}$$

To prove (4.2) we shall show that for any increasing sequence j_k there exists a subsequence j_{k_r} such that

$$\langle(\psi_{j_{k_r}}, \tilde{\mu}_{j_{k_r}}), \varphi\rangle \to \langle(\psi, \mu), \varphi\rangle\,.$$

For all j we set $f_j = Z(\operatorname{rot} \tilde{\mu}_j) \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, where Z is the operator defined in lemma 3.4, and, as in lemma 3.5, we have

$$\tilde{\mu}_j = f_j + Du_j$$

where $u_j \in BV_{\mathrm{loc}}(\boldsymbol{R}^n)$, and we may assume that $\int_Q u_j\, dx = 0$ for all j, where Q is some fixed cube containing the support of g. As the norms $\|\operatorname{rot} \tilde{\mu}_j\|$ are bounded, the sequence f_j is bounded and relatively compact in $L^1(Q, \boldsymbol{R}^n)$ (by lemma 3.4). As the norms $\|\mu_j\|$ and $\|f_j\|_{L^1(Q)}$ are bounded and $\int_Q u_j\, dx = 0$, we have that the sequence u_j is bounded in $BV(Q)$. We conclude that for any increasing sequence $j_k \in \boldsymbol{N}$ there exists a subsequence j_{k_r} and two functions $f \in L^1(Q)$, $u \in BV(Q)$ such that

$$\begin{aligned} &f_{j_{k_r}} \to f && \text{in } L^1(Q)\\ &u_{j_{k_r}} \to u && \text{in } L^p(Q), \text{ where } \frac{1}{p} + \frac{1}{n+\delta} = 1\\ &\operatorname{rot} f = \operatorname{rot} \tilde{\mu}\,, && \tilde{\mu} = Du + f\,. \end{aligned}$$

To conclude, we have that

$$\langle (\psi_{j_{k_r}}, \mu_{j_{k_r}}), \varphi\rangle = -\int_\Omega u_{j_{k_r}} \psi_{j_{k_r}} D\varphi\, dx - \int_\Omega u_{j_{k_r}} \operatorname{div} \psi_{j_{k_r}} \varphi\, dx + \int_\Omega f_{j_{k_r}} \varphi\, dx \to$$
$$\to - \int_\Omega u\psi\, D\varphi\, dx - \int_\Omega u \operatorname{div} \psi\varphi\, dx + \int_\Omega f\varphi\, dx = \langle (\psi, \tilde{\mu}), \varphi\rangle\,. \qquad \text{q.e.d.}$$

Under the hypotheses of theorem 4.1, the integrals $\int_\Omega (\psi_j, \mu_j)$ need not converge to $\int_\Omega (\psi, \mu)$. To ensure that, one needs the supplementary assumption $\int_\Omega |\mu_j| \to \int_\Omega |\mu|$, as it is shown in the next theorem.

THEOREM 4.2. – *Let μ, μ_j, ψ, ψ_j be as in theorem* 4.1, *and assume moreover that*

$$\int_\Omega |\mu| \to \int_\Omega |\mu| \tag{4.3}$$

then one has also

$$\int_\Omega (\psi_j, \mu_j)\varphi \to \int_\Omega (\psi, \mu)\varphi$$

for all $\varphi \in C^0(\Omega) \cap L^\infty(\Omega)$.

PROOF. – Take a fixed function $\varphi \in C^0(\Omega) \cap L^\infty(\Omega)$ and let $\varepsilon > 0$ be given. There exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$\int_{\Omega \setminus \Omega_\delta} |\mu| < \varepsilon$$

where $\Omega_\delta = \{x \in \Omega | \text{dist}(x, \partial\Omega) > \delta\}$. As $\mu_j \rightharpoonup \mu$ weakly, we have

$$\min_{j\to\infty} \lim \int_{\Omega_\delta} |\mu_j| \geq \int_{\Omega_\delta} |\mu|$$

and, recalling (4.3), we get

$$\max_{j\to\infty} \lim \int_{\Omega\setminus\Omega_\delta} |\mu_j| \leq \int_{\Omega\setminus\Omega_\delta} |\mu| < \varepsilon .$$

Now, we take a function $\eta \in C_0^0(\Omega)$ such that $\eta \equiv 1$ on Ω_δ and we write

$$(4.4)\qquad \int_\Omega (\psi_j, \mu_j)\varphi - \int_\Omega (\psi, \mu)\varphi =$$
$$= \left[\int_\Omega (\psi_j, \mu_j)\varphi\eta - \int_\Omega (\psi, \mu)\varphi\eta\right] + \left[\int_\Omega (\psi_j, \mu_j)\varphi(1-\eta) - \int_\Omega (\psi, \mu)\varphi(1-\eta)\right]$$

where the first term in brackets goes to zero, because of theorem 4.1, and the second terms in brackets, for j sufficiently big, is bounded by

$$\|\varphi\|_{\infty,\Omega}\|\psi\|_{\infty,\Omega}\int_{\Omega\setminus\Omega_\delta} |\mu_j| + \|\psi\|_{\infty,\Omega}\int_{\Omega\setminus\Omega_\delta} |\mu| \leq 2c_1\varepsilon\|\varphi\|_{\infty,\Omega} .$$

Taking the limit in (4.4) for $j \to \infty$ we get our result, as $\varepsilon > 0$ is arbitrary. q.e.d.

5. – Appendix.

LEMMA 5.1. – *Let Ω be any open set in $\mathbf{R}^n$, let $u \in BV(\Omega)$ be fixed and set $u_j = \tilde{\mu} * \eta_j$, where*

$$\tilde{\mu}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

and $\eta_j \in C_0^\infty(\mathbf{R}^n)$ is a sequence of mollifiers. Then one has

$$(5.1)\qquad u_j \to u \quad \text{in } L^1(\Omega) .$$

If A is an open set, $A \subset\subset \Omega$, one has also

$$(5.2)\qquad \int_A |Du_j| \to \int_A |Du| \quad \text{if } \int_{\partial A} |Du| = 0$$

$$(5.3)\qquad \int_A |Du - h(x)\, dx| \leq \min_{j\to\infty}\lim \int_A |Du_j - h|\, dx \leq \max_{j\to\infty}\lim \int_A |Du_j - h|\, dx \leq$$
$$\leq \int_{\bar{A}} |Du - h\, dx| \quad \text{for all } h \in L^1(\Omega) .$$

Moreover:

(5.4) *if* $u \in BV(\Omega) \cap L^q(\Omega)$, $q < +\infty$, *one has also* $u_j \to u$ *in* $L^q(\Omega)$

(5.5) *if* $u \in BV(\Omega) \cap L^\infty(\Omega)$, *one has* $\|u_j\|_{\infty,\Omega} \leqq \|u\|_{\infty,\Omega}$ $u_j \to u$ *in* $L^\infty(\Omega)$*-weak**

(5.6) *if* $u \in BV(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$, *one has also* $u_j \to u$ *in* $C^0_{\text{loc}}(\Omega)$.

PROOF. – (5.1), (5.4), (5.5), (5.6) are standard and (5.2) follows from (5.3). To prove (5.3), we notice that

$$Du_j = (Du) * \eta_j = (Du)^a * \eta_j + (Du)^s * \eta_j$$

where

$$(Du)^a * \eta_j \to (Du)^a \quad \text{in } L^1(\Omega)$$

hence

$$\max\lim_{j\to\infty}\int_A |Du_j - h|\, dx \leqq \lim_{j\to\infty}\int_A |(Du)^a * \eta_j - h|\, dx + \\ + \max\lim_{j\to\infty}\int_A |(Du)^s * \eta_j| \leqq \int_A |(Du)^a - h|\, dx + \int_{\bar{A}} |Du|^s = \int_{\bar{A}} |Du - h| .$$

On the other hand, we have $(Du_j - h) \rightharpoonup (Du - h)$, and, because of the semicontinuity of the total variation, (5.3) is proved. q.e.d.

LEMMA 5.2. – *Let* Ω *be any open set in* $\mathbf{R}^n$ *and let* $u \in BV(\Omega)$ *be fixed. Then there exists a sequence of functions* $u_j \in C^\infty(\Omega) \cap BV(\Omega)$ *such that*

(5.7) $$u_j \to u \quad \text{in } L^1(\Omega)$$

(5.8) $$\int_\Omega |Du_j| \to \int_\Omega |Du|$$

(5.9) $$\max\lim_{j\to\infty}\int_A |Du_j| \leqq \int_{\bar{A}} |Du| \quad \text{for all open sets } A \subset\subset \Omega$$

(5.10) $$\int_\Omega |Du_j - h|\, dx \to \int |Du - h\, dx| \quad \text{for all } h \in L^1(\Omega) .$$

Moreover:

(5.11) *if* $u \in BV(\Omega) \cap L^q(\Omega)$, $q < +\infty$, *one can find the functions* u_j *such that* $u_j \in L^q(\Omega)$, $u_j \to u$ *in* $L^q(\Omega)$

(5.12) *if* $u \in BV(\Omega) \cap L^\infty(\Omega)$, *one can find the* u_j *such that* $\|u_j\|_{\infty,\Omega} \leqq \|u\|_{\infty,\Omega}$ *and* $u_j \to u$ *in* $L^\infty(\Omega)$*-weak**

(5.13) *if* $u \in BV(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$ *one can find the* u_j *such that* $u_j \to u$ *in* $C^0_{\text{loc}}(\Omega)$ *also holds.*

Finally:

(5.14) *if* $\partial\Omega$ *is Lipschitz continuous one can find the* u_j *such that*

$$u_j|_{\partial\Omega} = u|_{\partial\Omega} \qquad \textit{for all } j\,.$$

PROOF. – (5.7) and (5.8) are proved in [4]; (5.11), (5.12), (5.13) follow easily by the same proof; (5.14) is proved in [7] and (5.9), (5.10) follow easily by adapting to the proof in [4] the argument given in the proof of lemma 5.1.

COROLLARY 5.3. – *If we take* $h = (Du)^a$ *in* (5.10) *we get*

$$\int_\Omega |Du_j - (Du)^a| \to \int_\Omega |Du|^s\,.$$

LEMMA 5.4. – *Assume that* $\partial\Omega$ *is Lipschitz continuous. If* $u \in H^{1,1}(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$ *and* $u|_{\partial\Omega} = 0$, *then there exists a sequence of functions* $g_j \in C^\infty_0(\Omega)$ *such that*

$$g_j \to u \qquad \textit{in } H^{1,1}(\Omega)$$

$$g_j \to u \qquad \textit{in } C^0_{\text{loc}}(\Omega)$$

$$\|g_j\|_{\infty,\Omega} \leqq \|u\|_{\infty,\Omega} \qquad \textit{for all } j\,.$$

The proof of lemma 5.4 can be obtained by standard techniques in Sobolev space theory.

LEMMA 5.5. – *Let* Ω *be a bounded open set in* $\mathbf{R}^n$ *with a Lipschitz boundary. Then, for any given function* $u \in L^1(\partial\Omega)$ *and for any given* $\varepsilon > 0$ *there exists a function* $w \in H^{1,1}(\Omega) \cap C^0(\Omega)$ *such that*

$$w|_{\partial\Omega} = u$$

$$\int_\Omega |Dw| \leqq \int_{\partial\Omega} |u| + \varepsilon$$

$$w(x) = 0 \qquad \textit{if } \operatorname{dist}(x, \partial\Omega) > \varepsilon\,.$$

Moreover, for any fixed number $q \geqq 1$, $q < +\infty$, *one can find the function* w *such that*

$$\|w\|_{L^q(\Omega)} \leqq \varepsilon\,.$$

Finally, if one has also $u \in L^\infty(\Omega)$, *one can find* w *such that*

$$\|w\|_{\infty,\Omega} \leqq \|u\|_{\infty,\partial\Omega} .$$

The proof of Lemma 5.5 is easily obtained by the same technique that GAGLIARDO [6] uses in proving his extension theorem $L^1(\partial\Omega) \to H^{1,1}(\Omega)$.

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