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# Integral Representation and Relaxation of Convex Local Functionals on $BV(\Omega)$

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## Introduction

Relaxation, homogenization, and $\Gamma$-convergence problems for functionals with linear growth (for example, area-type integrals) often lead to the following question: given a bounded open subset $\Omega$ of $\mathbb{R}^N$ and a functional $F(u, A)$, depending on a function $u : \Omega \to \mathbb{R}$ and on an open subset $A$ of $\Omega$, is it possible to give an integral representation of $F$?

If $F(u, \cdot)$ is a measure on $\Omega$ and $F(u, A) = F(v, A)$ whenever $u - v$ is constant on $A$, then it is known that, under very mild additional assumptions, there exists a convex integrand $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ such that for every open subset $A$ of $\Omega$ and for every function $u$ in the Sobolev space $W^{1,1}(\Omega)$, one can write

$$F(u, A) = \int_A f(x, \nabla u(x)) dx,$$

where $\nabla u$ denotes the gradient of $u$ (see the papers [17], [30], [13], [12], [9], [10], [2] and the books [5] and [8]).

These results are still not satisfactory for many applications, since all existence theorems for functionals with linear growth involve the space $BV(\Omega)$ of functions $u \in L^1_{\text{loc}}(\Omega)$ whose distributional gradient $Du$ is an $\mathbb{R}^N$-valued Radon measure with bounded variation in $\Omega$. Therefore, it is necessary to extend the above representation to functions $u$ which are not in the Sobolev space $W^{1,1}(\Omega)$. This problem includes the particular case of the integral representation on $BV(\Omega)$ of the lower semicontinuous envelope $\overline{J}$ of the functional

$$J(u, A) = \begin{cases} \displaystyle\int_A j(x, \nabla u(x)) dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{if } u \in BV(\Omega) \backslash W^{1,1}(\Omega), \end{cases} \tag{0.1}$$

whose properties were studied in [31] under very general hypotheses on the integrand $j$. When $j$ does not depend on $x$, an integral representation of $\overline{J}$ on $BV(\Omega)$ is given in [24] (see also [33], Section 5, for a similar problem on $BD(\Omega)$). More generally, when $j(x, z)$ is convex in $z$ and satisfies suitable continuity assumptions with respect to $x$, it is known (see [22] and [15]) that for every $u \in BV(\Omega)$ the lower semicontinuous envelope $\overline{J}$ of $J$ can be written as

$$\overline{J}(u, A) = \int_A j(x, \nabla u(x)) dx + \int_A j_\infty(x, \nu_u(x)) |D^s u|,$$

where $|D^s u|$ is the variation of the singular part of the measure $Du$ with respect to the Lebesgue measure $\mathcal{L}^N$, $\nabla u$ is the density of the absolutely continuous part, $\nu_u$ is the Radon-Nikodym derivative of the measure $Du$ with respect to its variation $|Du|$, and $j_\infty(x, z)$ is the recession function of $j(x, z)$ with respect to $z$.

Let us remark that the general case where $j(x,z)$ is just measurable in $x$ and convex in $z$ cannot be treated using the methods of [22] and [15]. The main difficulty, when dealing with a general lower semicontinuous functional $F$ on $BV(\Omega)$, is that $F(\cdot,A)$ is not uniquely determined by its restriction to any subspace of $BV(\Omega)$ consisting of smooth functions.

In this paper, denoting by $\mathcal{B}(\Omega)$ the $\sigma$-field of all Borel subsets of $\Omega$, we consider functionals $F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0,+\infty[$ such that:

(H1) for every $u \in BV(\Omega)$ the set function $F(u,\cdot)$ is a Borel measure on $\Omega$;

(H2) for every open subset $A$ of $\Omega$ the function $F(\cdot,A)$ is convex and $L^{N/(N-1)}_{\mathrm{loc}}(\Omega)$-lower semicontinuous on $BV(\Omega)$;

(H3) there exist a real constant $\gamma$ and a bounded Radon measure $\alpha$ on $\Omega$ such that

$$0 \leq F(u,B) \leq \alpha(B) + \gamma \int_B |Du|$$

for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$.

Our main result (Theorem 5.1) is that, under hypotheses (H1), (H2) and (H3) there exist a positive Radon measure $\mu$ on $\Omega$ and two Borel functions $f$, $h : \Omega \times \mathbb{R}^N \to [0,+\infty[$ such that for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$ we have

$$F(u,B) = \int_B f(x, \nabla_\mu u(x))d\mu + \int_B h(x, \nu_u(x))|D^s_\mu u|, \tag{0.2}$$

where $|D^s_\mu u|$ is the variation of the singular part of measure $Du$ with respect to $\mu$, and $\nabla_\mu u$ is the density of the absolutely continuous part with respect to $\mu$. Moreover, for every $x \in \Omega$, $f(x,\cdot)$ is convex and $h(x,\cdot)$ is positively homogeneous of degree one on $\mathbb{R}^N$.

When $F$ is the $L^{N/(N-1)}(\Omega)$-lower semicontinuous envelope $\overline{J}$ of the functional $J$ introduced in (0.1), then $\mu$ is the Lebesgue measure and $f$ and $h$ can be computed explicitly in terms of $j$ (Theorem 4.1). Then the necessary and sufficient conditions for the $L^{N/(N-1)}(\Omega)$-lower semicontinuity of $J$ on $W^{1,1}(\Omega)$, found in [21], follow easily from the equality $f = j$ (Theorem 4.4).

In the last section we consider the problem of the integral representation of $\Gamma$-limits of functionals with linear growth, extending to $BV(\Omega)$ part of the results of [17]. We prove that, under some natural conditions, the $\Gamma$-limit $F$ can be written in the form (0.2). An explicit example shows that the measure $\mu$ might be different from the Lebesgue measure $\mathcal{L}^N$. Finally, we determine a wide class of $\Gamma$-convergence problems for which the integral representation of the $\Gamma$-limit can be obtained with $\mu = \mathcal{L}^N$.

## 1. - Preliminaries

In this section we fix the notation and recall the main properties of measures and functions of bounded variation. Then we discuss some results of [3] and [4] about the duality between functions of bounded variation and bounded vector fields with divergence in $L^N(\Omega)$. This will allow us to study some properties of 1-homogeneous integrands.

### 1.1. - *Measures and functions of bounded variation*

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, let $\mathcal{B}(\Omega)$ be the $\sigma$-field of all Borel subsets of $\Omega$, let $\mathcal{A}(\Omega)$ be the family of all open subsets of $\Omega$, and let $\mathcal{A}_c(\Omega)$ be the family of all open subsets of $\Omega$ with a Lipschitz boundary such that $A \subset\subset \Omega$, i.e., $A$ is relatively compact in $\Omega$.

A Borel measure on $\Omega$ is a countably additive set function $\mu : \mathcal{B}(\Omega) \to ]-\infty, +\infty]$ such that $\mu(\emptyset) = 0$. Each positive Radon measure on $\Omega$ will be identified with the corresponding positive Borel measure $\mu$, which satisfies $\mu(K) < +\infty$ for every compact subset $K$ of $\Omega$. A real valued Radon measure on $\Omega$ is the difference of two positive Radon measures on $\Omega$ and an $\mathbb{R}^n$-valued Radon measure on $\Omega$ will be identified with an $n$-tuple of real valued Radon measures on $\Omega$. The total variation of a scalar or vector measure $\mu$ will be denoted by $|\mu|$. The Lebesgue measure on $\mathbb{R}^N$ will be denoted by $\mathcal{L}^N$ and the $(N-1)$-dimensional Hausdorff measure by $H^{N-1}$.

For any bounded positive Radon measure $\mu$ on $\Omega$, $\mathcal{B}_\mu(\Omega)$ is the $\sigma$-field of all $\mu$-measurable subsets of $\Omega$. A closed valued multifunction $\Phi : \Omega \to \mathbb{R}^n$ is said to be $\mu$-measurable if the set $\Phi^-(C) = \{x \in \Omega : \Phi(x) \cap C \neq \emptyset\}$ is $\mu$-measurable for every closed subset $C$ of $\mathbb{R}^n$. For the general properties of measurable multifunctions we refer to [14], Chapter III.

If $(\Phi_i)_{i\in I}$ is an arbitrary family of closed-valued $\mu$-measurable multifunctions from $\Omega$ into $\mathbb{R}^n$, there exists a closed-valued $\mu$-measurable multifunction $\Phi : \Omega \to \mathcal{P}(\mathbb{R}^n)$ with the following properties (see [34], Proposition 14):

i) for every $i \in I$ we have $\Phi_i(x) \subseteq \Phi(x)$ for $\mu$-a.e. $x \in \Omega$;

ii) if $\Psi : \Omega \to \mathbb{R}^n$ is a closed-valued $\mu$-measurable multifunction such that for every $i \in I$, $\Phi_i(x) \subseteq \Psi(x)$ for $\mu$-a.e. $x \in \Omega$, then $\Phi(x) \subseteq \Psi(x)$ for $\mu$-a.e. $x \in \Omega$.

This multifunction $\Phi$ is unique up to $\mu$-equivalence and will be denoted by

$$\Phi(x) = \mu\text{-}\operatorname*{ess\,sup}_{i\in I} \Phi_i(x). \tag{1.1}$$

The space $C^k_c(\Omega, \mathbb{R}^n)$ $(k = 0, 1, \dots, \infty)$ is the set of all $\mathbb{R}^n$-valued functions of class $C^k$ with compact support in $\Omega$, while $C^k(\overline{\Omega}, \mathbb{R}^n)$ is the set of all $\mathbb{R}^n$-valued functions of class $C^k$ in $\Omega$ whose derivatives up to order $k$ are uniformly continuous in $\Omega$. By $C_0^0(\Omega,\mathbb{R}^n)$ we denote the space of all continuous functions $u:\Omega\to\mathbb{R}^n$ such that the set $\{x\in\Omega : |u(x)|\geq t\}$ is compact in $\Omega$ for every $t>0$. It is a Banach space with the norm $\|u\|_{C_0^0}=\max_{x\in\Omega}|u(x)|$. The dual of $C_0^0(\Omega,\mathbb{R}^n)$ is the space $\mathcal{M}^b(\Omega,\mathbb{R}^n)$ of all $\mathbb{R}^n$-valued Radon measures with bounded total variation on $\Omega$, endowed with the norm $\|\mu\|_{\mathcal{M}^b}=|\mu|(\Omega)$. The duality pairing is given by $\langle\mu,\varphi\rangle=\int_\Omega \varphi d\mu$. The weak* topology on $\mathcal{M}^b(\Omega,\mathbb{R}^n)$ is defined as the weakest topology on $\mathcal{M}^b(\Omega,\mathbb{R}^n)$ for which the maps $\mu\mapsto\int_\Omega\varphi d\mu$ are continuous for every $\varphi\in C_0^0(\Omega,\mathbb{R}^n)$. In other words, a net $(\mu_h)$ in $\mathcal{M}^b(\Omega,\mathbb{R}^n)$ converges weakly to $\mu$ if and only if $\int_\Omega\varphi d\mu_h\to\int_\Omega\varphi d\mu$ for every $\varphi\in C_0^0(\Omega,\mathbb{R}^n)$.

If $n=1$, the previous spaces will be indicated by $C_c^k(\Omega)$, $C^k(\overline{\Omega})$, $C_0^0(\Omega)$ and $\mathcal{M}^b(\Omega)$. The cone of all bounded positive Radon measures on $\Omega$ will be denoted by $\mathcal{M}_+^b(\Omega)$.

Given $\mu\in\mathcal{M}^b(\Omega,\mathbb{R}^n)$ and $\nu\in\mathcal{M}_+^b(\Omega)$, we have a unique Lebesgue decomposition $\mu=\mu_a+\mu_s$, where $\mu_a$ is absolutely continuous and $\mu_s$ is singular with respect to $\nu$. The density of $\mu_a$ with respect to $\nu$ will be indicated by $\frac{d\mu}{d\nu}$ and will be called the Radon-Nikodym derivative of $\mu$ with respect to $\nu$. From the above definition it follows immediately that $\frac{d\mu}{d\nu}=\frac{d\mu_a}{d\nu}$ $\nu$-a.e. in $\Omega$ and $\mu(B)=\int_B\frac{d\mu}{d\nu}d\nu+\mu_s(B)$ for every $B\in\mathcal{B}(\Omega)$.

The space $BV(\Omega)$ of functions of bounded variation is defined as the space of all functions $u\in L^1_{\mathrm{loc}}(\Omega)$ whose distributional gradient $Du$ belongs to $\mathcal{M}^b(\Omega,\mathbb{R}^N)$. The total variation $|Du|$ of the measure $Du$ on a set $B\in\mathcal{B}(\Omega)$ will be indicated, as usual, by $\int_B|Du|$. Accordingly, the integral of a Borel function $f:B\to\mathbb{R}$ with respect to the measure $|Du|$ will be denoted by $\int_B f|Du|$. Similar notation is used for the integrals with respect to the vector measure $Du$. Given $\mu\in\mathcal{M}_+^b(\Omega)$ and $\mu\in BV(\Omega)$, the singular part of the measure $Du$ with respect to $\mu$ will be indicated by $D_\mu^s u$. Moreover we set $\nabla_\mu u=\frac{dDu}{d\mu}$ and $\nu_u=\frac{dDu}{d|Du|}$. Therefore

$$\int_B Du=\int_B\nabla_\mu u d\mu+\int_B\nu_u|D_\mu^s u|$$

for every $B\in\mathcal{B}(\Omega)$. If $\mu$ is the Lebesgue measure, $D_\mu^s u$ and $\nabla_\mu u$ will be denoted by $D^s u$ and $\nabla u$ respectively. The Sobolev space $W^{1,1}(\Omega)$ is, by definition, the set of all functions $u \in BV(\Omega) \cap L^1(\Omega)$ such that $Du \ll \mathcal{L}^N$, i.e. $Du$ is absolutely continuous with respect to the Lebesgue measure. For the general properties of $BV(\Omega)$ we refer to [23], [32], [35] and [26].

It is well known that $BV(\Omega)$ is contained in $L^{N/(N-1)}_{\text{loc}}(\Omega)$ and that, if $\Omega$ has a Lipschitz boundary, then $BV(\Omega)$ is contained in $L^{N/(N-1)}(\Omega)$ and

$$(1.2) \qquad \inf_{t \in \mathbb{R}} \|u - t\|_{L^{N/(N-1)}} \leq c \int_\Omega |Du| \qquad \forall u \in BV(\Omega),$$

where the constant $c$ depends only on $\Omega$ (see [26], Section 6.1.7).

The following proposition gives a useful characterization of the measures of $\mathcal{M}^b(\Omega, \mathbb{R}^N)$ which are gradients of functions of $BV(\Omega)$.

PROPOSITION 1.1. *The set* $\{Du : u \in BV(\Omega)\}$ *is closed in the weak* topology of* $\mathcal{M}^b(\Omega, \mathbb{R}^N)$. *Moreover, for every* $\mu \in \mathcal{M}^b(\Omega, \mathbb{R}^N)$ *the following conditions are equivalent:*

(i) *there exists* $u \in BV(\Omega)$ *such that* $\mu = Du$ *in* $\Omega$*;*

(ii) $\int_\Omega \varphi d\mu = 0$ *for every* $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$ *with* $\operatorname{div} \varphi = 0$ *in* $\Omega$.

PROOF. Let us prove first that $\mathcal{E} = \{Du : u \in BV(\Omega)\}$ is closed in the weak* topology of $\mathcal{M}^b(\Omega, \mathbb{R}^N)$. By the Krein-Šmulian theorem (see [19], Theorem V.5.7) it is enough to show that the intersection of $\mathcal{E}$ with every closed ball in $\mathcal{M}^b(\Omega, \mathbb{R}^N)$ is weakly*-closed. Since the weak* topology is metrizable on any closed ball of $\mathcal{M}^b(\Omega, \mathbb{R}^N)$, it is enough to prove that $\mathcal{E}$ is sequentially weakly*-closed. Let $(\mu_n)$ be a sequence in $\mathcal{E}$ which converges to some measure $\mu$ in the weak* topology of $\mathcal{M}^b(\Omega, \mathbb{R}^N)$. Then, for every $n \in \mathbb{N}$, there exists $u_n \in BV(\Omega)$ such that $\mu_n = Du_n$. Let $A \in \mathcal{A}_c(\Omega)$. By adding, if necessary, a constant to $u_n$, we may assume (see (1.2)) that

$$\|u_n\|_{L^{N/(N-1)}(A)} \leq c \int_\Omega |Du_n| = c\|\mu_n\|_{\mathcal{M}^b},$$

and, since $(\mu_n)$ is bounded in $\mathcal{M}^b(\Omega, \mathbb{R}^N)$, there exists a subsequence of $(u_n)$ which converges to some function $u_A$ weakly in $L^{N/(N-1)}(A)$. It is then easy to check that $\mu|_A$ is the gradient of $u_A$ in the distributional sense on $A$. Repeating this argument for every $A \in \mathcal{A}_c(\Omega)$, we can construct $u \in L^{N/N-1}_{\text{loc}}(\Omega)$ such that $\mu = Du$. This shows that $\mu \in \mathcal{E}$ and proves that $\mathcal{E}$ is weakly*-closed in $\mathcal{M}^b(\Omega, \mathbb{R}^N)$.

Let us prove now the equivalence between (i) and (ii). Since $\int_\Omega \varphi Du = -\int_\Omega u \operatorname{div} \varphi dx$ for every $u \in BV(\Omega)$ and for every $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$, it is clear that (i) implies (ii).

Assume now (ii) and let $A \in \mathcal{A}_c(\Omega)$. Using the standard approximation technique by convolutions, we can prove that

$$\int_\Omega \varphi d\mu = 0 \qquad \forall \varphi \in C_c^0(\Omega, \mathbb{R}^N),\ \operatorname{div}\varphi = 0 \text{ in } \Omega,\ \varphi = 0 \text{ in } \Omega\backslash A. \tag{1.3}$$

By the previous part of the proof the set $\mathcal{E}_A = \{Du : u \in BV(A)\}$ is weakly*-closed in $\mathcal{M}^b(A, \mathbb{R}^N)$. Therefore, by the Hahn-Banach Theorem, if $\mu|_A \notin \mathcal{E}_A$, then there exists $\psi \in C_0^0(A, \mathbb{R}^N)$ such that $\int_A \psi d\mu \neq 0$ and $\int_A \psi Du = 0$ for every $u \in BV(A)$. If $\varphi$ is the function defined by $\varphi = \psi$ on $A$ and $\varphi = 0$ on $\Omega\backslash A$, then $\varphi \in C_c^0(\Omega, \mathbb{R}^N)$,

$$\int_\Omega \varphi d\mu = \int_A \psi d\mu \neq 0, \tag{1.4}$$

and $\int_\Omega \varphi Du = \int_A \psi Du = 0$ for every $u \in BV(\Omega)$. By taking $u \in C_c^\infty(\Omega)$ we get $\operatorname{div}\varphi = 0$ in the sense of distributions in $\Omega$, hence (1.4) contradicts (1.3). This proves that $\mu|_A \in \mathcal{E}_A$. Therefore, for any $A \in \mathcal{A}_c(\Omega)$ there exists $u_A \in BV(A)$ such that $\mu = Du_A$ in $A$. This allows us to construct $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\mu = Du$ in the sense of distributions in $\Omega$. □

## 1.2. - *Duality and divergence theorem: the space $X(\Omega)$*

In order to obtain a good duality pairing involving $BV(\Omega)$, we introduce the Banach space

$$X(\Omega) = \{\sigma \in L^\infty(\Omega, \mathbb{R}^N) : \operatorname{div}\sigma \in L^N(\Omega)\}$$

endowed with the norm $\|\sigma\|_X = \|\sigma\|_{L^\infty} + \|\operatorname{div}\sigma\|_{L^N}$. The weak* topology(1) on

(1) The Banach space $X(\Omega)$ is (isometric to) the dual of the Banach space $Y/Z$, where $Y = L^1(\Omega, \mathbb{R}^N) \times L^{N/(N-1)}(\Omega)$ endowed with the norm $\|(\psi, u)\|_Y = \max\{\|\psi\|_{L^1}, \|u\|_{L^{N/(N-1)}}\}$, and $Z$ is the closure in $Y$ of the vector space $\{(\nabla u, u) : u \in C_c^\infty(\Omega)\}$. In fact, it is easy to check that $Z^\perp = \{(\sigma, \operatorname{div}\sigma) : \sigma \in X(\Omega)\} \subseteq L^\infty(\Omega, \mathbb{R}^N) \times L^N(\Omega) = Y^*$, and that $Z^\perp$ (endowed with the dual norm $\|(\sigma, f)\|_{Y^*} = \|\sigma\|_{L^\infty} + \|f\|_{L^N}$) is isometric to $(Y/Z)^*$ (see [19], Exercise II.4.18(b)). Denoting the $Z$-equivalence class of $(\psi, u) \in Y$ by $[\psi, u]$, the isometry $\sigma \mapsto \Phi_\sigma$ between $X(\Omega)$ and $(Y/Z)^*$ is defined by

$$\Phi_\sigma([\psi, u]) = \int_\Omega \psi\sigma\, dx + \int_\Omega u \operatorname{div}\sigma\, dx \qquad \forall \sigma \in X(\Omega),\ \forall \psi \in L^1(\Omega, \mathbb{R}^N),\ \forall u \in L^{N/(N-1)}(\Omega).$$

Having identified $X(\Omega)$ with $(Y/Z)^*$, the corresponding weak* topology is defined as the weakest topology on $X(\Omega)$ for which the maps $\sigma \mapsto \Phi_\sigma([\psi, u])$ are continuous for every $\psi \in L^1(\Omega, \mathbb{R}^N)$ and for every $u \in L^{N/(N-1)}(\Omega)$.

$X(\Omega)$ is defined as the weakest topology on $X(\Omega)$ for which the maps

$$\sigma \mapsto \int_{\Omega} \psi\sigma \, dx + \int_{\Omega} u \operatorname{div} \sigma \, dx$$

are continuous for every $\psi \in L^1(\Omega, \mathbb{R}^N)$ and for every $u \in L^{N/(N-1)}(\Omega)$. In other words, a net $(\sigma_h)$ in $X(\Omega)$ converges weakly* to $\sigma$ if and only if

$$\sigma_h \to \sigma \text{ weakly* in } L^\infty(\Omega, \mathbb{R}^N), \qquad \operatorname{div} \sigma_h \to \operatorname{div} \sigma \text{ weakly in } L^N(\Omega).$$

If we identify $X(\Omega)$ with the subspace $\{(\sigma, \operatorname{div} \sigma) : \sigma \in X(\Omega)\}$ of $L^\infty(\Omega, \mathbb{R}^N) \times L^N(\Omega)$, then the weak* topology of $X(\Omega)$ coincides with the product of the weak* topology of $L^\infty(\Omega, \mathbb{R}^N)$ and the weak topology of $L^N(\Omega)$. Therefore, the weak* topology of $X(\Omega)$ is metrizable on all bounded subsets of $X(\Omega)$, and, by the Banach-Alaoglu Theorem, all bounded weakly*-closed subsets of $X(\Omega)$ are weakly*- sequentially compact.

We now consider the notion of normal trace for vector fields $\sigma \in X(\Omega)$. For $H^{N-1}$-a.e $x \in \partial\Omega$ we denote by $\nu_\Omega(x)$ the outer unit normal to $\Omega$ at $x$. The following proposition is proved in [3], Section 1.

PROPOSITION 1.2. *Assume that $\Omega$ has a Lipschitz boundary. Then for every $\sigma$ in $X(\Omega)$, there exists a unique function $[\sigma \cdot \nu_\Omega]$ in $L^\infty_{H^{N-1}}(\partial\Omega)$ such that*

$$\int_{\partial\Omega} [\sigma \cdot \nu_\Omega] u \, dH^{N-1} = \int_{\Omega} u \operatorname{div} \sigma \, dx + \int_{\Omega} \sigma \nabla u \, dx \qquad \forall u \in C^1(\overline{\Omega}). \tag{1.5}$$

We can extend the previous identity to functions $u \in BV(\Omega)$ by giving a meaning to the integral $\int_{\Omega} (\sigma \cdot Du)$, which defines a duality product between $BV(\Omega)$ and $X(\Omega)$. For every $u \in BV(\Omega)$ and for every $\sigma \in X(\Omega)$ we define a measure $(\sigma \cdot Du)$ by setting

$$\int_{\Omega} \varphi(\sigma \cdot Du) = -\int_{\Omega} u\varphi \operatorname{div} \sigma \, dx - \int_{\Omega} u\sigma \nabla \varphi \, dx \qquad \forall \varphi \in C^1_c(\Omega). \tag{1.6}$$

The following proposition collects some properties of the measure $(\sigma \cdot Du)$ proved in [3].

PROPOSITION 1.3. *For every $u \in BV(\Omega)$ and for every $\sigma \in X(\Omega)$ formula (1.6) defines a Radon measure on $\Omega$, denoted by $(\sigma \cdot Du)$, which is absolutely continuous with respect to $|Du|$. Moreover*

(i) *for every $B \in \mathcal{B}(\Omega)$ we have $\int_B |(\sigma \cdot Du)| \leq c \int_B |Du|$, where $c = \|\sigma\|_{L^\infty}$ and the left-hand side denotes the variation of the measure $(\sigma \cdot Du)$ on the set $B$;*

(ii) *for every* $u \in BV(\Omega)$ *the linear operator* $\sigma \mapsto \dfrac{d(\sigma \cdot Du)}{d|Du|}$ *is sequentially continuous from* $X(\Omega)$ *into* $L^\infty_{|Du|}(\Omega)$, *when both spaces are endowed with the weak* topology;*

(iii) *for every* $\varphi \in C^1(\overline{\Omega})$, $u \in BV(\Omega)$, $\sigma \in X(\Omega)$ *one has* $(\varphi\sigma \cdot Du) = \varphi(\sigma \cdot Du)$ *and* $(\sigma \cdot D(\varphi u)) = \varphi(\sigma \cdot Du) + u\sigma\nabla\varphi dx$ *as measures in* $\Omega$*;*

(iv) *for every* $u \in BV(\Omega)$ *and for every* $\sigma \in X(\Omega)$ *the function* $\sigma\nabla u$ *is the Radon-Nikodym derivative of the measure* $(\sigma \cdot Du)$ *with respect to the Lebesgue measure;*

(v) *if* $\sigma \in C^1(\overline{\Omega}, \mathbb{R}^N)$, *then* $\int_B (\sigma \cdot Du) = \int_B \sigma Du$ *for every* $u \in BV(\Omega)$ *and for every* $B \in \mathcal{B}(\Omega)$*;*

(vi) *if* $\Omega$ *has a Lipschitz boundary, then one has*

$$\int_{\partial\Omega} [\sigma \cdot \nu_\Omega] u dH^{N-1} = \int_\Omega (\sigma \cdot Du) + \int_\Omega u \operatorname{div} \sigma \, dx$$

*for every* $u \in BV(\Omega)$ *and for every* $\sigma \in X(\Omega)$.

From assertion (iv) of Proposition 1.3 we deduce that $\sigma\nu_u$ is the Radon-Nikodym derivative of $(\sigma \cdot Du)$ with respect to $|Du|$ when $u \in W^{1,1}(\Omega)$. In order to obtain a representation of $\dfrac{d(\sigma \cdot Du)}{d|Du|}$ in the general case, we introduce the following definition (see [4], Definition (2.9)).

DEFINITION 1.4. For every $\sigma \in X(\Omega)$ let $q_\sigma : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be the Borel function defined by

$$q_\sigma(x, z) = \limsup_{\rho \to 0_+} \limsup_{r \to 0_+} \frac{1}{\mathcal{L}^N(C_{r,\rho}(x, \varsigma))} \int_{C_{r,\rho}(x,\varsigma)} \sigma(y) z dy, \tag{1.7}$$

where $\varsigma = z/|z|$ and $C_{r,\rho}(x, \varsigma)$ is the cylinder

$$C_{r,\rho}(x, \varsigma) = \{y \in \mathbb{R}^N : |(y - x)\varsigma| < r, |(y - x) - [(y - x)\varsigma]\varsigma| < \rho\}.$$

REMARK 1.5. It follows immediately from the definition that $q_\sigma(x, z)$ is positively 1-homogeneous with respect to $z$. Moreover, for every $(x, z) \in \Omega \times \mathbb{R}^N$ we have $|q_\sigma(x, z)| \leq |z| \, \|\sigma\|_{L^\infty}$ and $q_{\varphi\sigma}(x, z) = \varphi(x) q_\sigma(x, z)$, provided $\varphi \in C^1(\overline{\Omega})$ and $\varphi \geq 0$ on $\Omega$.

The following proposition is proved in [4].

PROPOSITION 1.6. *For every* $u \in BV(\Omega)$ *and for every* $\sigma \in X(\Omega)$ *we have*

$\dfrac{d(\sigma \cdot Du)}{d|Du|} = q_\sigma(x, \nu_u)$ *$|Du|$-a.e. in $\Omega$, i.e.,*

$$\int_B (\sigma \cdot Du) = \int_B q_\sigma(x,\nu_u)|Du| \qquad \forall u \in BV(\Omega).$$

*Moreover for $|Du|$-a.e. $x \in \Omega$ and for $z = \nu_u(x)$ the upper limits in* (1.7) *are actually limits.*

The following lemma will be used in the next section.

LEMMA 1.7. *Let $A \in \mathcal{A}_c(\Omega)$ and let $\sigma$ be an element of $X(\Omega)$ such that $\sigma = 0$ $\mathcal{L}^N$-a.e. on $\Omega\setminus A$. Then, for every $u \in BV(\Omega)$ the measure $(\sigma \cdot Du)$ is identically zero on $\Omega\setminus A$, hence $q_\sigma(x,\nu_u) = 0$ blank space $|Du|$-a.e. on $\Omega\setminus A$.*

PROOF. Since $\sigma = 0$ a.e. on $\Omega\setminus A$, we have $\operatorname{div}\sigma = 0$ in the sense of distributions on $\Omega\setminus\overline{A}$. As $\operatorname{div}\sigma \in L^N(\Omega)$ and $\mathcal{L}^N(\partial A) = 0$, we get $\operatorname{div}\sigma = 0$ $\mathcal{L}^N$-a.e. on $\Omega\setminus A$. Therefore, it follows immediately from (1.5) that $[\sigma \cdot \nu_A] = 0$ $H^{N-1}$-a.e. on $\partial A$. Let $u \in BV(\Omega)$ and $\varphi \in C^1_c(\Omega)$. If we apply Proposition 1.3(vi) with $\Omega$ replaced by $A$ and $u$ replaced by $\varphi u$, by (1.6) and by Proposition 1.3(iii) we obtain

$$\begin{aligned}\int_\Omega \varphi(\sigma \cdot Du) &= -\int_\Omega u\varphi \operatorname{div}\sigma\, dx - \int_\Omega u\sigma \nabla\varphi dx \\ &= -\int_A u\varphi \operatorname{div}\sigma\, dx - \int_A u\sigma \nabla\varphi dx = \int_A \varphi(\sigma \cdot Du),\end{aligned}$$

which implies that the measure $(\sigma \cdot Du)$ is identically zero on $\Omega\setminus A$. The conclusion follows now from Proposition 1.6. □

### 1.3. - *Homogeneous integrands*

A homogeneous integrand is, for us, a Borel function $h : \Omega \times \mathbb{R}^N \to ]-\infty, +\infty]$, satisfying the following conditions:

(a) $h(x, tz) = th(x, z)$ for every $x \in \Omega$, $z \in \mathbb{R}^N$, $t > 0$;

(b) there exists a negative constant $c \in \mathbb{R}$, such that $h(x, z) \geq c|z|$ for every $x \in \Omega$, $z \in \mathbb{R}^N$.

In what follows, we shall use a partial order between integrands. Given two homogeneous integrands, we define the relations $h_1 \lesssim h_2$ and $h_1 \sim h_2$ by

$$h_1 \lesssim h_2 \Leftrightarrow \forall u \in BV(\Omega),\ h_1(x,\nu_u) \leq h_2(x,\nu_u)\ |Du|\text{-a.e. in } \Omega, \tag{1.8}$$

$$h_1 \sim h_2 \Leftrightarrow \forall u \in BV(\Omega),\ h_1(x,\nu_u) = h_2(x,\nu_u)\ |Du|\text{-a.e. in } \Omega. \tag{1.9}$$

If the inequality in (1.8) (resp. the equality in (1.9)) occurs only $|Du|$-a.e. on some Borel subset $B$ of $\Omega$, we shall write $h_1 \lesssim h_2$ on $B$ (resp. $h_1 \sim h_2$ on $B$).

In the following proposition we associate a homogeneous integrand $h_K$ with any subset $K$ of $X(\Omega)$.

PROPOSITION 1.8. *Let $K$ be a subset of $X(\Omega)$. Then there exists a homogeneous integrand $h_K$ such that:*

(i) $q_\sigma \lesssim h_K$ *for every* $\sigma \in K$;

(ii) *if $h$ is a homogeneous integrand such that $q_\sigma \lesssim h$ for every $\sigma \in K$, then* $h_K \lesssim h$.

*Moreover, for every $u \in BV(\Omega)$ we have*

$$(1.10) \qquad h_K(x, \nu_u) = |Du|\text{-}\operatorname*{ess\,sup}_{\sigma \in K} q_\sigma(x, \nu_u) \qquad |Du|\text{-a.e. in } \Omega.$$

*If $h'$ is another homogeneous integrand satisfying* (i) *and* (ii), *then $h' \sim h_K$. Finally, if $D$ is countable and sequentially weakly*-dense in $K$, and $h_D$ is defined by*

$$(1.11) \qquad h_D(x, z) = \sup_{\sigma \in D} q_\sigma(x, z),$$

*then $h_D \sim h_K$.*

PROOF. Since $X(\Omega)$ is a countable union of compact metrizable subsets, we can choose a countable subset $D$ of $K$ which is sequentially weakly*-dense in $K$. Let $h_D$ be the homogeneous integrand defined by (1.11), let $\sigma \in K$, and let $u \in BV(\Omega)$. Since $D$ is sequentially weakly*-dense in $K$, there exists a sequence $(\sigma_n)$ in $D$ converging to $\sigma$ weakly* in $X(\Omega)$. By Proposition 1.3(ii) and 1.6 the sequence $(q_{\sigma_n}(x, \nu_u))$ converges to $q_\sigma(x, \nu_u)$ in the weak* topology of $L^\infty_{|Du|}(\Omega)$. Since $q_{\sigma_n}(x, \nu_u) \leq h_D(x, \nu_n)$ $|Du|$-a.e. in $\Omega$, we get $q_\sigma(x, \nu_u) \leq h_D(x, \nu_u)$ $|Du|$-a.e. in $\Omega$. As this inequality holds for every $u \in BV(\Omega)$, we have $q_\sigma \lesssim h_D$ for every $\sigma \in K$. If $h$ is a homogeneous integrand such that $q_\sigma \lesssim h$ for every $\sigma \in K$, then $q_\sigma(x, \nu_u) \leq h(x, \nu_u)$ $|Du|$-a.e. in $\Omega$ for every $\sigma \in D$. By (1.11) this implies that $h_D(x, \nu_u) \leq h(x, \nu_u)$ $|Du|$-a.e. in $\Omega$, hence $h_D \lesssim h$. This shows that $h_D$ satisfies conditions (i) and (ii) and proves the existence of $h_K$.

If $h'$ is another homogeneous integrand satisfying (i) and (ii), then $h' \lesssim h_K$ and $h_K \lesssim h'$, hence $h' \sim h_K$. In particular, since $h_D$ satisfies (i) and (ii), we have $h_D \sim h_K$.

If $u \in BV(\Omega)$, then the inequality

$$h_K(x, \nu_u) \geq |Du|\text{-}\operatorname*{ess\,sup}_{\sigma \in K} q_\sigma(x, \nu_u) \qquad |Du|\text{-a.e. in } \Omega$$

is a consequence of (i). Since the opposite inequality is trivial for $h_D$, (1.10) follows from the equivalence $h_D \sim h_K$. $\square$

We conclude this section with a technical lemma which will be very useful in Section 4.

LEMMA 1.9. *Let $A \in \mathcal{A}_c(\Omega)$, let $K$ and $K'$ be two subsets of $X(\Omega)$ with $K' \subseteq K$, and let $h$ and $h'$ be the corresponding homogeneous integrands. Suppose that $\sigma = 0$ $\mathcal{L}^N$-a.e. on $\Omega \backslash A$ for every $\sigma \in K'$ and that $\varphi\sigma \in K'$ for every $\sigma \in K$ and for every $\varphi \in C^1_c(A)$ with $0 \leq \varphi \leq 1$. Then $h' \sim h_A$, where $h_A(x, z) = h(x, z)$ if $x \in A$, and $h_A(x, z) = 0$ if $x \in \Omega \backslash A$.*

PROOF. By Lemma 1.7 and by (1.10) we have $h' \sim 0$ on $\Omega \backslash A$. Since $K' \subseteq K$, it is obvious that $h' \lesssim h$ on $A$. To prove the opposite inequality, let $A'$ be an open set with $A' \subset\subset A$ and let $\varphi \in C^1_c(A)$ with $\varphi = 1$ on $A'$ and $0 \leq \varphi \leq 1$ on $A$. For every $\sigma \in K$ we have $\varphi\sigma \in K'$ and, by Remark 1.5, we have $q_\sigma(x, z) = q_{\varphi\sigma}(x, z)$ for every $x \in A'$ and for every $z \in \mathbb{R}^N$. By (1.10) this implies $h(x, \nu_u) \leq h'(x, \nu_u)$ $|Du|$-a.e. in $A'$, for every $u \in BV(\Omega)$. Since $A'$ is an arbitrary open set with $A' \subset\subset A$, we obtain $h(x, \nu_u) \leq h'(x, \nu_u)$ $|Du|$-a.e. in $A$, which gives $h \lesssim h'$ on $A$. $\square$

## 2. - Representation with respect to a given measure

Let $F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$ be a functional satisfying the hypotheses (H1), (H2) and (H3) considered in the Introduction. Our aim in this section is to give an integral representation of $F(u, B)$ for those functions $u$ of $BV(\Omega)$ whose gradient $Du$ is absolutely continuous with respect to a prescribed measure $\mu$.

Throughout this section we fix a measure $\mu \in \mathcal{M}^b_+(\Omega)$, with $\mathcal{L}^N \ll \mu$, and we consider the space $W^{1,1}_\mu(\Omega)$ defined by

$$W^{1,1}_\mu(\Omega) = \{u \in BV(\Omega) : Du \ll \mu\}. \tag{2.1}$$

As the Lebesgue measure in absolutely continuous with respect to $\mu$, the space $W^{1,1}_\mu(\Omega)$ is not trivial, because of the inclusion $W^{1,1}(\Omega) \subseteq W^{1,1}_\mu(\Omega)$ (that justifies our assumption on $\mu$).

We shall also use the closed-valued multifunction $E_\mu : \Omega \to \mathcal{P}(\mathbb{R}^N)$ defined (see (1.1)) by

$$E_\mu(x) = \mu\text{-}\operatorname*{ess\,sup}_{u \in BV(\Omega)} \{\nabla_\mu u(x)\}. \tag{2.2}$$

Recall that, by definition, $E_\mu$ is the smallest closed-valued $\mu$-measurable multifunction such that $\nabla_\mu u(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$ for every $u \in BV(\Omega)$. An easy way to construct $E_\mu$ is to choose a countable subset $D$ of $BV(\Omega)$ such that the set $\{\nabla_\mu u : u \in D\}$ is dense in the subspace $\mathcal{E}_\mu = \{\nabla_\mu u : u \in BV(\Omega)\}$ of $L^1_\mu(\Omega, \mathbb{R}^N)$. Then one easily checks that

$$E_\mu(x) = \mathrm{cl}\{\nabla_\mu u(x) : u \in D\} \text{ for } \mu\text{-a.e. } x \in \Omega, \tag{2.3}$$

where cl denotes the closure in $\mathbb{R}^N$. Since $\mathcal{E}_\mu$ is a linear space, we may assume that the set $D$ is closed under finite linear combinations with rational coefficients. Therefore (2.3) shows that $E_\mu(x)$ is a linear subspace of $\mathbb{R}^N$ for $\mu$-a.e. $x \in \Omega$. By taking $u$ linear one easily checks that

$$E_\mu(x) = \mathbb{R}^N \qquad \text{for } \mathcal{L}^N\text{- a.e. } x \in \Omega. \tag{2.4}$$

REMARK 2.1. When $\mu$ is the Lebesgue measure $\mathcal{L}^N$, then $E_\mu(x) = \mathbb{R}^N$ for $\mu$-a.e. $x \in \Omega$. Let us underline the fact that this is not true for a general $\mu \in \mathcal{M}^b_+(\Omega)$. For instance, if

$$\mu(B) = \mathcal{L}^N(B) + H^{N-1}(B \cap \Sigma) \qquad \forall B \in \mathcal{B}(\Omega),$$

where $\Sigma$ is a smooth hypersurface in $\mathbb{R}^N$, then $E_\mu(x) = \mathbb{R}^N$ for $\mu$-a.e. $x \in \Omega \setminus \Sigma$ and $E_\mu(x) = \{t\nu(x) : t \in \mathbb{R}\}$ for $\mu$-a.e. $x \in \Omega \cap \Sigma$, where $\nu(x)$ is a unit vector normal to $\Sigma$ at $x$.

We shall use the following result due to G. Alberti.

THEOREM 2.2. *Let $\mu \in \mathcal{M}^b_+(\Omega)$, with $\mathcal{L}^N \ll \mu$, and let $\psi \in L^1_\mu(\Omega, \mathbb{R}^N)$ be a function such that $\psi(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Then there exists $u \in BV(\Omega)$ such that $\nabla_\mu u = \psi$ $\mu$-a.e. in $\Omega$.*

PROOF. See [1], Theorem 2.12. □

## 2.1. - *Representation of the absolutely continuous part of a local functional*

We now prove a general integral representation theorem for the absolutely continuous part (with respect to $\mu$) of a Lipschitz continuous local functional. This result will be used to study the Radon-Nikodym derivative (with respect to $\mu$) of the duality measure $(\sigma \cdot Du)$ defined by (1.6) (Section 2.2), and to obtain an integral representation of the functional $F$ on the space $W^{1,1}_\mu(\Omega)$ introduced in (2.1) (Section 2.3).

THEOREM 2.3. *Let $\mu \in \mathcal{M}^b_+(\Omega)$, with $\mathcal{L}^N \ll \mu$, and let $G : BV(\Omega) \times \mathcal{B}(\Omega) \to \mathbb{R}$ be a functional with the following properties:*

(a) $G(0, B) = 0$ *for every* $B \in \mathcal{B}(\Omega)$;

(b) *there exists a constant* $c > 0$ *such that* $|G(u,B) - G(v,B)| \leq c \int_B |Du - Dv|$ *for every* $u, v \in BV(\Omega)$ *and for every* $B \in \mathcal{B}(\Omega)$;

(c) $G(u, \cdot)$ *is a Borel measure on* $\Omega$ *for every* $u \in BV(\Omega)$.

*For every* $u \in BV(\Omega)$ *let* $G_a(u, \cdot)$ *be the absolutely continuous part of the measure* $G(u, \cdot)$ *with respect to* $\mu$. *Then there exists a Borel integrand* $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *such that*

(i) $j(x, 0) = 0$ *for* $\mu$*-a.e.* $x \in \Omega$;

(ii) *for* $\mu$*-a.e.* $x \in \Omega$ *and for every* $z_1, z_2 \in \mathbb{R}^N$ *we have* $|j(x, z_1) - j(x, z_2)| \leq c|p_x(z_1) - p_x(z_2)|$, *where* $p_x$ *denotes the orthogonal projection on* $E_\mu(x)$;

(iii) *for every* $u \in BV(\Omega)$ *and for every* $B \in \mathcal{B}(\Omega)$ *we have*

$$G_a(u, B) = \int_B j(x, \nabla_\mu u) d\mu.$$

*If* $j'$ *is another* $\mathcal{B}_\mu(\Omega) \times \mathcal{B}(\mathbb{R}^N)$*-measurable integrand satisfying* (i), (ii) *and* (iii), *then* $j(x, \cdot) = j'(x, \cdot)$ *on* $\mathbb{R}^N$ *for* $\mu$*-a.e.* $x \in \Omega$. *If* $G(\cdot, A)$ *is convex (resp. linear) for every open subset* $A$ *of* $\Omega$, *so is the function* $j(x, \cdot)$ *for* $\mu$*-a.e.* $x \in \Omega$.

PROOF. From (b) we obtain

$$|G_a(u, B) - G_a(v, B)| \leq c \int_B |\nabla_\mu u - \nabla_\mu v| d\mu \qquad \forall u, \ v \in BV(\Omega), \ \forall B \in \mathcal{B}(\Omega). \tag{2.5}$$

Using (a), we obtain also $|G_a(u, B)| \leq c \int_B |\nabla_\mu u| d\mu$ for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$. Therefore, by the Radon-Nikodym Theorem, for every $u \in BV(\Omega)$ there exists $f_u \in L^1_\mu(\Omega)$ such that

$$G_a(u, B) = \int_B f_u d\mu \qquad \forall B \in \mathcal{B}(\Omega). \tag{2.6}$$

By localization, (2.5) implies that for every $u, v \in BV(\Omega)$ we have

$$|f_u(x) - f_v(x)| \leq c|\nabla_\mu u(x) - \nabla_\mu v(x)| \qquad \text{for } \mu\text{-a.e. } x \in \Omega. \tag{2.7}$$

Choose now a countable subset $D$ of $BV(\Omega)$ containing 0 such that the set $\{\nabla_\mu u : u \in D\}$ is $L^1_\mu(\Omega, \mathbb{R}^N)$-dense in the set $\mathcal{E}_\mu = \{\nabla_\mu u : u \in BV(\Omega)\}$ and, consequently,

$$E_\mu(x) = \mathrm{cl}\{\nabla_\mu u(x) : u \in D\} \qquad \text{for } \mu\text{-a.e. } x \in \Omega. \tag{2.8}$$

As $D$ is countable, there exists $M \subseteq \Omega$, with $\mu(M) = 0$, such that (2.7) and (2.8) hold for every $x \in \Omega \backslash M$ and for every $(u, v) \in D \times D$. Moreover we may assume that $E_\mu(x)$ is a linear subspace of $\mathbb{R}^N$ for every $x \in \Omega \backslash M$.

For every $\delta > 0$ consider the function $\psi_{u,\delta} : \Omega \times \mathbb{R}^N \to ]-\infty, +\infty]$ defined by

$$\psi_{u,\delta}(x, z) = \begin{cases} f_u(x) & \text{if } x \in \Omega \backslash M \text{ and } |p_x(z) - \nabla_\mu u(x)| < \delta, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.9}$$

and define $j_\mu : \Omega \times \mathbb{R}^N \to \overline{\mathbb{R}}$ by

$$j_\mu(x, z) = \sup_{\delta > 0} \inf_{u \in D} \psi_{u,\delta}(x, z) = \liminf_{\delta \to 0_+} \inf_{u \in D} \psi_{u,\delta}(x, z) \tag{2.10}$$

(note that $\psi_{u,\delta}(x, z)$ is decreasing in $\delta$).

Since $D$ is countable and $(x, z) \mapsto p_x(z)$ is $\mathcal{B}_\mu(\Omega) \times \mathcal{B}(\mathbb{R}^N)$-measurable, (see [14], Theorem III.41), the integrand $j_\mu$ is $\mathcal{B}_\mu(\Omega) \times \mathcal{B}(\mathbb{R}^N)$-measurable.

Let us prove that for every $u \in D$ we have

$$j_\mu(x, \nabla_\mu u(x)) = f_u(x) \qquad \forall x \in \Omega \backslash M. \tag{2.11}$$

By (2.8) we have $\nabla_\mu u(x) \in E_\mu(x)$, hence $p_x(\nabla_\mu u(x)) = \nabla_\mu u(x)$ for every $x \in \Omega \backslash M$. This implies, by the definition (2.9) of $\psi_{u,\delta}$, that $\psi_{u,\delta}(x, \nabla_\mu u(x)) = f_u(x)$ for every $x \in \Omega \backslash M$ and for every $\delta > 0$, hence

$$j_\mu(x, \nabla_\mu u(x)) \leq f_u(x) \qquad \forall x \in \Omega \backslash M. \tag{2.12}$$

On the other hand (2.7) and (2.9) imply that for any $v \in D$ we have $\psi_{v,\delta}(x, \nabla_\mu u(x)) \geq f_u(x) - c\delta$ for every $x \in \Omega \backslash M$. Therefore, the definition (2.10) of $j_\mu$ gives $j_\mu(x, \nabla_\mu u(x)) \geq f_u(x)$ for every $x \in \Omega \backslash M$, which, together with (2.12), yields (2.11).

Let us prove that $j_\mu$ satisfies conditions (i), (ii) and (iii). Since $0 \in D$, by (2.6) and (2.11) we obtain $G_a(0, B) = \int_B j_\mu(x, 0) d\mu$ for every $B \in \mathcal{B}(\Omega)$. Condition (i) follows now from (a).

Let us prove (ii). For every $x \in \Omega \backslash M$ and for every $z \in \mathbb{R}^N$, by (2.9) and (2.10) there exists a sequence $(u_n)$ in $D$ such that

$$\nabla_\mu u_n(x) \to p_x(z), \qquad f_{u_n}(x) \to j_\mu(x, z). \tag{2.13}$$

Let us apply (2.13) with $z = z_1$ and $z = z_2$. Then there exist two sequences $(u_n)$ and $(v_n)$ in $D$ such that

$$|p_x(z_1) - p_x(z_2)| = \lim_{n \to \infty} |\nabla_\mu u_n(x) - \nabla_\mu v_n(x)|, \tag{2.14}$$

$$|j_\mu(x, z_1) - j_\mu(x, z_2)| = \lim_{n \to \infty} |f_{u_n}(x) - f_{v_n}(x)|. \tag{2.15}$$

By (2.7) we also have $|f_{u_n}(x) - f_{v_n}(x)| \leq c|\nabla_\mu u_n(x) - \nabla_\mu v_n(x)|$, which, together with (2.14) and (2.15), gives (ii).

By (2.6) and (2.11) we have

$$G_a(u, B) = \int_B j_\mu(x, \nabla_\mu u) d\mu \qquad \forall u \in D,\ \forall B \in \mathcal{B}(\Omega). \tag{2.16}$$

Let now $u \in BV(\Omega)$. Since $\nabla_\mu u \in \mathcal{E}_\mu$ and $D$ is dense in $\mathcal{E}_\mu$, there exists a sequence $(u_n)$ in $D$ such that $(\nabla_\mu u_n)$ converges to $\nabla_\mu u$ in $L^1_\mu(\Omega, \mathbb{R}^N)$. Since, by (2.16),

$$G_a(u_n, B) = \int_B j_\mu(x, \nabla_\mu u_n) d\mu \qquad \forall B \in \mathcal{B}(\Omega),\ \forall n \in N.$$

the representation formula (iii) for $u$ is a consequence of the continuity property of $G_a$ and $j_\mu$ stated in (2.5) and (ii).

Since $j_\mu$ satisfies (i) and (ii), there exist a Borel integrand $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ and a Borel set $N \subseteq \Omega$, with $\mu(N) = 0$, such that $j_\mu(x, z) = j(x, z)$ for every $x \in \Omega \backslash N$ and for every $z \in \mathbb{R}^N$. This implies that $j$ satisfies conditions (i), (ii) and (iii).

Let $j'$ be another $\mathcal{B}_\mu(\Omega) \times \mathcal{B}(\mathbb{R}^N)$-measurable integrand satisfying (i), (ii) and (iii). Then there exists $M' \subseteq \Omega$, with $\mu(M') = 0$, such that for every $x \in \Omega \backslash M'$ we have

$$\begin{aligned} & j(x, \nabla_\mu u(x)) = j'(x, \nabla_\mu u(x)) && \forall u \in D, \\ & |j(x, z_1) - j(x, z_1)| \leq c|p_x(z_1) - p_x(z_2)| && \forall z_1,\ z_2 \in \mathbb{R}^N, \\ & |j'(x, z_1) - j'(x, z_2)| \leq c|p_x(z_1) - p_x(z_2)| && \forall z_1,\ z_2 \in \mathbb{R}^N. \end{aligned}$$

Then, using the continuity of $j(x \cdot)$ and $j'(x, \cdot)$ and the density of $\{\nabla_\mu u(x) : u \in D\}$ in $E_\mu(x)$, one gets $j(x, z) = j(x, p_x(z)) = j'(x, p_x(z)) = j'(x, z)$ for every $x \in \Omega \backslash M'$ and for every $z \in \mathbb{R}^N$.

Assume now that $G(\cdot, A)$ is convex on $BV(\Omega)$ for every open subset $A$ of $\Omega$. Since $G(u, \cdot)$ is a bounded Radon measure, for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$ we have $G(u, B) = \inf\{G(u, A) : A \text{ open},\ B \subseteq A\}$. Therefore $G_a(\cdot, B)$ is convex on $BV(\Omega)$ for every $B \in \mathcal{B}(\Omega)$. By (ii), (iii), and by localization, there exists $M'' \subseteq \Omega$, with $\mu(M'') = 0$, such that $j(x, \cdot)$ is continuous on $\mathbb{R}^N$ for every $x \in \Omega \backslash M''$ and

$$\begin{aligned} j\left(x, \frac{\nabla_\mu u(x) + \nabla_\mu v(x)}{2}\right) \leq \frac{1}{2} j(x, \nabla_\mu u(x)) \\ + \frac{1}{2} j(x, \nabla_\mu v(x)) \qquad \forall x \in \Omega \backslash M'',\ \forall u,\ v \in D. \end{aligned}$$

Hence, by continuity, $j(x, \cdot)$ is convex on $E_\mu(x)$ for every $x \in \Omega \backslash M''$. The convexity on $\mathbb{R}^N$ follows from the fact that $j(x, z) = j(x, p_x(z))$ by (ii).

If $G(\cdot, A)$ is linear, we just apply the above argument to $G$ and $-G$. $\square$

2.2. - *Representation of the duality pairing*

We now give the application of Theorem 2.3 to the duality pairing $(\sigma \cdot Du)$ between $X(\Omega)$ and $BV(\Omega)$ defined in Section 1.2.

PROPOSITION 2.4. *Let $\mu \in \mathcal{M}_+^b(\Omega)$ with $\mathcal{L}^N \ll \mu$. Then for every $\sigma \in X(\Omega)$ there exists a unique $\sigma_\mu \in L^\infty_\mu(\Omega, \mathbb{R}^N)$ such that:*

(a) $\sigma_\mu(x) \in E_\mu(x)$ *for $\mu$-a.e. $x \in \Omega$;*

(b) $\dfrac{d(\sigma \cdot Du)}{d\mu} = \sigma_\mu \nabla_\mu u$ *$\mu$-a.e. in $\Omega$ for every $u \in BV(\Omega)$.*

*Moreover the lifting operator $\sigma \mapsto \sigma_\mu$ is sequentially continuous from the weak* topology of $X(\Omega)$ into the weak* topology of $L^\infty_\mu(\Omega, \mathbb{R}^N)$ and one has:*

(i) $\|\sigma_\mu\|_{L^\infty_\mu} \leq \|\sigma\|_{L^\infty}$ *for every $\sigma \in X(\Omega)$;*

(ii) $\sigma_\mu(x) = p_x(\sigma(x))$ *for $\mu$-a.e. $x \in \Omega$ and for every $\sigma \in C^1(\overline{\Omega}, \mathbb{R}^N)$;*

(iii) $\sigma_\mu = \sigma$ *$\mathcal{L}^N$-a.e. in $\Omega$ for every $\sigma \in X(\Omega)$;*

(iv) $(\varphi\sigma)_\mu = \varphi\sigma_\mu$ *$\mu$-a.e. in $\Omega$ for every $\varphi \in C^1(\overline{\Omega})$ and for every $\sigma \in X(\Omega)$;*

(v) *if $A \in \mathcal{A}_c(\Omega)$ and $\sigma = 0$ $\mathcal{L}^N$-a.e. on $\Omega \setminus A$, then $\sigma_\mu = 0$ $\mu$-a.e. on $\Omega \setminus A$.*

PROOF. Let $\sigma \in X(\Omega)$ and consider the functional $G : BV(\Omega) \times \mathcal{B}(\Omega) \to \mathbb{R}$ defined by $G(u, B) = \int_B (\sigma \cdot Du)$. Clearly $G$ satisfies conditions (a) and (c) of Theorem 2.3 and condition (b) with $c = \|\sigma\|_{L^\infty}$ (see Proposition 1.3(i)). Then there exists a Borel integrand $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ which satisfies conditions (i), (ii) and (iii) of Theorem 2.3. Since $G(\cdot, B)$ is linear on $BV(\Omega)$ for every $B \in \mathcal{B}(\Omega)$ (see (1.6)), the last assertion of Theorem 2.3 implies that there exists a Borel function $\sigma_\mu : \Omega \to \mathbb{R}^N$ such that $j(x, z) = \sigma_\mu(x) z$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$. Using condition (ii) of Theorem 2.3 we get $\sigma_\mu(x) z = \sigma_\mu(x) p_x(z)$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$, which implies (a), and also $|\sigma_\mu(x)| \leq c = \|\sigma\|_{L^\infty}$ for $\mu$- a.e. $x \in \Omega$, which proves (i). Equality (b) follows from condition (iii) of Theorem 2.3.

Let $\sigma'_\mu$ be another element of $L^\infty_\mu(\Omega, \mathbb{R}^N)$ which satisfies (a) and (b). Then the integrand $j'(x, z) = \sigma'_\mu(x) z$ satisfies conditions (i), (ii) and (iii) of Theorem 2.3, hence $j'(x, \cdot) = j(x, \cdot)$ on $\mathbb{R}^N$ for $\mu$-a.e. $x \in \Omega$, which yields $\sigma'_\mu = \sigma_\mu$ $\mu$-a.e. in $\Omega$.

Let us prove (ii). By (b) and by Proposition 1.3(i), for every $u \in BV(\Omega)$ we have

$$\int_B (\sigma \cdot Du) = \int_B \sigma_\mu \nabla_\mu u \, d\mu \tag{2.17}$$

for every $B \in \mathcal{B}(\Omega)$ such that $\int_B |D^s_\mu u| = 0$. If $\sigma \in C^1(\overline{\Omega}, \mathbb{R}^N)$, one has (see

Proposition 1.3(v)) $\int_B (\sigma \cdot Du) = \int_B \sigma Du = \int_B \sigma \nabla_\mu u d\mu$. Comparing with (2.17) and using localization one gets

$$\sigma \nabla_\mu u = \sigma_\mu \nabla_\mu u \quad \mu\text{-a.e. in } \Omega. \tag{2.18}$$

Since this holds for every $u \in BV(\Omega)$, we choose a countable subset $D$ of $BV(\Omega)$ such that the set $\{\nabla_\mu u(x) : u \in D\}$ is dense in $E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. By applying (2.18) we obtain that $\sigma(x) - \sigma_\mu(x)$ is orthogonal to $E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Together with (a), this gives (ii).

Take now $u \in W^{1,1}(\Omega)$. By Proposition 1.3(iv) and by (2.17), for every $\sigma \in X(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$ we have

$$\int_B \sigma \nabla u dx = \int_B (\sigma \cdot Du) = \int_B \sigma_\mu \nabla_\mu u d\mu = \int_B \sigma_\mu \nabla u dx.$$

Thus, using localization again, we get $(\sigma - \sigma_\mu)\nabla u = 0$ $\mathcal{L}^N$-a.e. in $\Omega$ for every $u \in W^{1,1}(\Omega)$. Taking $u$ affine, we finally obtain $\sigma = \sigma_\mu$ $\mathcal{L}^N$-a.e. in $\Omega$, that is (iii).

By Proposition 1.3(iii) we have $(\varphi\sigma)_\mu \nabla_\mu u = \varphi \sigma_\mu \nabla_\mu u$ $\mu$-a.e. in $\Omega \backslash A$ for every $u \in BV(\Omega)$. As in the case of (2.18), this implies that $(\varphi\sigma)_\mu(x) - \varphi(x)\sigma_\mu(x)$ is orthogonal to $E_\mu(x)$ for $\mu$-a.e $x \in \Omega$. Together with (a), this gives (iv).

If $A \in \mathcal{A}_c(\Omega)$ and $\sigma = 0$ $\mathcal{L}^N$-a.e. on $\Omega \backslash A$, then $(\sigma \cdot Du) = 0$ on $\Omega \backslash A$ by Lemma 1.7. This implies $\sigma_\mu \nabla_\mu u = 0$ $\mu$-a.e. in $\Omega \backslash A$ for every $u \in BV(\Omega)$. Choosing a countable subset $D$ of $BV(\Omega)$ such that the set $\{\nabla_\mu u(x) : u \in D\}$ is dense in $E_\mu(x)$ for $\mu$-a.e. $x \in \Omega \backslash A$, we obtain that $\sigma_\mu(x)$ is orthogonal to $E_\mu(x)$ for $\mu$-a.e. $x \in \Omega \backslash A$. Together with (a), this gives (v).

Let us prove, finally, that the lifting operator $\sigma \mapsto \sigma_\mu$ is sequentially continuous. Consider a sequence $(\sigma^n)$ converging weakly* to $\sigma$ in $X(\Omega)$. Then $(\sigma^n)$ is bounded in $L^\infty(\Omega, \mathbb{R}^N)$ and, by (i), $(\sigma^n_\mu)$ is bounded in $L^\infty_\mu(\Omega, \mathbb{R}^N)$. Passing, if necessary, to a subsequence, we can suppose that $(\sigma^n_\mu)$ converges to some function $\psi \in L^\infty_\mu(\Omega, \mathbb{R}^N)$ in the weak* topology of $L^\infty_\mu(\Omega, \mathbb{R}^N)$. Now we have to prove that $\psi = \sigma_\mu$ $\mu$-a.e. in $\Omega$. First, we observe that, by condition (a), we have

$$0 = \int_\Omega [\psi - p_x(\psi)]\sigma^n_\mu d\mu \to \int_\Omega [\psi - p_x(\psi)]\psi d\mu = \int_\Omega [\psi - p_x(\psi)]^2 d\mu,$$

hence $\psi(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Thanks to Proposition 1.3(ii), we have

$$\int_B (\sigma^n \cdot Du) \to \int_B (\sigma \cdot Du) \qquad \forall u \in BV(\Omega),\ \forall B \in \mathcal{B}(\Omega). \tag{2.19}$$

If $B \in \mathcal{B}(\Omega)$ is such that $\int_B |D^s_\mu u| = 0$, by Proposition 1.3(i) we have also

$$(2.20) \qquad \int_B (\sigma^n \cdot Du) = \int_B \sigma^n_\mu \nabla_\mu u d\mu, \qquad \int_B (\sigma \cdot Du) = \int_B \sigma_\mu \nabla_\mu u d\mu,$$

hence

$$(2.21) \qquad \int_B (\sigma^n \cdot Du) \to \int_B \psi \nabla_\mu u d\mu.$$

As the limits in (2.19) and (2.21) are equal, by (2.20) and by localization we obtain $\sigma_\mu \nabla_\mu u = \psi \nabla_\mu u$ $\mu$-a.e. in $\Omega$. Since this is true for any $u \in BV(\Omega)$, one deduces as before that $\psi(x) - \sigma_\mu(x)$ is orthogonal to $E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Since both $\psi(x)$ and $\sigma_\mu(x)$ belong to $E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, this gives $\sigma_\mu = \psi$ $\mu$-a.e. in $\Omega$. □

We are now in a position to prove that the multifunction $E_\mu$, introduced in (2.2), is the smallest closed-valued $\mu$-measurable multifunction such that $\nabla_\mu u(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$ and for every $u \in W^{1,1}_\mu(\Omega)$, where $W^{1,1}_\mu(\Omega)$ is the space defined by (2.1).

PROPOSITION 2.5. *Let* $\mu \in \mathcal{M}^b_+(\Omega)$, *with* $\mathcal{L}^N \ll \mu$. *Then*: $E_\mu(x) = \mu\text{-}\operatorname{ess\,sup}_{u \in W^{1,1}_\mu(\Omega)} \{\nabla_\mu u(x)\}$.

PROOF. Let us consider the closed-valued multifunction $E'_\mu : \Omega \to \mathbb{R}^N$ defined (see (1.1)) by

$$E'_\mu(x) = \mu\text{-}\operatorname*{ess\,sup}_{u \in W^{1,1}_\mu(\Omega)} \{\nabla_\mu u(x)\}.$$

We want to prove that $E'_\mu(x) = E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Recall that, by definition, $E'_\mu$ is the smallest closed-valued $\mu$-measurable multifunction such that $\nabla_\mu u(x) \in E'_\mu(x)$ for $\mu$-a.e. $x \in \Omega$ for every $u \in W^{1,1}_\mu(\Omega)$. As in the case of $E_\mu$, we can prove that $E'_\mu(x)$ is a linear subspace of $\mathbb{R}^N$ for $\mu$-a.e. $x \in \Omega$, and that $E'_\mu(x) = \mathbb{R}^N$ for $\mathcal{L}^N$-a.e. $x \in \Omega$.

As $W^{1,1}_\mu(\Omega) \subseteq BV(\Omega)$, we have $E'_\mu(x) \subseteq E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, thus we have only to prove that $E_\mu(x) \subseteq E'_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. By the Projection Theorem (see [14], Theorem III.23) the set $M = \{x \in \Omega : E_\mu(x) \backslash E'_\mu(x) \neq \emptyset\}$ is $\mu$-measurable. Our aim is to prove that $\mu(M) = 0$.

By the Measurable Selection Theorem (see [14], Theorem III.22) there exists a $\mu$-measurable function $\psi : \Omega \to \mathbb{R}^N$ such that $\psi(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$ and $\psi(x) \in E_\mu(x) \backslash E'_\mu(x)$ for $\mu$-a.e. $x \in M$. Replacing, if necessary, $\psi$ by $\psi/|\psi|$, we may assume that $\psi \in L^\infty_\mu(\Omega, \mathbb{R}^N)$.

Let $p_x$ and $p'_x$ be the orthogonal projections on $E_\mu(x)$ and $E'_\mu(x)$ respectively. We want to prove that $p'_x(\psi(x)) = \psi(x)$ for $\mu$-a.e. $x \in \Omega$, which implies that $\psi(x) \in E'_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, hence $\mu(M) = 0$. To prove this fact, it is enough to show that the function $p'_x(\psi) - \psi$ belongs to the linear subspace $\mathcal{F} = \{\sigma_\mu : \sigma \in X(\Omega), \operatorname{div}\sigma = 0\}$ of $L^\infty_\mu(\Omega, \mathbb{R}^N)$. Indeed, if $p'_x(\psi) - \psi \in \mathcal{F}$, then there exists $\sigma \in X(\Omega)$ such that $p'_x(\psi(x)) = \psi(x) + \sigma_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Since $p'_x$ is the identity for $\mathcal{L}^N$-a.e. $x \in \Omega$, by Proposition 2.4(iii) we have $\psi = p'_x(\psi) = \psi + \sigma_\mu = \psi + \sigma$ $\mathcal{L}^N$-a.e. in $\Omega$, hence $\sigma = 0$ $\mathcal{L}^N$-a.e. in $\Omega$, which implies $\sigma_\mu = 0$ $\mu$-a.e. in $\Omega$ (Proposition 2.4(i)). Therefore $p'_x(\psi(x)) = \psi(x) + \sigma_\mu(x) = \psi(x)$ for $\mu$-a.e. $x \in \Omega$.

In order to prove that $p'_x(\psi) - \psi \in \mathcal{F}$, we first show that $\mathcal{F}$ is closed in the weak* topology of $L^\infty_\mu(\Omega, \mathbb{R}^N)$. By the Krein-Šmulian theorem (see [19], Theorem V.5.7) it is enough to show that the intersection of $\mathcal{F}$ with every closed ball in $L^\infty_\mu(\Omega, \mathbb{R}^N)$ is weakly*-closed. Since the weak* topology is metrizable on any closed ball of $L^\infty_\mu(\Omega, \mathbb{R}^N)$, it is enough to prove that $\mathcal{F}$ is sequentially weakly*-closed. Let $(\sigma^n)$ be a sequence in $\mathcal{F}$ such that $(\sigma^n_\mu)$ converges to some function $\sigma$ in the weak* topology of $L^\infty_\mu(\Omega, \mathbb{R}^N)$. As $\mathcal{L}^N \ll \mu$, the sequence $(\sigma^n)$ converges to $\sigma$ also in the weak* topology of $L^\infty(\Omega, \mathbb{R}^N)$. Since $\operatorname{div}\sigma^n = 0$ for every $n \in \mathbb{N}$, we have $\operatorname{div}\sigma = 0$, hence $\sigma \in X(\Omega)$ and $(\sigma^n)$ converges to $\sigma$ weakly* in $X(\Omega)$. By the continuity of the lifting operator $\sigma \mapsto \sigma_\mu$ (Proposition 2.4), the sequence $(\sigma^n_\mu)$ converges to $\sigma_\mu$ in the weak* topology of $L^\infty_\mu(\Omega, \mathbb{R}^N)$. This gives $\sigma = \sigma_\mu$ $\mu$-a.e. in $\Omega$, hence $\sigma \in \mathcal{F}$, and proves that $\mathcal{F}$ is closed in the weak* topology of $L^\infty_\mu(\Omega, \mathbb{R}^N)$.

Suppose now, by contradiction, that $p'_x(\psi) - \psi \notin \mathcal{F}$. Then, by the Hahn-Banach Theorem, there exists $v \in L^1_\mu(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega p'_x(\psi) v \, d\mu \neq \int_\Omega \psi v \, d\mu \tag{2.22}$$

and $\int_\Omega \sigma_\mu v \, d\mu = 0$ for every $\sigma \in X(\Omega)$ with $\operatorname{div}\sigma = 0$. In particular, if $\sigma \in C^1_c(\Omega, \mathbb{R}^N)$ and $\operatorname{div}\sigma = 0$, we have $\sigma \in X(\Omega)$ and $\sigma_\mu(x) = p_x(\sigma(x))$ for $\mu$-a.e. $x \in \Omega$ (Proposition 2.4(ii)), hence

$$\int_\Omega \sigma p_x(v) \, d\mu = \int_\Omega p_x(\sigma) v \, d\mu = \int_\Omega v \, d\mu = 0.$$

This implies, by Proposition 1.1, that there exists $u \in W^{1,1}_\mu(\Omega)$ such that $\nabla_\mu u(x) = p_x(v(x))$ for $\mu$-a.e. $x \in \Omega$. Since $\nabla_\mu u(x) \in E'_\mu(x)$, we have $p_x(v(x)) \in E'_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Being $E'_\mu(x) \subseteq E_\mu(x)$, we conclude that $p'_x(v(x)) = p_x(v(x))$ for $\mu$-a.e. $x \in \Omega$. As $\psi(x) \in E_\mu(x)$, we have $p_x(\psi(x)) = \psi(x)$ for $\mu$-a.e. $x \in \Omega$, hence

$$\int_\Omega p'_x(\psi) v \, d\mu = \int_\Omega \psi p'_x(v) \, d\mu = \int_\Omega \psi p_x(v) \, d\mu = \int_\Omega \psi v \, d\mu,$$

which contradicts (2.22). This shows that $p'_x(\psi) - \psi \in \mathcal{F}$ and proves that $p'_x(\psi(x)) = \psi(x)$ for $\mu$-a.e. $x \in \Omega$, and hence $\mu(M) = 0$. Therefore $E_\mu(x) \subseteq E'_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, and the proposition is proved. □

The following proposition makes precise the link between $\sigma_\mu(x)$ and the function $q_\sigma(x, z)$ introduced in Definition 1.4.

PROPOSITION 2.6. *Let $\mu \in \mathcal{M}^b_+(\Omega)$, with $\mathcal{L}^N \ll \mu$. Then the following properties hold:*

(i) *for every $\sigma \in X(\Omega)$ and for $\mu$-a.e. $x \in \Omega$*

$$q_\sigma(x, z) = \sigma_\mu(x) z \qquad \forall z \in E_\mu(x);$$

(ii) *if $h_1,\ h_2 : \Omega \times \mathbb{R}^N \to ]-\infty, +\infty]$ are homogeneous integrands such that $h_1 \sim h_2$ on some Borel subset $B$ of $\Omega$, then*

$$h_1(x, z) = h_2(x, z) \qquad \forall z \in E_\mu(x)$$

*for $\mu$-a.e. $x \in B$.*

PROOF. Let $\sigma \in X(\Omega)$. As $\nu_u |\nabla_\mu u| = \nabla_\mu u$ $\mu$-a.e. in $\Omega$, by Proposition 1.6 and 2.4(b) we have

$$\int_B q_\sigma(x, \nabla_\mu u) d\mu = \int_B (\sigma \cdot Du) = \int_B \sigma_\mu \nabla_\mu u d\mu$$

for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$ with $\int_B |D^s_\mu u| = 0$. Therefore $q_\sigma(x, \nabla_\mu u) = \sigma_\mu \nabla_\mu u$ $\mu$-a.e. in $\Omega$. Let $M$ be the set of all points $x \in \Omega$ such that there exists $z \in E_\mu(x)$ with $q_\sigma(x, z) \neq \sigma_\mu(x) z$. By the Projection Theorem (see [14], Theorem III.23), the set $M$ is $\mu$-measurable. By the Measurable Selection Theorem (see [14], Theorem III.22) there exists a $\mu$-measurable function $\psi : \Omega \to \mathbb{R}^N$ such that $\psi(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in M$ and $q_\sigma(x, \psi(x)) \neq \sigma_\mu(x) \psi(x)$ for $\mu$-a.e. $x \in M$. Replacing, if necessary, $\psi$ by $\psi / |\psi|$, we may assume that $\psi \in L^\infty_\mu(\Omega, \mathbb{R}^N)$. By Theorem 2.2 there exists $u \in BV(\Omega)$ such that $\nabla_\mu u = \psi$ $\mu$-a.e. in $\Omega$. This implies that $q_\sigma(x, \nabla_\mu u) \neq \sigma_\mu \nabla_\mu u$ $\mu$-a.e. in $M$. As $q_\sigma(x, \nabla_\mu u) = \sigma_\mu \nabla_\mu u$ $\mu$-a.e. in $\Omega$, we conclude that $\mu(M) = 0$, which proves (i).

To prove (ii), we observe that, under our hypotheses on $h_1$ and $h_2$, we have

$$\int_{B'} h_1(x, \nabla_\mu u) d\mu = \int_{B'} h_2(x, \nabla_\mu u) d\mu$$

for every $u \in BV(\Omega)$ and for every $B' \in \mathcal{B}(\Omega)$ with $B' \subseteq B$ and $\int_{B'} |D^s_\mu u| = 0$; hence $h_1(x, \nabla_\mu u) = h_2(x, \nabla_\mu u)$ $\mu$-a.e. in $B$. The conclusion follows now, as in the previous case, using the Measurable Selection Theorem and Theorem 2.2. □

Let $K$ be a subset of $X(\Omega)$ and let $h_K$ be the homogeneous integrand associated with $K$ (Proposition 1.8). According to (1.1), we can consider the closed-valued $\mu$-measurable multifunction $\Gamma^\mu : \Omega \to \mathbb{R}^N$ defined by

$$\Gamma^\mu(x) = \mu\text{-}\operatorname*{ess\,sup}_{\sigma \in K} \{\sigma_\mu(x)\},$$

and its support function $h_\mu : \Omega \times \mathbb{R}^N \to ]-\infty, +\infty]$ defined by

$$h_\mu(x, z) = \sup\{zz^* : z^* \in \Gamma^\mu(x)\}. \tag{2.23}$$

The following proposition shows the connection between $h_K$ and $h_\mu$.

PROPOSITION 2.7. *Let $\mu \in \mathcal{M}_+^b(\Omega)$, with $\mathcal{L}^N \ll \mu$, let $K$ be a subset of $X(\Omega)$, let $h_K$ be the homogeneous integrand associated with $K$ according to Proposition 1.8, and let $h_\mu$ be the homogeneous integrand defined in (2.23). Then for $\mu$-a.e. $x \in \Omega$ we have*

$$h_\mu(x, z) = h_K(x, p_x(z)) \qquad \forall z \in \mathbb{R}^N,$$

*where $p_x$ denotes the orthogonal projection on $E_\mu(x)$.*

PROOF. Let us fix a countable subset $D$ of $K$ such that $D$ is sequentially dense in $K$ for the weak* topology of $X(\Omega)$ and $\{\sigma_x : \sigma \in D\}$ is dense in $\{\sigma_\mu : \sigma \in K\}$ for the strong topology of $L^1_\mu(\Omega, \mathbb{R}^N)$. Let us define

$$h_D(x, z) = \sup_{\sigma \in D} q_\sigma(x, z) \qquad \forall x \in \Omega, \ \forall z \in \mathbb{R}^N.$$

By Proposition 1.8 we have $h_D \sim h_K$. Therefore, by Proposition 2.6, for $\mu$-a.e. $x \in \Omega$ we obtain

$$h_K(x, p_x(z)) = h_D(x, p_x(z)) = \sup_{\sigma \in D} q_\sigma(x, p_x(z)) \qquad \forall z \in \mathbb{R}^N. \tag{2.24}$$

By the definition of $\Gamma^\mu$ we have $\Gamma^\mu(x) = \mathrm{cl}\{\sigma_\mu(x) : \sigma \in D\}$ for $\mu$-a.e. $x \in \Omega$, hence

$$h_\mu(x, z) = \sup_{\sigma \in D} \sigma_\mu(x) z \qquad \forall z \in \mathbb{R}^N \tag{2.25}$$

for $\mu$-a.e. $x \in \Omega$. Recalling that $\sigma_\mu(x) \in E_\mu(x)$, we deduce from Proposition 2.6 that

$$\sigma_\mu(x) z = \sigma_\mu(x) p_x(z) = q_\sigma(x, p_x(z)) \qquad \forall z \in \mathbb{R}^N, \ \forall \sigma \in D \tag{2.26}$$

for $\mu$-a.e. $x \in \Omega$. The conclusion follows now from (2.24), (2.25), (2.26). $\square$

### 2.3. - *Integral representation on* $W^{1,1}_\mu(\Omega)$

We prove now a representation theorem for the functional $F$ on the space $W^{1,1}_\mu(\Omega)$ introduced in (2.1).

THEOREM 2.8. *Let* $\alpha$, $\mu \in \mathcal{M}^b_+(\Omega)$, *with* $\alpha + \mathcal{L}^N \ll \mu$, *and let* $F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$ *be a functional satisfying the hypotheses* (H1), (H2) *and* (H3) *considered in the introduction. Then there exists a Borel integrand* $j : \Omega \times \mathbb{R}^N \to [0, +\infty[$ *such that*

(i) $j(x, \cdot)$ *is convex on* $\mathbb{R}^N$ *for* $\mu$*-a.e.* $x \in \Omega$;

(ii) *for* $\mu$*-a.e.* $x \in \Omega$ *and for every* $z \in \mathbb{R}^N$ *we have* $j(x, z) = j(x, p_x(z))$, *where* $p_x$ *denotes the orthogonal projection on the linear space* $E_\mu(x)$ *defined in* (2.2);

(iii) *for every* $u \in W^{1,1}_\mu(\Omega)$ *and for every* $B \in \mathcal{B}(\Omega)$ *we have*

$$F(u, B) = \int_B j(x, \nabla_\mu u) d\mu;$$

(iv) *for* $\mu$*-a.e.* $x \in \Omega$ *and for every* $z \in \mathbb{R}^N$ *we have* $0 \leq j(x, z) \leq \frac{d\alpha}{d\mu}(x) + \gamma |z|$.

*If* $j' : \Omega \times \mathbb{R}^N \to [0, +\infty[$ *is another* $\mathcal{B}_\mu(\Omega) \times \mathcal{B}(\mathbb{R}^N)$*-measurable integrand satisfying* (i), (ii) *and* (iii), *then* $j(x, \cdot) = j'(x, \cdot)$ *on* $\mathbb{R}^N$ *for* $\mu$*-a.e.* $x \in \Omega$.

PROOF. For every $A \in \mathcal{A}(\Omega)$ let us consider the recession function of $F(\cdot, A)$ defined by

$$F_\infty(u, A) = \lim_{t \to +\infty} \frac{F(tu, A)}{t} \tag{2.27}$$

(existence of the limit follows from the convexity assumption (H2)). It is well-known (see [27], Theorem 8.5) that

$$F_\infty(u, A) = \sup_{v \in BV(\Omega)} [F(v + u, A) - F(v, A)], \tag{2.28}$$

hence $F(u, A) \leq F(v, A) + F_\infty(u - v, A)$ for every $u$, $v \in BV(\Omega)$. But, by (H3), we obtain $0 \leq F_\infty(u - v, A) \leq \gamma \int_A |Du - Dv|$. Thus, interchanging $u$ and $v$ and extending the inequalities to all Borel sets thanks to (H1), we can write

$$|F(u, B) - F(v, B)| \leq \gamma \int_B |Du - Dv| \qquad \forall u,\ v \in BV(\Omega),\ \forall B \in \mathcal{B}(\Omega). \tag{2.29}$$

Let us consider the functional $G : BV(\Omega) \times \mathcal{B}(\Omega) \to \mathbb{R}$ defined by $G(u, B) = F(u, B) - F(0, B)$. Since $G$ satisfies the hypotheses of Theorem 2.3, there exists a Borel integrand $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ such that for $\mu$-a.e. $x \in \Omega$

$$g(x, 0) = 0, \tag{2.30}$$

$$|g(x, z_1) - g(x, z_2)| \leq \gamma |p_x(z_1) - p_x(z_2)| \qquad \forall z_1,\ z_2 \in \mathbb{R}^N, \tag{2.31}$$

$$G_a(u, B) = \int_B g(x, \nabla_\mu u) d\mu \qquad \forall u \in BV(\Omega),\ \forall B \in \mathcal{B}(\Omega), \tag{2.32}$$

where $G_a(u, \cdot)$ is the absolutely continuous part of the measure $G(u, \cdot)$ with respect to $\mu$. Since $G(\cdot\, A)$ is convex on $BV(\Omega)$ for every $A \in \mathcal{A}(\Omega)$, the function $g(x, \cdot)$ is convex on $\mathbb{R}^N$ for $\mu$-a.e. $x \in \Omega$.

From (2.29), by taking $v = 0$, one deduces that $|G(u, B)| \leq \gamma \int_B |Du|$, hence $G(u, \cdot) = G_a(u, \cdot)$ on $\mathcal{B}(\Omega)$ as soon as $u$ belongs to $W^{1,1}_\mu(\Omega)$. So, according to (2.32), we can write

$$F(u, B) = \int_B g(x, \nabla_\mu u) d\mu + F(0, B) \qquad \forall u \in W^{1,1}_\mu(\Omega),\ \forall B \in \mathcal{B}(\Omega). \tag{2.33}$$

From (H3) it turns out that the measure $F(0, \cdot)$ is absolutely continuous with respect to $\alpha$; therefore there exists a functions $a \in L^1_\mu(\Omega)$, with $0 \leq a(x) \leq \dfrac{d\alpha}{d\mu}(x)$ for $\mu$-a.e. $x \in \Omega$, such that

$$F(0, B) = \int_B a\, d\mu \qquad \forall B \in \mathcal{B}(\Omega). \tag{2.34}$$

Define $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ by $j(x, z) = g(x, z) + a(x)$. Then (iii) is a consequence of (2.33) and (2.34). The convexity of $j(x, \cdot)$ follows from the convexity of $g(x, \cdot)$. Property (ii) is a consequence of (2.31) and (iv) follows from (2.30), (2.31) and from the inequality $a \leq \dfrac{d\alpha}{d\mu}$ $\mu$-a.e. in $\Omega$.

The uniqueness of $j$ can be proved as in Theorem 2.3, using now Proposition 2.5. □

## 3. - Representation of the conjugate functional

In this section we prove a representation formula for the conjugate of an integral functional defined on the space $W^{1,1}_\mu(\Omega)$ introduced in Section 2. Let $\mu \in \mathcal{M}^b_+(\Omega)$, with $\mathcal{L}^N \ll \mu$, and let $j : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Borel function such that:

(J1) for $\mu$-a.e. $x \in \Omega$ the function $j(x,\cdot)$ is convex on $\mathbb{R}^N$;

(J2) for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$ we have $j(x,z) = j(x,p_x(z))$, where $p_x$ denotes the orthogonal projection on the linear space $E_\mu(x)$ defined in (2.2);

(J3) there exist $\gamma \in \mathbb{R}$ and $a \in L^1_\mu(\Omega)$ such that $0 \le j(x,z) \le a(x) + \gamma|z|$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$.

Note that, if $\mu$ is the Lebesgue measure, then condition (J2) is always satisfied by (2.4).

We define the functional $J_\mu : BV(\Omega) \times \mathcal{B}(\Omega) \to [0,+\infty]$ by

$$J_\mu(u,B) = \begin{cases} \displaystyle\int_B j(x,\nabla_\mu u)d\mu & \text{if } u \in W^{1,1}_\mu(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.1}$$

and the conjugate functional $J^*_\mu : L^N(\Omega) \times \mathcal{A}(\Omega) \to ]-\infty,+\infty]$ by

$$J^*_\mu(f,A) = \sup\left\{ \int_\Omega fu\,dx - J_\mu(u,A) : u \in BV(\Omega) \cap L^{N/N-1}(\Omega) \right\}. \tag{3.2}$$

Note that, since $\nabla_\mu u(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, the functional $J_\mu$ depends only on the values of $j(x,\cdot)$ on $E_\mu(x)$. For this reason, the definition of $j(x,z)$ for $z \notin E_\mu(x)$ is irrelevant. Our convention (J2) simplifies the statements of Proposition 3.1 and of Theorem 4.4.

For every $A \in \mathcal{A}(\Omega)$ let $j_A : \Omega \times \mathbb{R}^N \to [0,+\infty[$ be the functional defined by

$$j_A(x,z) = \begin{cases} j(x,z) & \text{if } x \in A, \\ 0 & \text{if } x \in \Omega \backslash A, \end{cases} \tag{3.3}$$

and let $j^*_A(x,z^*)$ be the conjugate function of $j_A(x,z)$ with respect to $z$. An easy computation shows that

$$j^*_A(x,z^*) = \begin{cases} j^*(x,z^*) & \text{if } x \in A \text{ and } z^* \in \mathbb{R}^N, \\ 0 & \text{if } x \in \Omega\backslash A \text{ and } z^* = 0, \\ +\infty & \text{if } x \in \Omega\backslash A \text{ and } z^* \neq 0, \end{cases} \tag{3.4}$$

where $j^*(x,z^*)$ is the conjugate function of $j(x,z)$ with respect to $z$. Since $j(x,z) = j(x,p_x(z))$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$, we have $j^*_A(x,z^*) = j^*(x,z^*) = +\infty$ for $\mu$-a.e. $x \in \Omega$ and for every $z^* \notin E_\mu(x)$.

Our aim in this section is to prove the following proposition, which will be used in the representation theorem of Section 4.

PROPOSITION 3.1. *Let $\mu \in \mathcal{M}_+^b(\Omega)$, with $\mathcal{L}^N \ll \mu$, and let $j : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Borel function satisfying hypotheses* (J1), (J2) *and* (J3). *Then for every $f \in L^N(\Omega)$ and for every $A \in \mathcal{A}_c(\Omega)$ we have*

$$J_\mu^*(f, A) = \min_{\sigma \in K_{f,A}} \int_\Omega j_A^*(x, \sigma_\mu) d\mu,$$

*where $K_{f,A} = \{\sigma \in L^\infty(\Omega, \mathbb{R}^N) : -\operatorname{div} \sigma = f$ in $\Omega$, $\sigma = 0$ $\mathcal{L}^N$-a.e. in $\Omega \setminus A\}$ and $\sigma_\mu$ is given by Proposition* 2.4. *As usual, we make the convention* $\min \emptyset = +\infty$.

PROOF. let $\Lambda : D(\Lambda) \subseteq L^{N/N-1}(\Omega) \to L^1_\mu(\Omega, \mathbb{R}^N)$ be the unbounded closed linear operator defined by

(3.5) $D(\Lambda) = W_\mu^{1,1}(\Omega) \cap L^{N/N-1}(\Omega), \qquad \Lambda u = \nabla_\mu u \ \forall u \in W_\mu^{1,1}(\Omega) \cap L^{N/N-1}(\Omega),$

and let $\Lambda^* : L_\mu^\infty(\Omega, \mathbb{R}^N) \supseteq D(\Lambda^*) \to L^N(\Omega)$ be the adjoint operator. It follows immediately from the definition that for every $(q, f) \in L_\mu^\infty(\Omega, \mathbb{R}^N) \times L^N(\Omega)$ we have

$$\Lambda^* q = f \Leftrightarrow \int_\Omega q \nabla_\mu u d\mu = \int_\Omega f u dx \qquad \forall u \in W_\mu^{1,1}(\Omega) \cap L^{N/N-1}(\Omega). \tag{3.6}$$

Note that every regular function $\varphi \in C_c^1(\Omega, \mathbb{R}^N)$ belongs to $D(\Lambda^*)$ and satisfies $\Lambda^* \varphi = -\operatorname{div} \varphi$. Note also that the range $R(\Lambda)$ of $\Lambda$ is closed in $L_\mu^1(\Omega, \mathbb{R}^N)$ by Proposition 1.1.

For every $A \in \mathcal{A}(\Omega)$ let $J_A : L_\mu^1(\Omega, \mathbb{R}^N) \to [0, +\infty[$ be the convex function defined by

$$J_A(v) = \int_\Omega j_A(x, v) d\mu.$$

Let $G_A : L^{N/(N-1)}(\Omega) \to [0, +\infty]$ be the convex function defined by

$$G_A(u) = \begin{cases} J_A(\Lambda u) = J_\mu(u, A) & \text{if } u \in D(\Lambda), \\ +\infty & \text{otherwise.} \end{cases}$$

Finally, let $J_A^* : L_\mu^\infty(\Omega, \mathbb{R}^N) \to ]-\infty, +\infty]$ and $G_A^* : L^N(\Omega) \to ]-\infty, +\infty]$ be the conjugate functions of $J_A$ and $G_A$ respectively. As $J_\mu^*(f, A) = G_A^*(f)$ (see (3.2)), by a classical theorem of Convex Analysis (see [29], Theorem 19) we have

$$J_\mu^*(f, A) = G_A^*(f) = \min\{J_A^*(q) : q \in D(\Lambda^*), \Lambda^* q = f\} \qquad \forall f \in L^N(\Omega),$$

with the convention that the $\min \emptyset = +\infty$. By the Rockafellar Conjugation Theorem (see [28]), we have

$$J_A^*(q) = \int_\Omega j_A^*(x, q) d\mu \qquad \forall q \in L_\mu^\infty(\Omega, \mathbb{R}^N),$$

hence

$$\text{(3.7)} \quad J^*_\mu(f,A) = \min\left\{ \int_\Omega j^*_A(x,q)d\mu : q \in D(\Lambda^*), \Lambda^* q = f \right\} \qquad \forall f \in L^N(\Omega).$$

Let us consider the set $H_{f,A}$ of all functions $q \in D(\Lambda^*)$, with $\Lambda^* q = f$, such that $q(x) = 0$ for $\mu$-a.e. $x \in \Omega \backslash A$, and $q(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. Recalling that $j^*_A(x, z^*) = +\infty$ for $z^* \neq 0$ for $\mu$-a.e. $x \in \Omega\backslash A$, and that $j^*_A(x,z^*) = +\infty$ for $z^* \notin E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, from (3.7) it follows that

$$J^*_\mu(f,A) = \min_{q \in H_{f,A}} \int_\Omega j^*_A(x,q)d\mu \qquad \forall f \in L^N(\Omega).$$

The conclusion is now a consequence of the following lemma, which explains the link between the set $H_{f,A}$ and the set $K_{f,A}$ considered in the statement of the proposition. □

LEMMA 3.2. *Let* $\Lambda$ *be the operator defined by* (3.5) *and let* $\Lambda^*$ *be the adjoint operator. Given* $f \in L^N(\Omega)$ *and* $A \in \mathcal{A}(\Omega)$*, let* $K_{f,A}$ *and* $H_{f,A}$ *be the sets defined in the statement and in the proof of Proposition* 3.1 *respectively. If* $A \in \mathcal{A}_c(\Omega)$*, then* $H_{f,A} = \{\sigma_\mu : \sigma \in K_{f,A}\}$.

PROOF. Let us fix $f \in L^N(\Omega)$ and $A \in \mathcal{A}_c(\Omega)$. If $\sigma \in K_{f,A}$, then by Proposition 2.4(v) we have $\sigma_\mu(x) = 0$ for $\mu$-a.e. $x \in \Omega\backslash A$, and by Proposition 2.4(a) we have $\sigma_\mu(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. To prove that $\sigma_\mu \in H_{f,A}$, it remains to show that $\sigma_\mu \in D(\Lambda^*)$ and $\Lambda^*\sigma_\mu = f$. For every $u \in W^{1,1}_\mu(\Omega)$ the measure $(\sigma \cdot Du)$ is absolutely continuous with respect to $\mu$ (Proposition 1.3(i)), hence $\int_\Omega \sigma_\mu \nabla_\mu u d\mu = \int_\Omega (\sigma \cdot Du)$ (Proposition 2.4(b)). By Lemma 1.7 the measure $(\sigma \cdot Du)$ is identically zero on $\Omega\backslash A$. Let $\varphi \in C^1_c(\Omega)$ with $\varphi = 1$ on $A$. Being $\sigma\nabla\varphi = 0$ and $\varphi \operatorname{div}\sigma = \operatorname{div}\sigma$ $\mathcal{L}^N$-a.e. in $\Omega$, from (1.6) it follows that

$$\text{(3.8)} \quad \begin{aligned} \int_\Omega \sigma_\mu \nabla_\mu u d\mu = \int_\Omega (\sigma \cdot Du) = \int_\Omega \varphi(\sigma \cdot Du) = \\ -\int_\Omega u\varphi \operatorname{div}\sigma\, dx = -\int_\Omega u \operatorname{div}\sigma\, dx. \end{aligned}$$

By (3.6) this implies that $\sigma_\mu \in D(\Lambda^*)$ and $\Lambda^*\sigma_\mu = -\operatorname{div}\sigma = f$. Therefore $\sigma_\mu \in H_{f,A}$ whenever $\sigma \in K_{f,A}$.

Conversely, let $q \in H_{f,A}$ and let $\sigma = q$ $\mathcal{L}^N$-a.e. in $\Omega$. Then $\sigma \in L^\infty(\Omega, \mathbb{R}^N)$ and $\sigma = 0$ $\mathcal{L}^N$-a.e. in $\Omega\backslash A$. By (3.6) for every $\varphi \in C^1(\overline{\Omega})$ we have

$$\int_\Omega \sigma\nabla\varphi dx = \int_\Omega q\nabla_\mu\varphi d\mu = \int_\Omega f\varphi dx,$$

hence $-\operatorname{div}\sigma = f$ in $\Omega$. Together with the previous remarks, this shows that $\sigma \in K_{f,A}$. In order to prove that $q = \sigma_\mu$ $\mu$-a.e. in $\Omega$, it is enough to show that the function $q - \sigma_\mu$ belongs to the linear subspace $\mathcal{F} = \{\psi_\mu : \psi \in X(\Omega), \operatorname{div}\psi = 0\}$ of $L^\infty_\mu(\Omega, \mathbb{R}^N)$. Indeed, if $q - \sigma_\mu \in \mathcal{F}$, then there exists $\psi \in X(\Omega)$ such that $q = \sigma_\mu + \psi_\mu$ $\mu$-a.e. in $\Omega$. Since $\psi_\mu = \psi$ $\mathcal{L}^N$-a.e. in $\Omega$ (Proposition 2.4(iii)) and $\mathcal{L}^N \ll \mu$, we have $\sigma = q = \sigma_\mu + \psi_\mu = \sigma + \psi$ $\mathcal{L}^N$-a.e. in $\Omega$, hence $\psi = 0$ $\mathcal{L}^N$-a.e. in $\Omega$, which implies $\psi_\mu = 0$ $\mu$-a.e. in $\Omega$ (Proposition 2.4(i)). Therefore $q = \sigma_\mu + \psi_\mu = \sigma_\mu$ $\mu$-a.e. in $\Omega$.

Suppose now, by contradiction, that $q - \sigma_\mu \notin \mathcal{F}$. Since $\mathcal{F}$ is closed in the weak* topology of $L^\infty_\mu(\Omega, \mathbb{R}^N)$ (see the proof of Proposition 2.5), by the Hahn-Banach Theorem there exists $v \in L^1_\mu(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega qv\,d\mu \neq \int_\Omega \sigma_\mu v\,d\mu \tag{3.9}$$

and $\int_\Omega \psi_\mu v\,d\mu = 0$ for every $\psi \in X(\Omega)$ with $\operatorname{div}\psi = 0$. If $\psi \in C^1_c(\Omega, \mathbb{R}^N)$ and $\operatorname{div}\psi = 0$, then we have $\psi \in X(\Omega)$ and $\psi_\mu = p_x(\psi)$ $\mu$-a.e. in $\Omega$ (Proposition 2.4(ii)), where $p_x$ is the orthogonal projection on the linear space $E_\mu(x)$. Therefore

$$\int_\Omega \psi p_x(v)\,d\mu = \int_\Omega p_x(\psi) v\,d\mu = \int_\Omega \psi_\mu v\,d\mu = 0$$

for every $\psi \in C^1_c(\Omega, \mathbb{R}^N)$ with $\operatorname{div}\psi = 0$. This implies, by Proposition 1.1, that there exists $u \in W^{1,1}_\mu(\Omega)$ such that $\nabla_\mu u(x) = p_x(v(x))$ for $\mu$-a.e. $x \in \Omega$. Since $p_x(q(x)) = q(x)$ for $\mu$-a.e. $x \in \Omega$ and $\Lambda^* q = f$, from (3.6) we obtain

$$\int_\Omega qv\,d\mu = \int_\Omega q p_x(v)\,d\mu = \int_\Omega q \nabla_\mu u\,d\mu = \int_\Omega fu\,dx. \tag{3.10}$$

Being $\sigma \in K_{f,A}$ and $p_x(\sigma_\mu(x)) = \sigma_\mu(x)$ for $\mu$-a.e. $x \in \Omega$ (Proposition 2.4(a)), by (3.8) we have

$$\int_\Omega fu\,dx = \int_\Omega \sigma_\mu \nabla_\mu u\,d\mu = \int_\Omega \sigma_\mu p_x(v)\,d\mu = \int_\Omega \sigma_\mu v\,d\mu. \tag{3.11}$$

Putting (3.10) and (3.11) together, we obtain a contradiction to (3.9). This shows that $q - \sigma_\mu \in \mathcal{F}$ and proves that $q = \sigma_\mu$ $\mu$-a.e. in $\Omega$. Therefore, for every $q \in H_{f,A}$ there exists $\sigma \in K_{f,A}$ such that $q = \sigma_\mu$ $\mu$-a.e. in $\Omega$. □

The proof of Proposition 3.1 is now complete. □

## 4. - Representation of the relaxed functional

In this section we prove the integral representation formula (0.2) for the lower semicontinuous envelope of an integral functional defined on the space $W^{1,1}_\mu(\Omega)$ introduced in (2.1). Let $\mu \in \mathcal{M}^b_+(\Omega)$, with $\mathcal{L}^N \ll \mu$, let $j : \Omega \times \mathbb{R}^N \to [0,+\infty[$ be a Borel function satisfying conditions (J1), (J2) and (J3) of Section 3, and let $\overline{J}_\mu : (BV(\Omega) \cap L^{N/(N-1)}(\Omega)) \times \mathcal{A}(\Omega) \to [0,+\infty[$ be the functional defined by

$$\overline{J}_\mu(u,A) = \inf\left\{ \liminf_{n\to\infty} J_\mu(u_n,A) : u_n \in BV(\Omega) \cap L^{N/(N-1)}(\Omega), \right. \tag{4.1}$$
$$\left. u_n \to u \text{ in } L^{N/(N-1)}(\Omega) \right\},$$

where $J_\mu$ is the integral functional introduced in (3.1). It is well-known that for every $A \in \mathcal{A}(\Omega)$ the function $\overline{J}_\mu(\cdot\, A)$ is the greatest $L^{N/(N-1)}(\Omega)$-lower semicontinuous function less than or equal to $J_\mu(\cdot\,,A)$.

The integrands $f$ and $h$ which will appear in the integral representation (0.2) of $\overline{J}_\mu$ can be described explicitly in terms of the integrand $j$ and of the convex subset $K$ of $X(\Omega)$ defined by

$$K = \left\{ \sigma \in X(\Omega) : \int_\Omega j^*(x,\sigma_\mu)d\mu < +\infty \right\}, \tag{4.2}$$

where $\sigma_\mu$ is given by Proposition 2.4 and $j^*(x,z^*)$ is the conjugate function of $j(x,z)$ with respect to $z$. Note that, as $j^*(x,z^*) \geq -a(x)$, the integral in (4.2) makes sense for every $\sigma \in X(\Omega)$.

Let $\Gamma^\mu : \Omega \to \mathbb{R}^N$ be the closed-valued $\mu$-measurable multifunction defined (see (1.1)) by

$$\Gamma^\mu(x) = \mu\text{-}\operatorname*{ess\,sup}_{\sigma\in K}\{\sigma_\mu(x)\}, \tag{4.3}$$

and let $h_\mu : \Omega \times \mathbb{R}^N \to ]-\infty,+\infty]$ be its support function, defined by

$$h_\mu(x,z) = \sup\{zz^* : z^* \in \Gamma^\mu(x)\}. \tag{4.4}$$

Note that, for $\mu$-a.e. $x \in \Omega$, the convex set $\Gamma^\mu(x)$ is contained in the closure of the essential domain of $j^*(x,\cdot)$, which, by (J2), is contained in linear subspace $E_\mu(x)$ defined by (2.2).

We are now in a position to introduce the integrands $f$ and $h$ which will appear in the integral representation of $\overline{J}_\mu$.

Let $f : \Omega \times \mathbb{R}^N \to [0,+\infty[$ be the $\mathcal{B}_\mu(\Omega) \times \mathcal{B}(\mathbb{R}^N)$-measurable function defined by

$$f(x,z) = \sup_{z^*\in\Gamma^\mu(x)} [zz^* - j^*(x,z^*)], \tag{4.5}$$

and let $h$ be the homogeneous integrand associated with $K$ according to Proposition 1.8. Using the fact that $f(x,z) \leq j^{**}(x,z) = j(x,z)$, $j^*(x,0) \leq 0$ and $\Gamma^\mu(x) \subseteq E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, we obtain that

$$0 \leq f(x,z) = f(x,p_x(z)) \leq a(x) + \gamma|z| \tag{4.6}$$

for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$, where $p_x$ denotes the orthogonal projection on $E_\mu(x)$. As $j^*(x,0) \leq 0$, we have $0 \in K$. Since $j^*(x,z^*) = +\infty$ for $|z^*| > \gamma$, by Proposition 2.4(iii) we have $\|\sigma\|_{L^\infty} \leq \gamma$ for every $\sigma \in K$. Therefore, (1.11) and Remark 1.5 imply that we can choose $h$ in its equivalence class (see (1.9)) so that

$$0 \leq h(x,z) \leq \gamma|z| \qquad \forall x \in \Omega,\ \forall z \in \mathbb{R}^N. \tag{4.7}$$

Let $f_\infty : \Omega \times \mathbb{R}^N \to [0,+\infty[$ be the recession function of $f$, defined by

$$f_\infty(x,z) = \lim_{t\to+\infty} \frac{f(x,tz)}{t}. \tag{4.8}$$

As $j^*(x,z^*) \geq -a(x)$, from (4.5) we obtain

$$f(x,z) \leq \sup_{z^*\in\Gamma^\mu(x)} [zz^* + a(x)] = h_\mu(x,z) + a(x).$$

Since $h_\mu(x,\cdot)$ is positively 1-homogeneous, we conclude that

$$f_\infty(x,z) \leq h_\mu(x,z) \qquad \forall x \in \Omega,\ \forall z \in \mathbb{R}^N. \tag{4.9}$$

Let $D$ be a countable subset of $K$ such that $\{\sigma_\mu : \sigma \in D\}$ is dense in $\{\sigma_\mu : \sigma \in K\}$ for the strong topology of $L^1_\mu(\Omega,\mathbb{R}^N)$. By the definition of $\Gamma^\mu(x)$ we have $\Gamma^\mu(x) = \mathrm{cl}\{\sigma_\mu(x) : \sigma \in D\}$ for $\mu$-a.e. $x \in \Omega$, where cl denotes the closure in $\mathbb{R}^N$. Therefore, for $\mu$-a.e. $x \in \Omega$ we obtain

$$h_\mu(x,z) = \sup_{\sigma\in D} \sigma_\mu(x)z \qquad \forall z \in \mathbb{R}^N. \tag{4.10}$$

For every $\sigma \in D$ and for $\mu$-a.e. $x \in \Omega$ we have $\sigma_\mu(x) \in \Gamma^\mu(x)$, hence, by (4.5),

$$\frac{f(x,tz)}{t} \geq \sigma_\mu(x)z - \frac{j^*(x,\sigma_\mu(x))}{t} \qquad \forall x \in \Omega,\ \forall z \in \mathbb{R}^N,\ \forall t > 0.$$

Since $j^*(x,\sigma_\mu) < +\infty$ $\mu$-a.e. in $\Omega$ by (4.2), taking the limit as $t$ tends to $+\infty$ we get $f_\infty(x,z) \geq \sigma_\mu(x)z$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$. Taking (4.10) into account, we obtain $f_\infty(x,z) \geq h_\mu(x,z)$, which, together with (4.9), gives $f_\infty(x,z) = h_\mu(x,z)$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$. Therefore Proposition 2.7 gives

$$f_\infty(x,z) = h_\mu(x,z) = h(x,p_x(z)) \tag{4.11}$$

for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$.

We are now in a position to state the integral representation theorem for the relaxed functional $\overline{J}_\mu$.

THEOREM 4.1. *Let $\mu \in \mathcal{M}_+^b(\Omega)$, with $\mathcal{L}^N \ll \mu$, let $j : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a Borel function satisfying conditions* (J1), (J2) *and* (J3) *of Section* 3, *and let*

$$\overline{J}_\mu : (BV(\Omega) \cap L^{N/N-1}(\Omega)) \times \mathcal{A}(\Omega) \to [0, +\infty[$$

*be the functional defined by* (4.1). *Then for every $u \in BV(\Omega) \cap L^{N/N-1}(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$ we have*

$$\overline{J}_\mu(u, A) = \int_A f(x, \nabla_\mu u) d\mu + \int_A h(x, \nu_u(x)) |D_\mu^s u|, \tag{4.12}$$

*where $f$ is defined by* (4.5) *and $h$ is the homogeneous integrand associated (according to Proposition* 1.8*) with the set $K$ defined by* (4.2).

EXAMPLE. Take $\mu = \mathcal{L}^N$ and let $\Omega_1$ and $\Omega_2$ be two open subsets of $\Omega$ such that $\Sigma = \partial\Omega_1 \cup \partial\Omega_2$ is a smooth $(N-1)$-dimensional hypersurface. Denoting by $\nu$ the unit normal to $\Sigma$ pointing outwards from $\Omega_1$ to $\Omega_2$, we have for every $u \in BV(\Omega)$ and every $B \in \mathcal{B}(\Omega)$ that $Du(\Sigma \cap B) = (u_2 - u_1)\nu H^{N-1}(\Sigma \cap B)$ where $u_1$ and $u_2$ are the traces in $L^1(\Sigma)$ of $u|_{\Omega_1}$ and $u|_{\Omega_2}$.

Having in mind the model of two homogeneous media separated by an interface, we consider convex functions $j_1$ and $j_2$ on $\mathbb{R}^N$ such that $0 \le j_i(z) \le \gamma(1 + |z|)$ $(i = 1, 2)$ and define:

$$j(x, z) = \begin{cases} j_1(z) & \text{if } x \in \Omega_1, \\ j_2(z) & \text{if } x \in \Omega_2, \end{cases}$$

$$E(u) = J_\mu(u, \Omega) = \begin{cases} \displaystyle\int_\Omega j(x, \nabla u(x)) dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

A quite easy computation yields that the homogeneous integrand defined by Proposition 1.8 can be taken as:

$$h(x, z) = \begin{cases} (j_i)_\infty(z) & \text{if } x \in \Omega_i, \\ (j_1)_\infty \wedge (j_2)_\infty(z) & \text{if } x \in \Sigma. \end{cases}$$

From Theorem 4.1 one gets:

$$\overline{E}(u) = \sum_{i=1,2} \int_{\Omega_i} j_i(\nabla u(x)) dx + \int_{\Omega_i} (j_i)_\infty(D^s u))$$

$$+ \int_\Sigma (j_1)_\infty \wedge (j_2)_\infty(\nu(u_2 - u_1)) dH^{N-1}.$$

Note that a similar result holds in case of vector-valued functions $u \in BV(\Omega, \mathbb{R}^N)$ and can be applied to heterogeneous elasto-plastic materials.

To prove the theorem, we need the notions of $C^1$-stability and $C^1$-inf-stability introduced in the following definition.

DEFINITION 4.2. Given a measure $\lambda \in \mathcal{M}_+^b(\Omega)$ and a set $H$ of $\lambda$-measurable functions from $\Omega$ into $\mathbb{R}^N$, we say that $H$ is $C^1$-*stable* if for every finite family $(u_i)_{i\in I}$ of elements of $H$ and for every family $(\alpha_i)_{i\in I}$ of non-negative functions of $C^1(\overline{\Omega})$ such that $\Sigma_i \alpha_i = 1$ in $\Omega$, we have that $\Sigma_i \alpha_i u_i$ belongs to $H$. In the case $n = 1$, we say that $H$ is $C^1$-*inf-stable* if, under the same conditions for $(u_i)_{i\in I}$ and $(\alpha_i)_{i\in I}$, there exists $u \in H$ such that $u \leq \Sigma_i \alpha_i u_i$ $\lambda$-a.e. in $\Omega$.

Note that $C^1$ stability implies convexity.

LEMMA 4.3. *Let $\lambda \in \mathcal{M}_+^b(\Omega)$, let $K$ be a $C^1$-stable set of $\lambda$-measurable functions from $\Omega$ into $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^N \to ]+\infty, +\infty$ be a $\mathcal{B}_\lambda(\Omega) \times \mathcal{B}(\mathbb{R}^N)$-measurable function such that $f(x, \cdot)$ is convex on $\mathbb{R}^N$ for $\lambda$-a.e. $x \in \Omega$. Suppose that $f(x, u) \in L^1_\lambda(\Omega)$ for every $u \in K$ and let $\Gamma(x) = \lambda\text{-}\operatorname*{ess\,sup}_{u\in K} \{u(x)\}$ (see* (1.1)*) and $g(x) = \inf_{z\in\Gamma(x)} f(x, z)$. Then*

$$\inf_{u\in K} \int_\Omega f(x, u)d\lambda = \int_\Omega g d\lambda. \tag{4.13}$$

*Moreover*

$$\inf_{z\in\Gamma(x)} f(x, z) = \lambda\text{-}\operatorname*{ess\,inf}_{u\in K} f(x, u(x)) \qquad \lambda\text{-a.e. in } \Omega. \tag{4.14}$$

*If $H$ is a $C^1$-inf-stable subset of $L^1_\lambda(\Omega)$, then*

$$\inf_{u\in H} \int_\Omega u d\lambda = \int_\Omega w d\lambda, \tag{4.15}$$

*where $w = \lambda\text{-}\operatorname*{ess\,inf}_{u\in H} u$.*

PROOF. For (4.13) and (4.14) see [7], Theorem 1. Equality (4.15) is an easy consequence. Indeed, let us consider the subset $K$ of $L^1_\lambda(\Omega)$ obtained by taking all convex combinations of elements of $H$ whose coefficients are functions of $C^1(\overline{\Omega})$. From the definition of $C^1$-inf-stability one has

$$\inf_{u\in H} \int_\Omega u d\lambda = \inf_{u\in K} \int_\Omega u d\lambda.$$

As $K$ is $C^1$-stable, we may apply (4.13) with $f(x, z) = z$ for every $(x, z) \in \Omega \times \mathbb{R}$. If $v$ is $\lambda$-measurable and $v \leq u$ $\lambda$-a.e. in $\Omega$ for every $u \in H$, then we have also

$v \leq u$ $\lambda$-a.e. in $\Omega$ for every $u \in K$, hence

$$\lambda\text{-}\operatorname*{ess\,inf}_{u \in K} u = \lambda\text{-}\operatorname*{ess\,inf}_{u \in H} u \qquad \lambda\text{-a.e. in } \Omega,$$

and the conclusion follows from (4.13) and (4.14). □

PROOF OF THEOREM 4.1. Let us prove that for every $A \in \mathcal{A}(\Omega)$ we have

$$\overline{J}_\mu(u, A) = \sup\{\overline{J}_\mu(u, A') : A' \in \mathcal{A}_c(\Omega), A' \subset\subset A\}. \tag{4.16}$$

Let $A \in \mathcal{A}(\Omega)$ and let $\varepsilon > 0$. Then there exists a compact subset $C$ of $A$ such that

$$\int_{A \setminus C} a\,d\mu + \gamma \int_{A \setminus C} |Du| < \varepsilon. \tag{4.17}$$

Using Proposition 4.16 and Theorem 6.1 of [16], we obtain

$$\overline{J}_\mu(u, A) \leq \overline{J}_\mu(u, A') + \overline{J}_\mu(u, A \setminus C) \tag{4.18}$$

for every $A' \in \mathcal{A}_c(\Omega)$ such that $C \subseteq A' \subset\subset A$. Since, by (J3),

$$\overline{J}_\mu(u, A \setminus C) \leq \int_{A \setminus C} a\,d\mu + \gamma \int_{A \setminus C} |Du|,$$

(4.16) follows from (4.17) and (4.18). Therefore, it is enough to prove the theorem when $A \in \mathcal{A}_c(\Omega)$.

By Proposition 3.1, for every $f \in L^N(\Omega)$ and for every $A \in \mathcal{A}_c(\Omega)$ we have

$$J_\mu^*(f, A) = \min_{\sigma \in K_{f,A}} \int_\Omega j_A^*(x, \sigma_\mu)\,d\mu,$$

where $K_{f,A} = \{\sigma \in X(\Omega) : -\operatorname{div}\sigma = f \text{ in } \Omega,\ \sigma = 0\ \mathcal{L}^N\text{-a.e. in } \Omega \setminus A\}$ and $j_A$ is defined by (3.3). Let us fix $u \in BV(\Omega) \cap L^{N/(N-1)}(\Omega)$ and $A \in \mathcal{A}_c(\Omega)$. Since $\overline{J}_\mu(u, A) = J_\mu^{**}(u, A)$ (see [29], Theorem 5), we have

$$\begin{aligned} \overline{J}_\mu(u, A) &= \sup_{f \in L^N(\Omega)} \left[ \int_\Omega fu\,dx - J_\mu^*(f, A) \right] \\ &= \sup_{f \in L^N(\Omega)} \max_{\sigma \in K_{f,A}} \left[ \int_\Omega fu\,dx - \int_\Omega j_A^*(x, \sigma_\mu)\,d\mu \right]. \end{aligned}$$

If $\sigma \in X(\Omega)$ and $\sigma = 0$ $\mathcal{L}^N$-a.e. in $\Omega \setminus A$, then $[\sigma \cdot \nu_\Omega] = 0$ $H^{N-1}$-a.e. on $\partial A$ (see (1.5)). This implies that, if $f = -\operatorname{div}\sigma$, then $f \in L^N(\Omega)$ and $\int_\Omega fu\,dx = \int_\Omega (\sigma \cdot Du)$

(see Proposition 1.3(vi) and Lemma 1.7). Therefore, we can eliminate $f$ in the supremum and we obtain

$$\overline{J}_\mu(u,A) = \sup_{\sigma\in K_A}\left[\int_\Omega (\sigma\cdot Du) - \int_\Omega j_A^*(x,\sigma_\mu)d\mu\right], \tag{4.19}$$

where $K_A = \{\sigma \in L^\infty(\Omega,\mathbb{R}^N) : \sigma = 0\ \mathcal{L}^N\text{-a.e. in } \Omega\backslash A, \int_\Omega j_A^*(x,\sigma_\mu)d\mu < +\infty\}$.

Note that, since $j^*(x,z^*) \geq -a(x)$, the integral $\int_\Omega j_A^*(x,\sigma_\mu)d\mu$ is well-defined (possibly $+\infty$) for every $\sigma \in X(\Omega)$ (see 3.4)). In particular $j^*(x,\sigma_\mu) \in L^1_\mu(\Omega)$ for every $\sigma \in K_A$. By Proposition 2.4((iii) and (v)) we have

$$K_A = \{\sigma \in X(\Omega) : \sigma_\mu = 0\ \mu\text{-a.e. in } \Omega\backslash A, \int_\Omega j_A^*(x,\sigma_\mu)d\mu < +\infty\}.$$

Since $j(x,z) \geq 0$, one checks easily that $j^*(x,0) \leq 0$. Using Proposition 2.4(iv) and the convexity of $j^*(x,\cdot)$, it is easy to prove that $\varphi\sigma \in K_A$ for every $\sigma \in K$ and for every $\varphi \in C^1_c(A)$ with $0 \leq \varphi \leq 1$. Since $K_A \subseteq K$ and $\sigma = 0$ $\mathcal{L}^N$-a.e. on $\Omega\backslash A$ for every $\sigma \in K_A$, from Lemma 1.9 we obtain that, if $h$ is the homogeneous integrand associated with $K$, then the homogeneous integrand associated with $K_A$ is equivalent to the function $h_A$ defined by $h_A(x,z) = h(x,z)$, if $x \in A$, and $h_A(x,z) = 0$, if $x \in \Omega\backslash A$. Therefore, Proposition 1.8 yields

$$h_A(x,\nu_u) = |Du|\text{-}\operatorname*{ess\,sup}_{\sigma\in K_A} q_\sigma(x,\nu_u) \qquad |Du|\text{-a.e. in } \Omega. \tag{4.20}$$

Let $\Gamma_A^\mu : \Omega \to \mathbb{R}^N$ be the multifunction defined by $\Gamma_A^\mu(x) = \Gamma^\mu(x)$, if $x \in A$, and $\Gamma_A^\mu(x) = \{0\}$, if $x \in \Omega\backslash A$. Using Proposition 2.4(iv) it is easy to see that

$$\Gamma_A^\mu(x) = \mu\text{-}\operatorname*{ess\,sup}_{\sigma\in K_A}\{\sigma_\mu(x)\} \qquad \mu\text{-a.e. in } \Omega. \tag{4.21}$$

Finally, let $f_A : \Omega\times\mathbb{R}^N \to [0,+\infty[$ be defined by $f_A(x,z) = f(x,z)$, if $x \in A$, and $f_A(x,z) = 0$, if $x \in \Omega\backslash A$, where $f$ is given by (4.5). By (3.4) it is easy to see that

$$f_A(x,z) = \sup_{z^*\in\Gamma_A^\mu(x)} [zz^* - j_A^*(x,z^*)] \qquad \forall x\in\Omega,\ \forall z\in\mathbb{R}^N.$$

Since the set $\{\sigma_\mu : \sigma \in K_A\}$ is $C^1$-stable (see Proposition 2.4(iv)), by Lemma 4.3 and by (4.21) we have

$$-f_A(x,\nabla_\mu u) = \mu\text{-}\operatorname*{ess\,inf}_{\sigma\in K_A}[j_A^*(x,\sigma_\mu) - \sigma_\mu\nabla_\mu u] \qquad \mu\text{-a.e. on } \Omega. \tag{4.22}$$

Let us consider now the measure $\lambda = \mu + |D^s_\mu u|$. Clearly $|Du| \ll \lambda$, hence $(\sigma \cdot Du) \ll \lambda$ for every $\sigma \in X(\Omega)$ (Proposition 1.3(i)). Let $M$ be a Borel subset of $\Omega$ such that $\mu(M) = 0$ and $|D^s_\mu u|(\Omega \backslash M) = 0$. Thanks to Proposition 1.6 and 2.4 we can write

$$\frac{d(\sigma \cdot Du)}{d\lambda} = \begin{cases} \sigma_\mu \nabla_\mu u & \lambda\text{-a.e. on } \Omega \backslash M, \\ q_\sigma(x, \nu_\mu) & \lambda\text{-a.e. on } M. \end{cases}$$

Let $T_u : K_A \to L^1_\lambda(\Omega)$ be the operator defined by

$$T_u(\sigma) = \begin{cases} j^*_A(x, \sigma_\mu) - \sigma_\mu \nabla_\mu u & \text{on } \Omega \backslash M, \\ q_\sigma(x, \nu_\mu) & \text{on } M, \end{cases}$$

and let $H = \{T_u(\sigma) : \sigma \in K_A\}$. Notice that, by (4.19),

$$\overline{J}_\mu(u, A) = -\inf_{\sigma \in K_A} \int_\Omega T_u(\sigma) d\lambda. \tag{4.23}$$

If $(\sigma^i)_{i \in I}$ is a finite subset of $H$ and $(\alpha^i)_{i \in I}$ is a family of non-negative functions of $C^1(\overline{\Omega})$ such that $\Sigma_i \alpha_i = 1$ in $\Omega$, then $\sigma = \Sigma_i \alpha^i \sigma^i$ belongs to $K_A$ and $\sigma_\mu = \Sigma_i \alpha^i \sigma^i_\mu$ $\mu$-a.e. in $\Omega$ (see Proposition 2.4(iv)). Therefore, by the convexity of $j^*_A(x, \cdot)$, we have

$$T_u(\sigma) \le \Sigma_i \alpha^i T_u(\sigma^i) \qquad \lambda\text{a.e. in } \Omega \backslash M. \tag{4.24}$$

On the other hand, by Remark 1.5 we have

$$T_u(\sigma) \le \Sigma_i \alpha^i T_u(\sigma^i) \qquad \lambda\text{a.e. in } M. \tag{4.25}$$

From (4.24) and (4.25) it follows that $H$ is $C^1$-inf-stable. Let

$$g = \lambda\text{-}\operatorname*{ess\,inf}_{\sigma \in K_A} T_u(\sigma).$$

By Lemma 4.3 and by (4.23) we have $\overline{J}_\mu(u, A) = -\int_\Omega g d\lambda$. By (4.20) and (4.22) we have $g = -f_A(x, \nabla_\mu u)$ $\lambda$-a.e. on $\Omega \backslash M$ and $g = -h_A(x, \nu_u)$ $\lambda$-a.e. on $M$, hence

$$\overline{J}_\mu(u, A) = \int_\Omega f_A(x, \nabla_\mu u) d\mu + \int_\Omega h_A(x, \nu_u) |D^s_\mu u|.$$

By the definition of $f_A$ and $h_A$, the last equality is equivalent to (4.12). $\square$

We conclude this section with a lower semicontinuity theorem on $W^{1,1}_\mu(\Omega)$, which extends the results for $W^{1,1}(\Omega)$ obtained in [21]. We recall that $\Gamma^\mu$ is the closed valued $\mu$-measurable multifunction defined by (4.3) and that $j^*(x, z^*)$ is the conjugate function of $j(x,z)$ with respect to $z$. The (effective) domain of a convex function $g : \mathbb{R}^N \to \overline{\mathbb{R}}$ will be denoted by $\operatorname{dom} g$, and the closure of a subset $A$ of $\mathbb{R}^N$ will be denoted by $\operatorname{cl} A$.

THEOREM 4.4. *Let $\mu \in \mathcal{M}_+^b(\Omega)$, with $\mathcal{L}^N \ll \mu$, and let $j : \Omega \times \mathbb{R}^N \to [0,+\infty[$ be a Borel function satisfying conditions* (J1), (J2) *and* (J3) *of Section* 3, *and let $J : W_\mu^{1,1}(\Omega) \to [0,+\infty[$ be the functional defined by*

$$J(u) = \int_\Omega j(x, \nabla_\mu u) d\mu.$$

*Then the following conditions are equivalent:*

(i) *$J$ is $L^{N/(N-1)}(\Omega)$-lower semicontinuous on $W_\mu^{1,1}(\Omega) \cap L^{N/N-1}(\Omega)$;*

(ii) $\operatorname{cl}(\operatorname{dom} j^*(x,\cdot)) = \Gamma^\mu(x)$ *for $\mu$-a.e. $x \in \Omega$;*

(iii) $\operatorname{dom} j^*(x,\cdot) \subseteq \Gamma^\mu(x)$ *for $\mu$-a.e. $x \in \Omega$;*

(iv) $j(x,z) = \sup\limits_{z^* \in \Gamma^\mu(x)} [zz^* - j^*(x,z^*)]$ *for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$;*

(v) *there exists a countable subset $D$ of $X(\Omega)$ such that*

$$\int_\Omega j^*(x, \sigma_\mu) d\mu < +\infty \qquad \forall \sigma \in D$$

*and*

$$j(x,z) = \sup_{\sigma \in D} [\sigma_\mu(x) z - j^*(x, \sigma_\mu(x))] \qquad \forall z \in \mathbb{R}^N$$

*for $\mu$-a.e. $x \in \Omega$.*

PROOF. (i) $\Rightarrow$ (iv). If $J$ is $L^{N/(N-1)}(\Omega)$-lower semicontinuous on $W_\mu^{1,1}(\Omega) \cap L^{N/(N-1)}(\Omega)$, then $J(u) = \overline{J}_\mu(u,\Omega)$ for every $u \in W_\mu^{1,1}(\Omega) \cap L^{N/(N-1)}(\Omega)$. By Theorem 4.1 this implies that

$$\int_\Omega j(x, \nabla_\mu u) d\mu = \int_\Omega f(x, \nabla_\mu u) d\mu \qquad \forall u \in W_\mu^{1,1}(\Omega) \cap L^{N/(N-1)}(\Omega),$$

where $f$ is the function defined by (4.5). As $f(x,z) \le j^{**}(x,z) = j(x,z)$, we obtain $j(x, \nabla_\mu u) = f(x, \nabla_\mu u)$ $\mu$-a.e. in $\Omega$ for every $u \in W_\mu^{1,1}(\Omega) \cap L^{N/(N-1)}(\Omega)$. Let $D$ be a countable subset of $W_\mu^{1,1}(\Omega) \cap L^{N/(N-1)}(\Omega)$ such that $\{\nabla_\mu u : u \in D\}$ is dense in $\{\nabla_\mu u : u \in W_\mu^{1,1}(\Omega)\}$ for the strong topology of $L_\mu^1(\Omega, \mathbb{R}^N)$. By Proposition 2.5 we have $E_\mu(x) = \operatorname{cl}\{\nabla_\mu u(x) : u \in D\}$ for $\mu$-a.e. $x \in \Omega$. Since the functions $f(x,\cdot)$ and $j(x,\cdot)$ are continuous, and for every $u \in D$ we have $j(x, \nabla_\mu u(x)) = f(x, \nabla_\mu u(x))$ for $\mu$-a.e. $x \in \Omega$, we conclude that $j(x,z) = f(x,z)$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in E_\mu(x)$. By (J2) and (4.6), the same equality holds for every $z \in \mathbb{R}^N$. This concludes the proof of (iv).

(iv) $\Rightarrow$ (i). Let $f$ be the function defined by (4.5). If (iv) holds, then $j(x,z) = f(x,z)$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$. Therefore, using Theorem 4.1 we get $J(u) = \overline{J}_\mu(u,\Omega)$ for every $u \in W^{1,1}_\mu(\Omega) \cap L^{N/(N-1)}(\Omega)$, and, by (4.1), this implies that $J$ is $L^{N/(N-1)}(\Omega)$-lower semicontinuous on $W^{1,1}_\mu(\Omega) \cap L^{N/(N-1)}(\Omega)$.

(ii) $\Leftrightarrow$ (iii). Recall that, by the definition of $\Gamma^\mu(x)$, we have $\Gamma^\mu(x) \subseteq \mathrm{cl}(\mathrm{dom}\, j^*(x,\cdot))$ for $\mu$-a.e. $x \in \Omega$.

(iii) $\Leftrightarrow$ (iv). For every $x \in \Omega$, let $\chi_\mu(x,\cdot)$ be the indicator function of $\Gamma^\mu(x)$, defined by $\chi_\mu(x,z^*) = 0$ if $z^* \in \Gamma^\mu(x)$, and $\chi_\mu(x,z^*) = +\infty$ if $z^* \notin \Gamma^\mu(x)$. Then (iii) can be written as

$$j^*(x,\cdot) = j^*(x,\cdot) + \chi_\mu(x,\cdot) \qquad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{4.26}$$

while (iv) can be written as

$$j(x,\cdot) = (j^*(x,\cdot) + \chi_\mu(x,\cdot))^* \qquad \text{for } \mu\text{-a.e. } x \in \Omega. \tag{4.27}$$

Since $j(x,\cdot)$, $j^*(x,\cdot)$ and $\chi_\mu(x,\cdot)$ are convex and lower semicontinuous on $\mathbb{R}^N$, conditions (4.26) and (4.27) are equivalent.

(iii) $\Rightarrow$ (v). By the definition of $\Gamma^\mu(x)$ there exists a countable subset $D$ of $K$ such that $\{\sigma_\mu(x) : \sigma \in D\}$ is dense in $\Gamma^\mu(x)$ for $\mu$-a.e. $x \in \Omega$. From (iii) and from the definition of $K$ it follows that the set $\{\sigma_\mu(x) : \sigma \in D\}$ is dense also in $\mathrm{dom}\, j^*(x,\cdot)$ for $\mu$-a.e. $x \in \Omega$.

We recall that the relative interior of a convex subset $A$ of $\mathbb{R}^N$, denoted by $\mathrm{ri}\, A$, is defined as the interior of $A$ in the relative topology of the affine hull of $A$. For every $x \in \Omega$ we set $D(x) = \{\sigma_\mu(x) : \sigma \in D\} \cap \mathrm{ri}(\mathrm{dom}\, j^*(x,\cdot))$. Then $D(x)$ is dense in $\mathrm{ri}(\mathrm{dom}\, j^*(x,\cdot))$ for $\mu$-a.e. $x \in \Omega$. As the restriction of $j^*(x,\cdot)$ is continuous on $\mathrm{ri}(\mathrm{dom}\, j^*(x,\cdot))$ (see [27], Theorem 10.1), we obtain

$$\sup_{z^* \in D(x)} [zz^* - j^*(x,z^*)] = \sup_{z^* \in \mathrm{ri}(\mathrm{dom}\, j^*(x,\cdot))} [[zz^* - j^*(x,z^*)] = j^{**}(x,z)$$

(see [27], Corollary 12.2.2), hence

$$\begin{aligned} j^{**}(x,z) = \sup_{z^* \in D(x)} [zz^* - j^*(x,z^*)] &\le \sup_{\sigma \in D} [\sigma_\mu(x) z - j^*(x,\sigma_\mu(x))] \\ &\le j^{**}(x,z) = j(x,z), \end{aligned}$$

which proves (v).

(v) $\Rightarrow$ (iv). If (v) holds, then $\sigma_\mu(x) \in \Gamma^\mu(x)$ for $\mu$-a.e. $x \in \Omega$ and for every $\sigma \in D$. Therefore

$$j(x,z) \le \sup_{z^* \in \Gamma^\mu(x)} [zz^* - j^*(x,z^*)] \qquad \forall z \in \mathbb{R}^N$$

for $\mu$-a.e. $x \in \Omega$. As $j(x,z) = j^{**}(x,z)$, the opposite inequality is trivial. $\square$

## 5. - The main representation theorem

In this section we prove an integral representation theorem for a functional

$$F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$$

satisfying the following hypotheses:

(H1) for every $u \in BV(\Omega)$ the set function $F(u, \cdot)$ is a Borel measure on $\Omega$;

(H2) for every $A \in \mathcal{A}(\Omega)$ the function $F(\cdot, A)$ is convex and $L^{N/(N-1)}_{\mathrm{loc}}(\Omega)$-lower semicontinuous on $BV(\Omega)$;

(H3) there exists $\gamma \in \mathbb{R}$ and $\alpha \in \mathcal{M}^b_+(\Omega)$ such that

$$0 \leq F(u, B) \leq \alpha(B) + \gamma \int_B |Du|$$

for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$.

THEOREM 5.1. *Assume that the functional* $F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$ *satisfies hypotheses* (H1), (H2) *and* (H3). *Then there exist a measure* $\mu \in \mathcal{M}^b_+(\Omega)$, *with* $\alpha + \mathcal{L}^N \ll \mu$, *and two Borel functions* $f,\ h : \Omega \times \mathbb{R}^N \to [0, +\infty[$ *such that*

(i) *for $\mu$-a.e. $x \in \Omega$ the function $f(x, \cdot)$ is convex on $\mathbb{R}^N$ and satisfies*

$$0 \leq f(x, z) = f(x, p_x(z)) \leq \frac{d\alpha}{d\mu}(x) + \gamma |z| \qquad \forall z \in \mathbb{R}^N,$$

*where $p_x$ denotes the orthogonal projection on the linear space $E_\mu(x)$ defined in* (2.2)*;*

(ii) *for every $x \in \Omega$ the function $h(x, \cdot)$ is positively 1-homogeneous on $\mathbb{R}^N$ and*

$$0 \leq h(x, z) \leq \gamma |z| \qquad \forall z \in \mathbb{R}^N;$$

(iii) *for $\mu$-a.e. $x \in \Omega$ we have*

$$f_\infty(x, z) = h(x, p_x(z)) \qquad \forall z \in \mathbb{R}^N,$$

*where $f_\infty$ is the recession function of $f$ with respect to $z$ defined in* (4.8)*;*

(iv) *for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$ we have*

$$F(u, B) = \int_B f(x, \nabla_\mu u(x)) d\mu + \int_B h(x, \nu_u(x)) |D^s_\mu u|.$$

We begin with some lemmas concerning the conjugate functional $F^* : L^N(\Omega) \times \mathcal{A}(\Omega) \to ]-\infty, +\infty]$ defined by

$$F^*(f, A) = \sup \left\{ \int_\Omega f u dx - F(u, A) : u \in BV(\Omega) \cap L^{N/(N-1)}(\Omega) \right\}. \tag{5.1}$$

LEMMA 5.2. *Assume that $F$ satisfies* (H1), (H2) *and* (H3). *Then*

$$F(u,A) = \sup_{f\in L^N(\Omega)} \left[ \int_\Omega fu\,dx - F^*(f,A) \right], \tag{5.2}$$

*for every $u \in BV(\Omega) \cap L^{N/(N-1)}(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$.*

PROOF. Given $A \in \mathcal{A}(\Omega)$, let $\Phi : L^{N/(N-1)}(\Omega) \to [0,+\infty]$ be the convex function defined by

$$\Phi(u) = \begin{cases} F(u,A) & \text{if } u \in BV(\Omega) \cap L^{N/(N-1)}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{5.3}$$

and let $\Phi^* : L^N(\Omega) \to ]-\infty,+\infty]$ be the conjugate function of $\Phi$. Then $\Phi^*(f) = F^*(f,A)$ for every $f \in L^N(\Omega)$ and $\Phi(u) = \Phi^{**}(u)$ at every $u \in L^{N/(N-1)}(\Omega)$ where $\Phi$ is lower semicontinuous (see [29], Theorem 5). The conclusion follows now from the fact that $\Phi$ is lower semicontinuous at each point $u$ of $BV(\Omega) \cap L^{N/N-1}(\Omega)$. □

Following an idea from [6], we construct now a measure $\mu$ in order to obtain $F = \overline{J}_\mu$, where $\overline{J}_\mu$ is the functional defined by (4.1).

LEMMA 5.3. *Assume that $F$ satisfies* (H1), (H2) *and* (H3). *Then for every countable subset $\mathcal{D}$ of $\mathcal{A}(\Omega)$ there exists a measure $\mu \in \mathcal{M}_+^b(\Omega)$, with $\alpha + \mathcal{L}^N \ll \mu$, such that*

$$F^*(f,A) = \sup \left\{ \int_\Omega fu\,dx - F(u,A) : u \in W_\mu^{1,1}(\Omega) \cap L^{N/N-1}(\Omega) \right\} \tag{5.4}$$

*for every $f \in L^N(\Omega)$ and for every $A \in \mathcal{D}$.*

PROOF. Let $A \in \mathcal{D}$. By the definition (5.1), $F^*(\cdot,A)$ is the supremum of a family of continuous functions on $L^N(\Omega)$. Since $L^N(\Omega)$ is a separable Banach space, by the Lindelöf property this supremum is reached by taking a countable subfamily. Doing this for every element $A$ of the countable set $\mathcal{D}$, we finally obtain the existence of a sequence $(u_k)$ in $BV(\Omega) \cap L^{N/(N-1)}(\Omega)$ such that

$$F^*(f,A) = \sup_{k\in\mathbb{N}} \left[ \int_\Omega fu_k\,dx - F(u_k,A) \right] \qquad \forall f \in L^N(\Omega),\ \forall A \in \mathcal{D}. \tag{5.5}$$

Let $c_k = 2^{-k} \left[ \int_\Omega |Du_k| \right]^{-1}$ and let $\mu \in \mathcal{M}_+^b(\Omega)$ be the measure defined by $\mu = \alpha + \mathcal{L}^N + \Sigma_k c_k |Du_k|$. Then (5.4) follows from (5.5), noticing that $u_k \in W_\mu^{1,1}(\Omega) \cap L^{N/(N-1)}(\Omega)$ for every $k \in \mathbb{N}$. □

LEMMA 5.4. *Assume that $F$ satisfies* (H1), (H2) *and* (H3). *Then there exist a measure $\mu \in \mathcal{M}_+^b(\Omega)$, with $\alpha+\mathcal{L}^N \ll \mu$, and a Borel function $j : \Omega\times\mathbb{R}^N \to [0,+\infty[$, satisfying conditions* (J1), (J2) *and* (J3) *of Section* 3 *with $a(x) = \dfrac{d\alpha}{d\mu}(x)$, such that for every $u \in BV(\Omega)\cap L^{N/(N-1)}(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$ we have*

$$F(u, A) = \overline{J}_\mu(u, A), \tag{5.6}$$

*where $\overline{J}_\mu$ is defined by* (4.1).

PROOF. Let $\mathcal{D}$ be a countable subset of $\mathcal{A}_c(\Omega)$ with the following density property: for every pair $(A_1, A_2) \in \mathcal{A}(\Omega) \times \mathcal{A}(\Omega)$, with $A_1 \subset\subset A_2$, there exists $A \in \mathcal{D}$ such that $A_1 \subset\subset A \subset\subset A_2$.

Thanks to Lemma 5.3 there exists some measure $\mu \in \mathcal{M}_+^b(\Omega)$, with $\alpha + \mathcal{L}^N \ll \mu$, such that (5.4) holds for every $f \in L^N(\Omega)$ and for every $A \in \mathcal{D}$. By Theorem 2.8, there exists a Borel function $j : \Omega \times \mathbb{R}^N \to [0, +\infty[$, satisfying conditions (J1), (J2) and (J3) of Section 3 with $a(x) = \dfrac{d\alpha}{d\mu}(x)$, such that

$$F(u, A) = \int_A j(x, \nabla_\mu u) d\mu = J_\mu(u, A) \qquad \forall u \in W_\mu^{1,1}(\Omega).$$

Therefore, by (5.4), $F^*(f, A) = J_\mu^*(f, A)$ for every $f \in L^N(\Omega)$ and for every $A \in \mathcal{D}$, where $J_\mu^*$ is the functional definition by (3.2). Since $\overline{J}_\mu(\,\cdot\,, A) = J_\mu^{**}(\,\cdot\,, A)$ on $BV(\Omega)\cap L^{N/(N-1)}(\Omega)$ (see [29], Theorem 5), from Lemma 5.2 we obtain that (5.6) holds for every $u \in BV(\Omega) \cap L^{N/(N-1)}(\Omega)$ and for every $A \in \mathcal{D}$. The extension of this equality to the general case $A \in \mathcal{A}(\Omega)$ follows from the fact that both $F(u, \cdot\,)$ and $\overline{J}_\mu(u, \cdot\,)$ are measures (see (H1) and (4.12)) and from the density property of $\mathcal{D}$. □

PROOF OF THEOREM 5.1. By Theorem 4.1 and Lemma 5.4, for every $u \in BV(\Omega)\cap L^{N/(N-1)}(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$ we have

$$F(u, A) = \int_A f(x, \nabla_\mu u) d\mu + \int_A h(x, \nu_u(x))|D_\mu^s u|,$$

where $f$ is defined by (4.5) and $h$ is the homogeneous integrand associated with the set $K$ defined by (4.2). Conditions (i), (ii) and (iii) follow from (4.6), (4.7) and (4.11). The extension of this representation formula to the case $u \in BV(\Omega)$ follows from the locality property (2.29). The extension to an arbitrary $B \in \mathcal{B}(\Omega)$ is a trivial consequence of the fact that $F(u, \cdot\,)$ is a Borel measure (hypothesis (H1)).

In general, the function $f$ defined by (4.5) is not Borel measurable, but it can be replaced by a Borel function, still denoted by $f$, which coincides with the previous one for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$ (the existence of such a function follows easily from the fact that $f$ is $\mathcal{B}_\mu(\Omega)$-measurable in $x$ and continuous in $z$). After this modification, the pair $f$, $h$ satisfies all conditions of the theorem. $\square$

The following corollary deals with the case of positively 1-homogeneous functionals.

COROLLARY 5.5. *Assume that the functional* $F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$ *satisfies the hypotheses* (H1), (H2) *and* (H3), *and that for every* $A \in \mathcal{A}(\Omega)$ *the function* $F(\cdot, A)$ *is positively* 1*-homogeneous on* $BV(\Omega)$. *Then there exist a Borel function* $h : \Omega \times \mathbb{R}^N \to [0, +\infty[$ *satisfying the inequalities*

$$0 \leq h(x, z) \leq \gamma |z| \qquad \forall z \in \Omega, \ \forall z \in \mathbb{R}^N$$

*such that* $h(x, \cdot)$ *is positively* 1*-homogeneous on* $\mathbb{R}^N$ *for every* $x \in \Omega$ *and*

$$F(u, B) = \int_B h(x, \nu_u(x)) |Du| \tag{5.7}$$

*for every* $u \in BV(\Omega)$ *and for every* $B \in \mathcal{B}(\Omega)$.

PROOF. Since $F$ satisfies hypotheses (H1), (H2) and (H3), there exist a measure $\mu \in \mathcal{M}_+^b(\Omega)$, with $\alpha + \mathcal{L}^N \ll \mu$, and two Borel functions $f$, $h : \Omega \times \mathbb{R}^N \to [0, +\infty[$ satisfying conditions (i), (ii), (iii) and (iv) of Theorem 5.1. From (H1) and from our hypotheses it follows that $F(\cdot, B)$ is positively 1-homogeneous on $BV(\Omega)$ for every $B \in \mathcal{B}(\Omega)$. By Theorem 5.1(iv) we have

$$\int_B f(x, t\nabla_\mu u(x)) d\mu = F(tu, B) = tF(u, B) = t \int_B f(x, \nabla_\mu u(x)) d\mu$$

for every $t > 0$, for every $u \in BV(\Omega)$, and for every $B \in \mathcal{B}(\Omega)$ with $\int_B |D_\mu^s u| = 0$.

Therefore, for every $u \in BV(\Omega)$ we have $f(x, t\nabla_\mu u(x)) = tf(x, \nabla_\mu u(x))$ for $\mu$-a.e. $x \in \Omega$. Arguing as in the last part of the proof of Theorem 2.3, we can prove that the function $f(x, \cdot)$ is positively 1-homogeneous on $E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$. As $f(x, z) = f(x, p_x(z))$ (Theorem 5.1(i)), we conclude that $f(x, \cdot)$ is positively 1-homogeneous on $\mathbb{R}^N$ for $\mu$-a.e. $x \in \Omega$. Therefore, from Theorem 5.1(iii) we obtain that $f(x, z) = h(x, p_x(z))$ for $\mu$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$. As $\nabla_\mu u(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, using Theorem 5.1(iv) we get

$$F(u, B) = \int_B h(x, \nabla_\mu(x)) d\mu + \int_B h(x, \nu_u(x)) |D_\mu^s u|$$

for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$. Since $\nu_u |\nabla_\mu u| = \nabla_\mu u$ $\mu$-a.e. in $\Omega$, (5.7) follows from the homogeneity of $h$. $\square$

In Theorem 5.1 the measure $\mu$ depends on the functional $F$, as well as the functions $f$ and $h$. We consider now the problem of the integral representation with respect to a prescribed measure $\lambda$.

THEOREM 5.6. *Assume that the functional* $F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$ *satisfies hypotheses* (H1), (H2) *and* (H3), *and let* $\lambda \in \mathcal{M}_+^b(\Omega)$ *enjoy* $\alpha + \mathcal{L}^B \ll \lambda$. *Then the following conditions are equivalent:*

(i) *for every* $B \in \mathcal{B}(\Omega)$ *with* $\lambda(B) = 0$ *the function* $F(\cdot, B)$ *is positively* 1*-homogeneous on* $BV(\Omega)$*;*

(ii) *there exist two Borel functions* $f$, $h : \Omega \times \mathbb{R}^N \to [0, +\infty[$ *which satisfy conditions* (i), (ii), (iii) *and* (iv) *of Theorem* 5.1 *with* $\mu = \lambda$.

PROOF. It is clear that (ii) implies (i). Conversely, if (i) holds, then there exist a measure $\mu \in \mathcal{M}_+^b(\Omega)$, with $\alpha + \mathcal{L}^N \ll \mu$, and two Borel functions $g$, $h : \Omega \times \mathbb{R}^N \to [0, +\infty[$ satisfying conditions (i), (ii), (iii) and (iv) of Theorem 5.1 (here $g$ plays the role of $f$). It is not restrictive to assume also that $\lambda \ll \mu$ (see the proof of Lemma 5.3). Let $M$ be a Borel subset of $\Omega$ such that $\lambda(M) = 0$ and $\mu_s(\Omega \backslash M) = 0$, where $\mu_s$ is the singular part of $\mu$ with respect to $\lambda$. By our hypotheses (i), for every $B \in \mathcal{B}(\Omega)$, with $B \subseteq M$, we have

$$\int_B g(x, t\nabla_\mu u(x)) d\mu + \int_B h(x, \nu_u(x)) |tD_\mu^s u|$$

$$= t \int_B g(x, \nabla_\mu u(x)) d\mu + t \int_B h(x, \nu_u(x)) |D_\mu^s u|$$

for every $t > 0$ and for every $u \in BV(\Omega)$. As in Corollary 5.5, this implies that the function $g(x, \cdot)$ is positively 1-homogeneous on $\mathbb{R}^N$ for $\mu$-a.e. $x \in M$. Therefore, from Theorem 5.1(iii) we obtain that $g(x, z) = h(x, p_x(z))$ for $\mu$-a.e. $x \in M$ and for every $z \in \mathbb{R}^N$. As $\nabla_\mu u(x) \in E_\mu(x)$ for $\mu$-a.e. $x \in \Omega$, using Theorem 5.1(iv) we get

$$\begin{aligned} F(u, B) = \int_{B \backslash M} g(x, \nabla_\mu u(x)) d\mu + \int_{B \cap M} h(x, \nabla_\mu u(x)) d\mu \\ + \int_B h(x, \nu_u(x)) |D_\mu^s u| \end{aligned} \tag{5.8}$$

for every $u \in BV(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$. Let us define $f : \Omega \times \mathbb{R}^N \to [0, +\infty[$ by

$$f(x, z) = g\left(x, \frac{d\lambda}{d\mu}(x) z\right) \frac{d\mu}{d\lambda}(x).$$

Then, for $\lambda$-a.e. $x \in \Omega$ the function $g(x, \cdot)$ is convex on $\mathbb{R}^N$. As $\lambda \ll \mu$, we have $\dfrac{d\lambda}{d\mu} \dfrac{d\mu}{d\lambda} = 1$ $\lambda$-a.e. in $\Omega$ and $\dfrac{d\alpha}{d\mu} \dfrac{d\mu}{d\lambda} = \dfrac{d\alpha}{d\lambda}$ $\lambda$-a.e. in $\Omega$, hence

$0 \le f(x,z) \le \dfrac{d\alpha}{d\lambda}(x) + \gamma|z|$ for $\lambda$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$. Moreover, being $\nabla_\mu u = \nabla_\lambda u \dfrac{d\lambda}{d\mu}$ $\mu$-a.e. in $\Omega$ and $\dfrac{d\lambda}{d\mu} > 0$ $\mu$-a.e. in $\Omega$ and $\dfrac{d\lambda}{d\mu} > 0$ $\lambda$-a.e. in $\Omega$, we have $E_\lambda(x) = E_\mu(x)$ for $\lambda$-a.e. $x \in \Omega$. Therefore the recession function $f_\infty$ satisfies $f_\infty(x,z) = h(x, p_x^\lambda(z))$ for $\lambda$-a.e. $x \in \Omega$ and for every $z \in \mathbb{R}^N$, where $p_x^\lambda$, denotes the orthogonal projection on the linear space $E_\lambda(x)$.

Let us fix $u \in BV(\Omega)$, and let $N$ be a Borel subset of $\Omega$, with $M \subseteq N$, such that $\lambda(N) = 0$ and $|D_\lambda^s u|(\Omega \backslash N) = 0$. Being $\lambda \ll \mu$, we have $|D_\mu^s u|(\Omega\backslash N) = 0$. As $\mu_s(N\backslash M) = 0$ and $\lambda(N\backslash M) = 0$, we also have $\mu(N\backslash M) = 0$. Since $\nabla_\mu u = \nabla u \dfrac{d\lambda}{d\mu}$ $\mu$-a.e. in $\Omega\backslash M$ and $\nu_u|\nabla_\mu u| = \nabla_\mu u$ $\mu$-a.e. in $\Omega$, from (5.8) we obtain

$$\begin{aligned}
F(u,B) &= \int_B f(x, \nabla_\lambda u(x))d\lambda + \int_{B\cap N} h(x, \nu_u(x))|\nabla_\mu u| d\mu \\
&\qquad + \int_{B\cap N} h(x,\nu_u(x))|D_\mu^s u| \\
&= \int_B f(x, \nabla_\lambda u(x))d\lambda + \int_{B\cap N} h(x,\nu_u(x))|Du| \\
&= \int_B f(x, \nabla_\lambda u(x))d\lambda + \int_B h(x,\nu_u(x))|D_\lambda^s u|,
\end{aligned}$$

which proves (ii). □

The following corollary describes a situation where Theorem 5.6. can be applied.

COROLLARY 5.7. *Assume that the functional* $F : BV(\Omega) \times \mathcal{B}(\Omega) \to [0, +\infty[$ *satisfies hypotheses* (H1), (H2) *and* (H3). *Let* $\beta$, $\lambda \in \mathcal{M}_+^b(\Omega)$, *with* $\alpha + \beta + \mathcal{L}^N \ll \lambda$. *Assume, in addition, that*

$$F(u+v, A) \le F(u,A) + F(v,A) + \beta(A) \tag{5.9}$$

*for every* $u$, $v \in BV(\Omega)$ *and for every* $A \in \mathcal{A}(\Omega)$. *Then there exist two Borel functions* $f$, $h : \Omega \times \mathbb{R}^N \to [0, +\infty[$ *which satisfy conditions* (i), (ii), (iii) *and* (iv) *of Theorem* 5.1 *with* $\mu = \lambda$.

PROOF. For every $A \in \mathcal{A}(\Omega)$ let $F_\infty(\cdot, A)$ be the recession function of $F(\cdot, A)$ defined in (2.27). By (2.28) and (5.9) we have

$$F(u,A) - \alpha(A) \le F_\infty(u,A) \le F(u,A) + \beta(A)$$

for every $u \in BV(\Omega)$ and for every $A \in \mathcal{A}(\Omega)$. Using the fact that $F(u, \cdot)$ is a bounded Radon measure, for every $B \in \mathcal{B}(\Omega)$ we obtain

$$F(u,B) - \alpha(B) \leq \inf\{F_\infty(u,A) : A \in \mathcal{A}(\Omega), B \subseteq A\} \leq F(u,B) + \beta(B).$$

If $\lambda(B) = 0$, then $\alpha(B) = \beta(B) = 0$, hence

$$F(u,B) = \inf\{F_\infty(u,A) : A \in \mathcal{A}(\Omega), B \subseteq A\}.$$

As $F_\infty(\cdot\,,A)$ is positively 1-homogeneous for every $A \in \mathcal{A}(\Omega)$, we obtain that $F(\cdot\,,B)$ is positively 1-homogeneous on $BV(\Omega)$ for every $B \in \mathcal{B}(\Omega)$ with $\lambda(B) = 0$. The conclusion follows now from Theorem 5.6. □

REMARK 5.8. It is clear from the proofs that all the results of this section still hold if we replace $BV(\Omega)$ by $BV(\Omega) \cap L^1(\Omega)$, or by $BV(\Omega) \cap L^{N/(N-1)}(\Omega)$, both in the hypotheses (H1), (H2) and (H3) and in the statements of the theorems. Moreover, the $L^{N/(N-1)}_{\mathrm{loc}}(\Omega)$-lower semicontinuity of $F(\cdot\,,A)$ was used only to prove the lower semicontinuity of the functional defined by (5.3) at each point of $BV(\Omega) \cap L^{N/(N-1)}(\Omega)$. Therefore, it may be replaced by the $L^{N/(N-1)}(\Omega)$-lower semicontinuity of $F(\cdot\,,A)$ on $BV(\Omega) \cap L^{N/(N-1)}(\Omega)$, or by any stronger assumption, like, for instance, $L^1(\Omega)$-lower semicontinuity on $BV(\Omega) \cap L^1(\Omega)$.

## 6. - Integral representation of Γ-limits

In this section we prove an integral representation theorem for Γ-limits of area-type functionals. Let us begin by recalling the definition of Γ-limit (see [18]). Given a metric space $M$, we say that a sequence of functions $F_n : M \to \overline{\mathbb{R}}$ is Γ-convergent in $M$ to a function $F : M \to \overline{\mathbb{R}}$ if both the following conditions are satisfied for every $u \in M$:

(a) for every sequence $(u_n)$ converging to $u$ in $M$ we have $F(u) \leq \liminf\limits_{n\to\infty} F_n(u_n)$;

(b) there exists a sequence $(v_n)$ converging to $u$ in $M$ such that $F(u) = \lim\limits_{n\to\infty} F_n(v_n)$.

This notion of convergence is called also epi-convergence, because it is equivalent to the convergence, in the sense of Kuratowski, of the epigraph of the functions $F_n$. Together with other similar notions, Γ-convergence has been used by several authors to investigate asymptotic properties of variational problems. We refer to the book [5] for a general treatment of this subject and for a wide bibliography on related topics.

Let $j_n : \Omega \times \mathbb{R}^N \to [0, +\infty[$ be a sequence of (possibly non-convex) Borel functions such that

$$\gamma_1|z| \leq j_n(x,z) \leq a(x) + \gamma_2|z| \qquad \forall n \in \mathbb{N},\ \forall x \in \Omega,\ \forall z \in \mathbb{R}^N, \tag{6.1}$$

where the function $a \in L^1(\Omega)$ and the constants $\gamma_1$ and $\gamma_2$, with $0 < \gamma_1 \leq \gamma_2$, are independent of $n$. Let us consider the corresponding integral functionals $J_n : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ defined by

$$(6.2) \qquad J_n(u, A) = \begin{cases} \displaystyle\int_A j_n(x, \nabla u)dx & \text{if } u|_A \in W^{1,1}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

A general compactness result about Γ-limits of integral functionals, whose main ideas go back to [17], states that there exist a subsequence of $(J_n)$, still denoted by $(J_n)$, and a functional $J : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$, such that for every $A \in \mathcal{A}(\Omega)$:

$$(6.3) \qquad J_n(\cdot, A) \ \Gamma\text{-converges to } J(\cdot, A) \text{ in } L^1(\Omega),$$

$$(6.4) \qquad J(u, A) = \sup\{J(u, A') : A' \in \mathcal{A}(\Omega), A' \subset\subset A\} \qquad \forall u \in L^1(\Omega),$$

$$(6.5) \qquad \gamma_1 \int_A |Du| \leq J(u, A) \leq \int_A a dx + \gamma_2 \int_A |Du| \qquad \forall u \in L^1(\Omega),$$

with the usual convention that $\int_A |Du| = +\infty$ if $u|_A \notin BV(A)$ (see [23], Definition 1.1). For the proof we refer to [9] (see, in particular, Proposition 2.4, Theorem 3.8, and the first lines of the proof of Theorem 4.3). The same result can be obtained also by a slightly different argument, similar to the proof of (4.16), based on Proposition 4.16 and Theorem 6.1 of [16].

The following theorem provides an integral representation of $J$. It can be considered as an extension to $BV(\Omega)$ of some of the results of [17].

THEOREM 6.1. *Let $(j_n)$ be a sequence of Borel integrands satisfying* (6.1) *and let $(J_n)$ be the corresponding sequence of integral functionals defined by* (6.2). *Assume that* (6.3) *holds for every $A \in \mathcal{A}(\Omega)$. Then there exist a measure $\mu \in \mathcal{M}_+^b(\Omega)$, with $\mathcal{L}^N \ll \mu$, and two Borel functions $f, h : \Omega \times \mathbb{R}^N \to [0, +\infty]$, such that*

(i) *for $\mu$-a.e. $x \in \Omega$ the function $f(x, \cdot)$ is convex on $\mathbb{R}^N$ and satisfies*

$$\gamma_1|z| \leq f(x, z) = f(x, p_x(z)) \leq a(x)\frac{d\mathcal{L}^N}{d\mu}(x) + \gamma_2|z| \qquad \forall z \in \mathbb{R}^N,$$

*where $p_x$ denotes the orthogonal projection on the linear space $E_\mu(x)$ defined in* (2.2);

(ii) *for every $x \in \Omega$ the function $h(x, \cdot)$ is positively 1-homogeneous on $\mathbb{R}^N$ and*

$$\gamma_1|z| \leq h(x, z) \leq \gamma_2|z| \qquad \forall z \in \mathbb{R}^N;$$

(iii) *for $\mu$-a.e. $x \in \Omega$ we have*

$$f_\infty(x,z) = h(x, p_x(z)) \qquad \forall z \in \mathbb{R}^N,$$

*where* $f_\infty$ *is the recession function of* $f$ *with respect to* $z$ *defined in* (4.8)*;*

(iv) *for every* $A \in \mathcal{A}(\Omega)$ *and for every* $u \in L^1(\Omega)$ *we have*

$$J(u,A) = \begin{cases} \displaystyle\int_A f(x, \nabla_\mu u(x))d\mu + \int_A h(x, \nu_u(x))|Du^s_\mu u| & \text{if } u|_A \in BV(A), \\ +\infty & \text{otherwise.} \end{cases}$$

PROOF. Let $F : (BV(\Omega) \cap L^1(\Omega)) \times \mathcal{B}(\Omega) \to [0, +\infty[$ be the functional defined by

$$F(u,B) = \inf\{J(u,A) : A \in \mathcal{A}(\Omega), B \subseteq A\}. \tag{6.6}$$

Since $J(u,\cdot)$ is increasing on $\mathcal{A}(\Omega)$, we have

$$J(u,A) = F(u,A) \qquad \forall u \in BV(\Omega) \cap L^1(\Omega),\ \forall A \in \mathcal{A}(\Omega). \tag{6.7}$$

Moreover, it is possible to prove that:

(6.8) for every $u \in BV(\Omega) \cap L^1(\Omega)$ the set function $F(u,\cdot)$ is a Borel measure on $\Omega$;

(6.9) for every $A \in \mathcal{A}(\Omega)$ the function $F(\cdot, A)$ is convex and $L^1(\Omega)$-lower semicontinuous on $BV(\Omega) \cap L^1(\Omega)$;

(6.10) for every $u \in BV(\Omega) \cap L^1(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$ we have

$$\gamma_1 \int_B |Du| \leq F(u,B) \leq \int_B a\,dx + \gamma_2 \int_B |Du|.$$

Property (6.8) is proved in [9], Theorem 3.8. Since $J(u,\cdot)$ satisfies (6.4), property (6.8) follows also from Theorems 4.18 and 6.1 of [16]. The $L^1(\Omega)$-lower semicontinuity of $F(\cdot, A)$ is a consequence of (6.7) and of a general property of $\Gamma$-limits (see [18], Proposition 1.8, or [5], Theorem 2.1).

We now prove the convexity of $F(\cdot, A)$. For every $A \in \mathcal{A}(\Omega)$ let $\overline{J}_n(\cdot, A)$ be the greatest $L^1(\Omega)$-lower semicontinuous function less than or equal to $J_n(\cdot A)$. It can be proved that there exists a Borel function $f_n(x,z)$, convex in $z$, such that

$$\overline{J}_n(u,A) = \int_A f_n(x, \nabla u)dx$$

for every $A \in \mathcal{A}(\Omega)$ and for every $u \in L^1(\Omega)$ with $u|_A \in W^{1,1}(A)$ (see [10], Corollary 2.3). See also [20], Chapter X, and [25] for similar results under slightly different hypotheses on $j_n(x,z)$. Let $F_n : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be defined by

$$F_n(u,A) = \begin{cases} \displaystyle\int_A f_n(x,\nabla u)dx & \text{if } u|_A \in W^{1,1}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

Since, for every $A \in \mathcal{A}(\Omega)$, $\overline{J}_n(\cdot\,,A)$ $\Gamma$-converges to $J(\cdot\,,A)$ in $L^1(\Omega)$ (see [18], Proposition 1.11, or [5], Corollary 2.7), and $\overline{J}_n(\cdot\,,A) \leq F_n(\cdot\,,A) \leq J_n(\cdot\,,A)$, by comparison we conclude that $F_n(\cdot\,,A)$ $\Gamma$-converges to $J(\cdot\,,A)$ in $L^1(\Omega)$. Since $F_n(\cdot\,,A)$ is convex, and convexity is preserved by $\Gamma$-limits, we obtain that $J(\cdot\,,A)$ is convex on $L^1(\Omega)$, hence $F(\cdot\,,A)$ is convex on $BV(\Omega) \cap L^1(\Omega)$ and the proof of (6.9) is complete.

Finally (6.10) is an easy consequence of (6.5) and (6.6).

By Theorem 5.1 and Remark 5.8, from (6.8), (6.9) and (6.10) it follows that there exist $\mu$, $f$ and $h$ satisfying conditions (i), (ii) and (iii) of Theorem 5.1, such that

$$F(u,B) = \int_B f(x,\nabla_\mu u(x))d\mu + \int_B h(x,\nu_u(x))|D^s_\mu u| \tag{6.11}$$

for every $u \in BV(\Omega)\cap L^1(\Omega)$ and for every $B \in \mathcal{B}(\Omega)$. The additional inequalities involving $\gamma_1|z|$ required in Theorem 6.1 are an easy consequence of (6.10). If we take $B = A \in \mathcal{A}(\Omega)$ in (6.11), by (6.7) we obtain

$$J(u,A) = \int_A f(x,\nabla_\mu u(x))d\mu + \int_A h(x,\nu_u(x))|D^s_\mu u| \tag{6.12}$$

for every $u \in BV(\Omega)\cap L^1(\Omega)$. We now prove that (6.12) holds under the weaker assumption $u|_A \in BV(A)$.

Let $A \in \mathcal{A}(\Omega)$ and let $u \in L^1(\Omega)$ with $u|_A \in BV(A)$. For every $A' \in \mathcal{A}(\Omega)$ with $A' \subset\subset A$ there exists $v \in BV(\Omega) \cap L^1(\Omega)$ such that $v = u$ a.e. on $A'$. By the definition of $\Gamma$-limit, we have $J(u,A') = J(v,A')$. Moreover $\nabla_\mu u = \nabla_\mu v$, $\nu_u = \nu_v$, and $D^s_\mu u = D^s_\mu v$ on $A'$. Therefore (6.12) implies that

$$J(u,A') = \int_{A'} f(x,\nabla_\mu u(x))d\mu + \int_{A'} h(x,\nu_u(x))|D^s_\mu u|.$$

By (6.4) the same equality holds with $A'$ replaced by $A$. This proves (iv) in the case $u|_A \in BV(A)$. If $u|_A \notin BV(A)$, then (6.5) implies that $J(u,A) = +\infty$.□

In Theorem 6.1 the assumption that $J_n$ is an integral functional is not essential. It can be replaced by more qualitative hypotheses as, for instance, the following set of assumptions:

(6.13) if $u$, $v \in L^1(\Omega)$, $A \in \mathcal{A}(\Omega)$, and $u = v$ a.e. on $A$, then $J_n(u,A) = J_n(v,A)$;

(6.14) for every $u \in L^1(\Omega)$ the set function $J_n(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on $\Omega$;

(6.15) there exist two constants $\gamma_1,\ \gamma_2 \in ]0,+\infty[$ and a function $a \in L^1(\Omega)$ such that

$$\gamma_1 \int_A |Du| \le J_n(u,A) \le \int_A a\,dx + \gamma_2 \int_A |Du| \qquad \forall u \in L^1(\Omega),\ \forall A \in \mathcal{A}(\Omega),$$

where $\int_A |Du| = +\infty$ whenever $u|_A \notin BV(A)$.

By using Proposition 4.16 and Theorem 6.4 of [16], we can still prove that there exist a subsequence of $(J_n)$ and a functional $J : L^1(\Omega) \times \mathcal{A}(\Omega) \to [0,+\infty]$ such that (6.3), (6.4) and (6.5) hold for every $A \in \mathcal{A}(\Omega)$. Moreover, by Theorem 4.18 and 6.4 of [16], one obtains that $J(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on $\Omega$. Repeating, with obvious modifications, the proof of Theorem 6.1, we obtain the following result:

THEOREM 6.2. *The conclusions* (i), (ii), (iii) *and* (iv) *of Theorem* 6.1 *hold for an arbitrary sequence of functionals* $(J_n)$ *satisfying* (6.3), (6.13), (6.14) *and* (6.15).

REMARK 6.3. Condition (6.15) in Theorem 6.2 can be replaced by the following assumptions:

$$\gamma_1 \int_A |\nabla u| dx \le J_n(u,A) \le \int_A (a + \gamma_2 |\nabla u|) dx \qquad \text{if } u|_A \in W^{1,1}(A),$$

$$J_n(u,A) = +\infty \qquad \text{if } u|_A \notin W^{1,1}(A).$$

With these modifications, Theorem 6.2 includes Theorem 6.1.

The following example shows that, in general, we cannot take $\mu = \mathcal{L}^N$ in the integral representation of $\Gamma$-limits provided by Theorem 6.1.

EXAMPLE 6.4. Let $N = 1$ and $\Omega = ]-1,1[$. Let us consider a convex function $g : \mathbb{R} \to [0,+\infty[$ such that

$$|z| \le g(z) \le 2|z| \qquad \forall z \in \mathbb{R}, \tag{6.16}$$

$$g(tz) \ne t g(z) \qquad \forall z > 0, \forall t > 0. \tag{6.17}$$

An example is given by $g(z) = |z| - 1 + \sqrt{1+|z|^2}$. Let $a_n(x) = 1$ for $|x| \ge 1/n$, and $a_n(x) = n/2$ for $|x| < 1/n$. Then the integrand $j_n : \Omega \times \mathbb{R} \to [0,+\infty[$, defined by $j_n(x,z) = g(z/a_n(x))a_n(x)$, is convex in $z$ and, by (6.16), it satisfies the inequalities

$$|z| \le j_n(x,z) \le 2|z| \qquad \forall x \in \Omega,\ \forall z \in \mathbb{R}.$$

If $\lambda_n$ is the Radon measure on $\Omega$ given by $\lambda_n(B) = \int_B a_n dx$, then the integral functional $J_n$ defined by (6.2) can be written as

$$J_n(u, A) = \begin{cases} \int_A g(\nabla_{\lambda_n} u) d\lambda_n & \text{if } u|_A \in W^{1,1}_{\lambda_n}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

The sequence $(\lambda_n)$ converges in the weak* topology of $\mathcal{M}^b(\Omega)$ to the measure $\lambda = \delta + \mathcal{L}^1$, where $\delta$ denotes the Dirac mass at the origin. Therefore, the results of [11] (see also [6]) show that $J_n(\cdot, A)$ $\Gamma$-converges to $J(\cdot, A)$ in $L^1(\Omega)$ for every $A \in \mathcal{A}(\Omega)$, where

$$J(u, A) = \begin{cases} \int_A g(\nabla_\lambda u) d\lambda + \int_A g_\infty(\nu_u) |D^s_\lambda u| & \text{if } u|_A \in BV(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{6.18}$$

$g_\infty$ being the recession function of $g$.

Suppose, by contradiction, that $J$ can also be represented as in Theorem 6.1 with $\mu = \mathcal{L}^1$, that is to say

$$J(u, A) = \int_A f(x, \nabla u(x)) dx + \int_A h(x, \nu_u(x)) |D^s u| \qquad \forall u \in BV(\Omega), \quad \forall A \in \mathcal{A}(\Omega). \tag{6.19}$$

Let $w \in BV(\Omega)$ be the function defined by $w(x) = 0$ for $x < 0$, and $w(x) = 1$ for $x > 0$. For every $t > 0$ from (6.18) we get $J(tw, \Omega) = g(t)$, while (6.19) yields $J(tw, \Omega) = ct$, with $c = h(0, 1)$. This implies that $g(t) = ct$ for every $t > 0$, which contradicts (6.17).

The following theorem describes a situation where the $\Gamma$-limit can be represented by an integral with $\mu = \mathcal{L}^N$. Note that the additional hypothesis on $(j_n)$ is always satisfied if $\gamma_1 = \gamma_2$ in (6.1).

THEOREM 6.5. *In addition to the hypotheses of Theorem* 6.1, *assume that for every* $n \in \mathbb{N}$

$$j_n(x, z_1 + z_2) \leq j_n(x, z_1) + j_n(x, z_2) + b(x) \qquad \forall x \in \Omega, \ \forall z_1, \ z_2 \in \mathbb{R}^N,$$

*where* $b \in L^1(\Omega)$ *is a function independent of* $n$. *Then there exist two Borel functions* $f, h : \Omega \times \mathbb{R}^N \to [0, +\infty]$ *which satisfy conditions* (i), (ii), (iii) *and* (iv) *of Theorem* 6.1 *with* $\mu = \mathcal{L}^N$.

PROOF. It is easy to see that under our hypotheses we have

$$J_n(u + v, A) \leq J_n(u, A) + J_n(v, A) + \int_A b dx \qquad \forall u, \ v \in L^1(\Omega), \ \forall A \in \mathcal{A}(\Omega).$$

Since this inequality is preserved by $\Gamma$-convergence, we obtain

$$J(u+v,A) \leq J(u,A)+J(v,A)+\int_A bdx \qquad \forall u,\ v \in L^1(\Omega),\ \forall A \in \mathcal{A}(\Omega).$$

The conclusion follows now from (6.7) and from Corollary 5.7. $\square$

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