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Integral Representation of Convex Functionals on a Space of Measures

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In duality pairs such as $(\mathcal{M}^b, \mathscr{C}_0)$ and $(W^{-1,p'}, W_0^{1,p})$, a convex integral functional on the space of functions has a polar which admits an integral representation. This representation is the sum of a first term involving the absolutely continuous component of the measure and of a second one which is a positively homogeneous function of the singular part. The duality is useful in plasticity theory. In the Sobolev case the study of non-parametric integrands is new. A description of the sub-differential is obtained.

INTRODUCTION

Our motivations arise from two kinds of problems.

FIRST PROBLEM. In the mathematical theory of plasticity the energy can be expressed by

$$\int_\Omega f(x, Du(x))\, dx,$$

where $f(x, \cdot)$ is convex with linear growth. The function u can be discontinuous so its gradient (more precisely its deformation) Du has to be taken in the distribution sense. With some appropriate hypotheses (see [36]), Du

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belongs to the space $\mathcal{M}^b$ of bounded measures, hence the idea of extending the functional

$$I_f: v \mapsto \int_\Omega f(x, v(x))\, dx$$

from L^1 to $\mathcal{M}^b$ by taking the $\sigma(\mathcal{M}^b, \mathscr{C}_0)$ lower semi-continuous hull

$$\bar{F}: \lambda \mapsto \varliminf_{v \to \lambda} I_f(v).$$

Let us point out that the $\sigma(\mathcal{M}^b, \mathscr{C}_0)$ topology is the one which provides relative compactness of the sequence Du_ε when u_ε approaches the equilibrium.

When I_f is convex and proper one has

$$\bar{F}(\lambda) = \sup\{\langle \lambda, \varphi \rangle - I_{f^*}(\varphi) \mid \varphi \in \mathscr{C}_0\}.$$

The problem is to give an integral expression of $\bar{F}(\lambda)$.

SECOND PROBLEM. In the variational approach of semi-linear elliptic equations involving measures such as the Thomas–Fermi problem (see Brezis [13, 14] and Attouch, Bouchitté, and Mabrouk [2]), the Euler equation is obtained by computing the sub-differential on the Sobolev space $W_0^{1,p}$ of an integral functional $\int j(x, u(x))\, dx$. Usually the domain of the polar functional is contained in $\mathcal{M}^b \cap W^{-1,p'}$.

Thus the two problems lead to the calculus on a space of measures of the polar of an integral functional. When f or j do not depend on x, the expression of the polar is due to Temam [37] and Demengel and Temam [19] for the first problem (but already in Valadier [40, 41]), and Brezis [11] completed by Grun–Rehomme [23] for the second one.

In the two previous problems it is important to allow f and j to depend on x (non-homogeneous media in the first situation and second member measure in the second one). In this direction the duality $(\mathcal{M}^b, \mathscr{C}_0)$ has been considered by several authors (Rockafellar [32], Olech [28, 29], Valadier [41]). In the same way Giaquinta, Modica, and Soucek [21] and Dal Maso [16], using a result of Reschetniak [30], obtain the integral representation of $\bar{F}$ under hypotheses implying the continuity of f in (x, z) and its linear growth in z. Since 1985 this problem has been intensively studied by Hadhri [24], Valadier [42] (using Tran cao Nguyen [38, 39]), and De Giorgi, Ambrosio, and Buttazzo [17].

Our approach is new. It reduces the calculus of

$$\sup\left\{\int \varphi \cdot d\lambda - \int f^*(\cdot, \varphi)\, d\mu \mid \varphi \in \mathscr{C}_0\right\}$$

to the calculus of

$$[J+\delta(\cdot\,|\,\mathscr{C}_0)]^*\left(\frac{d\lambda}{dm}\right),$$

where m is a positive measure such that $\mu \ll m$ and $\lambda \ll m$, and J $(=I_{f^*})$ is an integral functional with respect to m. The basic result (Theorem 1 of Section 2) may seem rather abstract but it contains almost all difficulties. On the whole the proof is shorter than those of all previous paper.

In Section 3 we recover the formula (already in Valadier [40])

$$\bar{F}(\lambda)=\int g\left(\cdot,\frac{d\lambda_a}{d\mu}\right)d\mu+\int h\left(\cdot,\frac{d\lambda_s}{d\,|\lambda_s|}\,d\,|\lambda_s|\right),$$

where $\lambda_a+\lambda_s$ is the Lebesgue decomposition (with respect to μ) of λ and the integrands h and g derive from f. The situation is quite different from the non-parametric case where $g=f$ and $h=f_\infty$ the recession function of f. Indeed as shown in the examples of Section 5, g can be different from f. Nevertheless, under some regularity assumptions which are set in Section 4, the equality $h=f_\infty(x,\cdot)$ may occur μ-a.e. (which implies $g(x,\cdot)=f(x,\cdot)$ a.e.) or everywhere. A comparison is then possible with the results of [1, 16, 21].

The application to the duality $(W_0^{1,p}, W^{-1,p'})$ (second problem) is studied in [5, 7, 8]; the results of Brezis [11] and Grun–Rehomme [23] are extended.

The present paper follows and improves in some details on Bouchitté [4, 5, 6]. Sections 2 to 4 include the results of Valadier [42], with new proofs, and some other results (especially in Section 4).

1. Notations

Throughout this paper Ω denotes a locally compact metrizable space which is σ-compact, that is, a union of a countable sequence of compact subsets. This allows Ω to be compact metrizable (which from the mathematical standpoint would be simpler). This also allows Ω to be an open subset of $\mathbb{R}^N$.

A positive Radon measure μ on Ω is given. When Ω is an open subset of $\mathbb{R}^N$ it may be the Lebesgue measure. We will denote by m an auxiliary positive measure.

The space of continous functions tending to 0 at infinity is denoted by $\mathscr{C}_0(\Omega)$, abbreviated as $\mathscr{C}_0$. The space of $\mathbb{R}^d$-valued functions $\mathscr{C}_0(\Omega;\mathbb{R}^d)$ is also denoted by $[\mathscr{C}_0]^d$ and d will often be omitted. By $\mathscr{C}_c$ we denote the space of continuous functions with compact supports. When Ω is an open subset of $\mathbb{R}^N$, $\mathscr{C}^\infty$ is the space of infinitely differentiable functions and $\mathscr{C}_c^\infty$ or $\mathscr{D}$ is the subspace of functions with compact supports.

By $\mathscr{M}$ and $\mathscr{M}^b$ we denote respectively the spaces of Radon measures on Ω and of bounded measures. The spaces of $\mathbb{R}^d$-valued measures are denoted by $\mathscr{M}(\Omega; \mathbb{R}^d)$, $\mathscr{M}^b(\Omega; \mathbb{R}^d)$ or $[\mathscr{M}]^d$, $[\mathscr{M}^b]^d$ (d will often be omitted).

Most of the paper uses one of the duality pairs $(\mathscr{M}, \mathscr{C}_c)$ or $(\mathscr{M}^b, \mathscr{C}_0)$. The bilinear form is denoted with brackets (for example $\langle \lambda, \varphi \rangle$) but the scalar product of $z, z' \in \mathbb{R}^d$ is denoted by $z \cdot z'$. If F is a function on a vector space E, F^* denotes its polar

$$F^*(x') = \sup\{\langle x', x\rangle - F(x) \mid x \in E\}$$

and $\operatorname{dom} F = \{x \mid F(x) < \infty\}$. If C is a subset of E, $\delta(\cdot \mid C)$ denotes its indicator function (taking value 0 on C, $+\infty$ outside) and $\delta^*(\cdot \mid C)$ its support function.

A *normal integrand* f is a measurable function $f: \Omega \times \mathbb{R}^d \to \bar{\mathbb{R}}$. We say that f is a convex normal integrand if moreover, $\forall x$, $f(x, \cdot)$ is convex l.s.c.

Other notation: $\mathbb{N}$ is the set of integers $n \geqslant 0$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\bar{B}(x, r)$ is the closed ball with center x and radius r, and δ_a is the Dirac measure at a.

2. Preliminary Results

2.1. We denote by $\mathscr{L}^0(\Omega, m)$ the vector space of real measurable functions.

Definition. A subset $\mathscr{H}$ of $[\mathscr{L}^0]^d$ is said to be *PCU-stable* if for any continuous partition of unity $(\alpha_0, ..., \alpha_n)$ such that $\alpha_1, ..., \alpha_n$ belong to $\mathscr{C}_c$ (variant, when Ω is an open subset of $\mathbb{R}^N$, $\alpha_1, ..., \alpha_n \in \mathscr{D}(\Omega)$, $\alpha_0 \in \mathscr{C}^\infty(\Omega)$), for every $u_0, ..., u_n$ in $\mathscr{H}$, $\sum_{i=0}^n \alpha_i u_i$ belongs to $\mathscr{H}$.

Remark. In the main applications $\mathscr{H}$ will be $[\mathscr{C}_0]^d$ or $[\mathscr{C}_c]^d$ and, in other papers [5, 7, 8], $\{\tilde{u} \mid u \in [W_0^{1,p} \cap L^\infty]^d\}$, where $\tilde{u}$ denotes all quasi-continuous elements of the Lebesgue equivalence class of u ([3, 12]).

2.2. Recall the following result [43, Proposition 1.14] (for a more recent paper see Fougères [20]). For any subset $\mathscr{H}_1$ of $[\mathscr{L}^0]^d$ there exists a smallest closed-valued measurable multifunction Γ such that $\forall u \in \mathscr{H}_1$, $u(x) \in \Gamma(x)$ m-a.e. (smallest refers to inclusion a.e.). We write $\Gamma = \operatorname{ess\,sup}_{u \in \mathscr{H}_1}\{u(\cdot)\}$ and say that Γ is the *essential supremum* of the multifunctions $x \mapsto \{u(x)\}$ $(u \in \mathscr{H}_1)$. Moreover there exists a sequence (u_n) in $\mathscr{H}_1$ such that a.e. $\Gamma(x) = \operatorname{cl}\{u_n(x) \mid n \in \mathbb{N}\}$. If (v_n) is any other sequence in $\mathscr{H}_1$ we can add the v_n to the u_n. Thus if $\mathscr{H}_1 \subset [\mathscr{C}_0(\Omega)]^d$, since $\mathscr{C}_0$ is separable (for the uniform convergence norm), we can add a dense sequence and this proves $\Gamma(x) = \mathrm{cl}\{u(x) \mid u \in \mathscr{H}_1\}$. If $\mathscr{H}_1$ is convex it is easy to see that Γ is (a.e.) convex valued. This remains true if $\mathscr{H}_1$ is PCU stable. Indeed for any compact subset K of Ω and $r_0, ..., r_n \geqslant 0$ such that $\sum r_i = 1$, there exists a continuous partition of unity $(\alpha_0, ..., \alpha_n)$ with $\alpha_1, ..., \alpha_n \in \mathscr{C}_c$ and $\forall i$, $\alpha_i(x) = r_i$ on K. Then adding to the u_n, all the $\sum \alpha_i u_i$ for $(\alpha_0, ..., \alpha_n)$ corresponding to rational r_i and K running through a countable family of compacts (K_p) such that $\bigcup K_p = \Omega$, one can easily check that $\Gamma(x)$ is convex.

2.3. Let $j: \Omega \times \mathbb{R}^d \to]-\infty, \infty]$ be a normal convex integrand. For any $u \in [\mathscr{L}^0]^d$, $j(\cdot, u)$ denotes the function $x \mapsto j(x, u(x))$. Denote J the functional

$$\left| \begin{array}{l} u \mapsto \int_\Omega j(\cdot, u)\, dm \\ [\mathscr{L}^0]^d \to \overline{\mathbb{R}}, \end{array} \right.$$

where, as usual in convex analysis, $\int j(\cdot, u)\, dm = +\infty$ as soon as $\int j(\cdot, u)^+\, dm = +\infty$.

THEOREM 1. *Let $\mathscr{H}$ be a PCU-stable subset of $[\mathscr{L}^0]^d$. Suppose $\exists u_0 \in \mathscr{H}$ with $J(u_0) \in \mathbb{R}$. Then $\Gamma = \operatorname{ess\,sup}_{u \in \mathscr{H} \cap \operatorname{dom} J} \{u(\cdot)\}$ is convex valued,*

$$\inf_{u \in \mathscr{H}} J(u) = \int_\Omega [\inf_{z \in \Gamma(x)} j(x, z)]\, m(dx)$$

and

$$\inf_{z \in \Gamma(x)} j(x, z) = \operatorname*{ess\,inf}_{u \in \mathscr{H} \cap \operatorname{dom} J} j(\cdot, u).$$

Commentary. Classical results about commutativity of $\int$ and inf assume that $\mathscr{H}$ is a decomposable vector space or the set of measurable selectors of a multifunction: see Rockafellar [31, 33], Hiai and Umegaki [25], and Bourass and Valadier [9].

Remark/Example. We cannot take $\Gamma = \operatorname{ess\,sup}_{u \in \mathscr{H}} \{u(\cdot)\}$. Indeed let $\Omega = \mathbb{R}$, m the Lebesgue measure, $d = 1$, K a compact subset of $\mathbb{R}$ such that $\operatorname{int}(K) = \varnothing$ and $m(K) > 0$ (one can construct K analogously to the Cantor set). Let

$$j(x, z) = \begin{cases} z & \text{if } x \in K \\ \delta(z \mid \{0\}) & \text{otherwise.} \end{cases}$$

Let $\mathscr{H} = \mathscr{C}_c$. Then $\inf_{u \in \mathscr{H}} J(u) = 0$ because, if $u \not\equiv 0$, the set $\{x \mid u(x) \neq 0$

and $x \notin K\}$ is open and non-empty, so has >0 measure and $J(u) = +\infty$. But $\operatorname{ess\,sup}_{u \in \mathscr{C}_c} \{u(\cdot)\}$ is the constant multifunction $x \mapsto \mathbb{R}$ and

$$\inf_{z \in \mathbb{R}} j(x, z) = \begin{cases} -\infty & \text{if } \quad x \in K \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) First $\mathscr{H} \cap \operatorname{dom} J$ is still PCU-stable (because $j(\cdot, \sum \alpha_i u_i) \leqslant \sum \alpha_i j(\cdot, u_i)^+$), hence Γ is convex valued.

(2) Prove the first equality.

Let $\gamma(x) = \inf_{z \in \Gamma(x)} j(x, z)$ (γ is μ-measurable; Castaing and Valadier [15, Lemma III.39]). First $\geqslant$ holds because, $\forall u \in \mathscr{H} \cap \operatorname{dom} J$, $u(x) \in \Gamma(x)$ a.e. so

$$j(x, u(x)) \geqslant \gamma(x) \qquad \text{a.e.}$$

Prove now $\leqslant$. Let $r \in \mathbb{R}$, $r > \int \gamma \, dm$. Thanks to Bourbaki [10] or Dellacherie and Meyer [18, Théorème 48, pp. 107–108] there exists α l.s.c. integrable such that $\forall x$, $\alpha(x) \geqslant \gamma(x)$ and $\int \alpha \, dm < r$ (as $\gamma^+ \leqslant j(\cdot, u_0)^+$, γ^+ is integrable and can be approached upper by a l.s.c. function, and γ^- can be approached below by an u.s.c. function). We may modify slightly α to obtain $\forall x$, $\alpha(x) > \gamma(x)$.

Let $(u_n)_{n \geqslant 1}$ be a sequence in $\mathscr{H} \cap \operatorname{dom} J$ such that $\Gamma(x) = \operatorname{cl}\{u_n(x) \mid n \in \mathbb{N}^*\}$. Let N be a negligible set such that $\forall n$, $\forall x \in \Omega \backslash N$, $j(x, u_n(x)) \in \mathbb{R}$ (recall that $u_n \in \operatorname{dom} J$ implies $j(\cdot, u_n)^+$ is integrable and that $j(x, z) > -\infty$). Let $\varepsilon > 0$. There exists K compact, $K \subset \Omega \backslash N$ such that $\int_{\Omega \backslash K} [|j(\cdot, u_0)| + |\alpha|] \, dm < \varepsilon$. There exists $\eta > 0$ such that $m(A) < \eta$ implies $\int_A [|j(\cdot, u_0)| + |\alpha|] \, dm < \varepsilon$. Let K^ε be a compact such that $K^\varepsilon \subset K$, $m(K \backslash K^\varepsilon) < \eta$ and $\forall n$, $j(\cdot, u_n)$ is continuous on K^ε.

Let $A_n = \{x \in K^\varepsilon \mid j(x, u_n(x)) < \alpha(x)\}$. It is an open subset of K^ε. From Lemma A1 (see Appendix 1) applied with $D = \{u_n(x) \mid n \in \mathbb{N}^*\}$ (so $\bar{D} = \Gamma(x)$), for any $x \in K^\varepsilon$, $\gamma(x) = \inf_{n \geqslant 1} j(x, u_n(x))$, hence $\bigcup_{n \geqslant 1} A_n = K^\varepsilon$. By compactness there exists p such that $K^\varepsilon = \bigcup_{n=1}^{p} A_n$. There exists an open subset V^ε of Ω such that $V^\varepsilon \supset K^\varepsilon$ and

$$\forall n, \qquad 0 \leqslant n \leqslant p \Rightarrow \int_{V^\varepsilon \backslash K^\varepsilon} j(\cdot, u_n)^+ \, dm < \frac{\varepsilon}{p+1}.$$

Let V_n be a relatively compact open subset of Ω such that $V_n \cap K^\varepsilon = A_n$. We may suppose $V_n \subset V^\varepsilon$. There exists a continuous partition of unity $(\alpha_0, ..., \alpha_p)$ such that $\forall i = 1, ..., p$, $\operatorname{supp} \alpha_i \subset V_i$ and $\operatorname{supp} \alpha_0 \subset \Omega \backslash K^\varepsilon$ (see, for example, Bourbaki [10, Chap. III.1, n° 2, Lemme 1, p. 43]; when Ω is an open subset of $\mathbb{R}^N$ it is possible to get $\forall i$, $\alpha_i \in \mathscr{C}^\infty(\Omega)$, see L. Schwartz [35, Chap. I, Théorème II]).

Let $u=\sum_{n=0}^{p}\alpha_n u_n$. As $\mathscr{H}$ is PCU-stable, $u\in\mathscr{H}$. One has

$$j(x,u(x))\leqslant\sum_{n=0}^{p}\alpha_n(x)\,j(x,u_n(x))\leqslant\begin{cases}\alpha(x) & \text{if} \quad x\in K^{\varepsilon}\\ \displaystyle\sum_{n=0}^{p} j(x,u_n(x))^{+} & \text{if} \quad x\in V^{\varepsilon}\backslash K^{\varepsilon}\end{cases}$$

$$j(x,u(x))=j(x,u_0(x)) \qquad \text{if} \quad x\in\Omega\backslash V^{\varepsilon}.$$

Then

$$\int_{\Omega} j(\cdot,u)\,dm\leqslant\int_{K^{\varepsilon}}\alpha\,dm+\int_{V^{\varepsilon}\backslash K^{\varepsilon}}\sum_{0}^{p} j(\cdot,u_n)^{+}\,dm+\int_{\Omega\backslash V^{\varepsilon}}|j(\cdot,u_0)|\,dm.$$

We have

$$\int_{K^{\varepsilon}}\alpha\,dm=\int_{\Omega}\alpha\,dm-\left(\int_{\Omega\backslash K}\alpha\,dm+\int_{K\backslash K^{\varepsilon}}\alpha\,dm\right)$$

$$\leqslant\int_{\Omega}\alpha\,dm+2\varepsilon\leqslant r+2\varepsilon$$

$$\int_{V^{\varepsilon}\backslash K^{\varepsilon}}\sum_{0}^{p} j(\cdot,u_n)^{+}\,dm\leqslant\varepsilon$$

$$\int_{\Omega\backslash V^{\varepsilon}}|j(\cdot,u_0)|\,dm\leqslant\int_{\Omega\backslash K^{\varepsilon}}\cdots$$

$$=\int_{\Omega\backslash K}\cdots+\int_{K\backslash K^{\varepsilon}}\cdots\leqslant 2\varepsilon.$$

Finally, $\int_{\Omega} j(\cdot,u)\,dm\leqslant r+5\varepsilon$.

(3) As shown in (2), $\gamma(x)=\inf_{n\geqslant 1} j(x,u_n(x))$ a.e. Hence $\gamma\geqslant\operatorname{ess\,inf}_{u\in\mathscr{H}\cap\operatorname{dom}J} j(\cdot,u)$. Conversely there exists a sequence (v_k) in $\mathscr{H}\cap\operatorname{dom}J$ such that

$$\operatorname*{ess\,inf}_{u\in\mathscr{H}\cap\operatorname{dom}J} j(\cdot,u)=\inf_k j(\cdot,v_k).$$

But $v_k(x)\in\Gamma(x)$ a.e. so

$$\gamma(x)\leqslant\inf_k j(x,v_k(x)).$$

THEOREM 2. *We keep the hypotheses of Theorem* 1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be vector spaces of $\mathbb{R}^d$-valued measurable functions such that $\forall u\in\mathscr{X}$, $\forall v\in\mathscr{Y}$, $u(\cdot)\cdot v(\cdot)$ is m-integrable and $\mathscr{H}\subset\mathscr{X}$. Then, in the duality $(\mathscr{X},\mathscr{Y})$*

$$\forall v\in\mathscr{Y},\qquad [J+\delta(\cdot\mid\mathscr{H})]^{*}(v)=\int_{\Omega} k(\cdot,v)\,dm,$$

where $k(x,\cdot)=[j^{*}(x,\cdot)\nabla\delta^{*}(\cdot\mid\Gamma(x))]^{**}$ (here ∇ denotes the infimum convolution [27]).

Remark. It is possible with a minoration hypothesis to obtain that the $\sigma(\mathscr{X}, \mathscr{Y})$ l.s.c. hull of $J + \delta(\cdot \mid \mathscr{H})$ is $u \mapsto J(u) + \int_\Omega \delta(u(x) \mid \Gamma(x))\, m(dx)$ (see Bouchitté [5, Théorème 2]).

Proof.

$$[J + \delta(\cdot \mid \mathscr{H})]^*(v) = \sup_{u \in \mathscr{X}} [\langle u, v\rangle - J(u) - \delta(u \mid \mathscr{H})]$$

$$= \sup_{u \in \mathscr{H}} \int [u(\cdot) \cdot v(\cdot) - j(\cdot, u)]\, dm$$

$$= - \inf_{u \in \mathscr{H}} \int j'(\cdot, u)\, dm$$

with $j'(x, z) = j(x, z) - z \cdot v(x)$. Since $\operatorname{dom} J' \cap \mathscr{X} = \operatorname{dom} J \cap \mathscr{X}$, the multifunction $\operatorname{ess\,sup}_{u \in \mathscr{H} \cap \operatorname{dom} J'} \{u(\cdot)\}$ is still Γ. Moreover $J'(u_0) \in \mathbb{R}$.

By Theorem 1,

$$[J + \delta(\cdot \mid \mathscr{H})]^*(v) = -\int \inf_{z \in \Gamma(x)} [j(x, z) - z \cdot v(x)]\, m(dx)$$

$$= \int [j(x, \cdot) + \delta(\cdot \mid \Gamma(x))]^*(v(x))\, m(dx).$$

Since $j(x, \cdot)$ and $\delta(\cdot \mid \Gamma(x))$ are l.s.c.

$$j(x, \cdot) + \delta(\cdot \mid \Gamma(x)) = [j^*(x, \cdot)\, \nabla \delta^*(\cdot \mid \Gamma(x))]^*$$

(see, for example, Castaing and Valadier [15, Proposition I.19]).

It is possible to choose classical spaces for $\mathscr{X}$ and $\mathscr{Y}$.

PROPOSITION 3. *Let j be a normal convex integrand. Suppose $\mathscr{H}$ is a vector subspace of $[\mathscr{L}^\infty]^d$ such that $\forall u \in \mathscr{H}$, $\forall \alpha \in \mathscr{C}_c(\Omega)$ (variant, when Ω is an open subset of $\mathbb{R}^N$, $\forall \alpha \in \mathscr{D}(\Omega)$), αu belongs to $\mathscr{H}$. Suppose $\exists u_0 \in \mathscr{H}$ such that $J(u_0) \in \mathbb{R}$. Let $\Gamma = \operatorname{ess\,sup}_{u \in \mathscr{H} \cap \operatorname{dom} J} \{u(\cdot)\}$.*

(1) *Consider the functional on $[L^\infty]^d$, $J + \delta(\cdot \mid \mathscr{H})$. Then its polar on $[L^1]^d$ verifies*

$$[J + \delta(\cdot \mid \mathscr{H})]^*(v) = \int_\Omega k(\cdot, v)\, dm,$$

where $k(x, \cdot) = [j^*(x, \cdot)\, \nabla \delta^*(\cdot \mid \Gamma(x))]^{**}$.

(2) *If $\mathscr{H} \subset [\mathscr{C}_0]^d$ then $\Gamma(x) = \operatorname{cl}\{u(x) \mid u \in \mathscr{H} \cap \operatorname{dom} J\}$ a.e.*

Proof. Remark that $\mathscr{H}$ is PCU-stable because $\sum_{i=0}^n \alpha_i u_i = u_0 + \sum_{i=1}^n \alpha_i (u_i - u_0)$.

(1) This results from Theorem 2 applied with $\mathscr{X} = [\mathscr{L}^\infty]^d$ and $\mathscr{Y} = [\mathscr{L}^1]^d$.

(2) This has been said in 2.2.

Remark. It is possible to give a variant with $\mathscr{Y} = [\mathscr{L}^1_{\mathrm{loc}}]^d$ and for $\mathscr{X}$ the space of $\mathscr{L}^\infty$-functions with compact supports.

3. Description of $\bar{F}$

Let $f\colon \Omega \times \mathbb{R}^d \to]-\infty, \infty]$ be a convex normal integrand. We suppose

(H1) $\exists \varphi_0 \in \mathscr{C}_c$, $\exists a \in L^1$ such that μ-a.e. in x, $\forall z$, $f(x, z) \geqslant \varphi_0(x) \cdot z - a(x)$ (equivalently $\exists \varphi_0 \in \mathscr{C}_c$ such that $I_{f^*}(\varphi_0) < \infty$).

(H2) $\exists u_0 \in [L^1_{\mathrm{loc}}(\Omega, \mu)]^d$ such that $I_f(u_0) < \infty$ (equivalently $\exists u_0 \in [L^1_{\mathrm{loc}}]^d$, $\exists b \in L^1$ such that μ-a.e., $\forall z$, $f^*(x, z) \geqslant z \cdot u_0(x) - b(x)$).

Here, for any $u \in [\mathscr{L}^0(\mu)]^d$, $I_f(u) = \int_\Omega f(\cdot, u)\, d\mu$. Let $F\colon [\mathscr{M}]^d \to]-\infty, \infty]$ be defined as

$$F(\lambda) = \begin{cases} I_f\left(\dfrac{d\lambda}{d\mu}\right) & \text{if } \lambda \ll \mu \\ +\infty & \text{otherwise.} \end{cases}$$

(Note that $d\lambda/d\mu \in L^1_{\mathrm{loc}}$ and, by (H1), $f(\cdot, d\lambda/d\mu) \geqslant \varphi_0(\cdot) \cdot (d\lambda/d\mu)(\cdot) - a$, hence $F(\lambda) > -\infty$.)

Theorem 4. *Let*

$$h(x, z) = \sup\{\varphi(x) \cdot z \mid \varphi \in \mathscr{C}_c \cap \operatorname{dom} I_{f^*}\}$$

$$g(x, \cdot) = [f(x, \cdot)\, \nabla h(x, \cdot)]^{**},$$

$\lambda \in [\mathscr{M}]^d$, $\lambda_a + \lambda_s$ its Lebesgue decomposition with respect to μ, θ any positive measure such that $\lambda_s \ll \theta$. Then the $\sigma(\mathscr{M}, \mathscr{C}_c)$ l.s.c. hull of F is

$$\bar{F}(\lambda) = \int_\Omega g\left(\cdot, \frac{d\lambda_a}{d\mu}\right) d\mu + \int_\Omega h\left(\cdot, \frac{d\lambda_s}{d\theta}\right) d\theta,$$

and the $\sigma(L^1_{\mathrm{loc}}, \mathscr{C}_c)$ l.s.c. hull of I_f is I_g.

With

(H2)′ $\exists u_0 \in [L^1]^d$ such that $I_f(u_0) < \infty$, and $F_1\colon [\mathscr{M}^b]^d \to]-\infty, \infty]$ defined by

$$F_1(\lambda) = \begin{cases} I_f\left(\dfrac{d\lambda}{d\mu}\right) & \text{if } \lambda \ll \mu \\ +\infty & \text{otherwise} \end{cases}$$

we obtain

THEOREM 4'. *The $\sigma(\mathcal{M}^b, \mathscr{C}_0)$ l.s.c. hull $\bar{F}_1$ of F_1 is*

$$\bar{F}_1(\lambda) = \int_\Omega g\left(\cdot, \frac{d\lambda_a}{d\mu}\right) d\mu + \int_\Omega h\left(\cdot, \frac{d\lambda_s}{d\theta}\right) d\theta$$

with g and h defined as in Theorem 4. Moreover the $\sigma(L^1, \mathscr{C}_0)$ l.s.c. hull of I_f is I_g.

Remarks. (1) If (H1) were replaced by

(H1)' $\exists \varphi_0 \in \mathscr{C}_0$ such that $I_{f^*}(\varphi_0) < \infty$

one would have to redefine h and g.

(2) If μ is non-atomic one can start from a measurable integrand f not necessarily convex, and the l.s.c. hulls $\bar{F}$ and $\bar{F}_1$ are the same as those obtained starting from f^{**}; this results from the Liapunov theorem. See Valadier [41] and Bouchitté [5].

(3) As h is sublinear the choice of θ is immaterial as soon as $\lambda_s \ll \theta$. See Goffman and Serrin [22].

Proof of Theorem 4. First, since L^1_{loc} is decomposable and $I_f(u_0) < \infty$, thanks to a famous theorem by Rockafellar, the polar F^* of F in the duality $(\mathcal{M}, \mathscr{C}_c)$ is

$$F^*(\varphi) = \sup_{u \in L^1_{\mathrm{loc}}} [\langle u, \varphi\rangle - I_f(u)] = I_{f^*}(\varphi).$$

Thanks to minoration (H1) and convexity, $\bar{F} = F^{**}$, hence

$$\bar{F}(\lambda) = \sup_{\varphi \in \mathscr{C}_c} [\langle \lambda, \varphi\rangle - I_{f^*}(\varphi)].$$

Consider now a fixed $\lambda \in [\mathcal{M}]^d$. There exists a Borel set A such that

$$\mu(\Omega \setminus A) = |\lambda_s|(A) = 0.$$

Let $m = \mu + |\lambda_s|$. Then $\lambda \ll m$ and

$$\frac{d\lambda}{dm}(x) = \begin{cases} \dfrac{d\lambda_a}{d\mu}(x) & \text{if } x \in A \\[2ex] \dfrac{d\lambda_s}{d|\lambda_s|}(x) & \text{if } x \in \Omega \setminus A. \end{cases}$$

Thus $d\lambda/dm \in L^1_{\mathrm{loc}}(m)$. Setting

$$j(x, z) = \begin{cases} f^*(x, z) & \text{if } x \in A \\ 0 & \text{if } x \in \Omega \setminus A \end{cases}$$

one has

$$\langle \lambda, \varphi \rangle - \int_\Omega f^*(\cdot, \varphi)\, d\mu = \int_\Omega \frac{d\lambda}{dm} \cdot \varphi\, dm - \int_\Omega j(\cdot, \varphi)\, dm.$$

Now we can apply Theorem 2 with $\mathscr{X} = \mathscr{H} = \mathscr{C}_c$ and $\mathscr{Y} = [L^1_{\mathrm{loc}}]^d$. Indeed, by (H1) and (H2), $J(\varphi_0) \in \mathbb{R}$ (remark $J = I_{f^*}$). Thus

$$\begin{aligned} \bar{F}(\lambda) &= \sup_{\varphi \in \mathscr{C}_c} \int_\Omega \left[\frac{d\lambda}{dm} \cdot \varphi - j(\cdot, \varphi) \right] dm \\ &= [J + \delta(\cdot \mid \mathscr{C}_c)]^* \left(\frac{d\lambda}{dm} \right) \\ &= \int_\Omega k\left(\cdot, \frac{d\lambda}{dm} \right) dm \end{aligned}$$

with $k(x, \cdot) = [j^*(x, \cdot)\, \nabla \delta^*(\cdot \mid \Gamma(x))]^{**}$.

Since $\Gamma(x) = \mathrm{cl}\{\varphi(x) \mid \varphi \in \mathscr{C}_c \cap \mathrm{dom}\, I_{f^*}\}$ (in fact Γ is defined up to equality m-a.e. but this expression is independent of m),

$$\delta^*(z \mid \Gamma(x)) = h(x, z).$$

Since

$$j^*(x, z) = \begin{cases} f(x, z) & \text{if } x \in A \\ \delta(z \mid \{0\}) & \text{if } x \in \Omega \setminus A, \end{cases}$$

$$k(x, \cdot) = \begin{cases} g(x, \cdot) & \text{if } x \in A \\ h(x, \cdot) & \text{if } x \in \Omega \setminus A. \end{cases}$$

Finally,

$$\begin{aligned} \bar{F}(\lambda) &= \int_A g\left(\cdot, \frac{d\lambda}{dm} \right) dm + \int_{\Omega \setminus A} h\left(\cdot, \frac{d\lambda}{dm} \right) dm \\ &= \int_\Omega g\left(\cdot, \frac{d\lambda_a}{d\mu} \right) d\mu + \int_\Omega h\left(\cdot, \frac{d\lambda_s}{d|\lambda_s|} \right) d|\lambda_s|. \end{aligned}$$

Proof of Theorem 4′. We still have, for $\varphi \in \mathscr{C}_0$, $F_1^*(\varphi) = I_{f^*}(\varphi)$ and

$$\bar{F}_1(\lambda) = \sup_{\varphi \in \mathscr{C}_0} [\langle \lambda, \varphi \rangle - I_{f^*}(\varphi)].$$

For a given $\lambda \in [\mathscr{M}^b]^d$, let A, m, and j be as in the proof of Theorem 4. Here $d\lambda/dm \in L^1(m)$.

We apply Theorem 2 with $\mathscr{Y} = \mathscr{L}^1$, $\mathscr{H} = \mathscr{C}_0$, and $\mathscr{X} = \mathscr{C}_0$ (or $\mathscr{L}^\infty$) (we may also apply Proposition 3). We get $\bar{F}_1(\lambda) = \int_\Omega k(\cdot, d\lambda/dm)\, dm$. Here the only difference is that

$$\Gamma(x) = \mathrm{cl}\{\varphi(x) \mid \varphi \in \mathscr{C}_0 \cap \mathrm{dom}\, I_{f^*}\}.$$

A priori, using $\mathscr{C}_0$ in place of $\mathscr{C}_c$ should give a greater function h. But let $\varphi \in \mathscr{C}_0 \cap \operatorname{dom} I_{f^*}$. There exists $\beta_n \in \mathscr{C}_c$, $\beta_n \geqslant 0$, $\beta_n \nearrow \chi_\Omega$, then $\psi_n = \beta_n \varphi + (1-\beta_n)\varphi_0$ (where φ_0 satisfies (H1)) belongs to $\mathscr{C}_c \cap \operatorname{dom} I_{f^*}$. Hence, for any x, $\psi_n(x) \to \varphi(x)$ and the function $\delta^*(z \mid \Gamma(x))$ is the same h as in Theorem 4.

THEOREM 5. *Under* (H1), *with h and g defined in Theorem* 4 *one has, for any bounded positive Borel function* ψ, $\forall \lambda \in [\mathscr{M}]^d$ (*or* $[\mathscr{M}^b]^d$),

$$\int_\Omega \psi g\left(\cdot, \frac{d\lambda_a}{d\mu}\right) d\mu + \int_\Omega \psi h\left(\cdot, \frac{d\lambda_s}{d\theta}\right) d\theta$$
$$= \sup\left\{\int_\Omega \psi\varphi \cdot d\lambda - \int_\Omega \psi f^*(\cdot, \varphi)\, d\mu \mid \varphi \in \operatorname{dom} I_{f^*} \cap \mathscr{C}_c \,(\text{resp. } \mathscr{C}_0)\right\}.$$

Moreover, if ψ *is continuous, the supremum can be taken on the whole space* $\mathscr{C}_c$ *or* $\mathscr{C}_0$.

Comment. Consider the measure $G(\lambda)$ with values in $]-\infty, \infty]$ defined by, $\forall B$ Borel set,

$$[G(\lambda)](B) = \int_B g\left(\cdot, \frac{d\lambda_a}{d\mu}\right) d\mu + \int_B h\left(\cdot, \frac{d\lambda_s}{d\theta}\right) d\theta.$$

The first member in the statement is $\int \psi\, dG(\lambda)$. When $G(\lambda)$ is a Radon measure (equivalently takes finite values on compact sets) it is characterized by the knowledge of the values $\int \psi\, dG(\lambda)$, ψ continuous. The formula has been given by Temam [36, 37], Demengel and Temam [19], Hadhri [24], and Valadier [42, 45]. The continuity of ψ is necessary to take the supremum on $\mathscr{C}_0$.

Proof. (a) Consider for a fixed λ, $\lambda' = \psi\lambda$ and $m = \psi\mu + \psi\, |\lambda_s|$. Then $\lambda' \ll m$ and, if A is a Borel set such that $\mu(\Omega \setminus A) = |\lambda_s|\,(A) = 0$, one has

$$\frac{d\lambda'}{dm}(x) = \begin{cases} \dfrac{d\lambda_a}{d\mu}(x) & \text{if } x \in A \\[2ex] \dfrac{d\lambda_s}{d|\lambda_s|}(x) & \text{if } x \in \Omega \setminus A, \end{cases}$$

and, since ψ is bounded, $d\lambda'/dm \in L^1_{\mathrm{loc}}(m)$ (resp. $L^1(m)$). Set also

$$j(x, z) = \begin{cases} f^*(x, z) & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_\Omega \psi f^*(\cdot, \varphi)\, d\mu = \int_\Omega j(\cdot, \varphi)\, dm$, which will be denoted by $J(\varphi)$. Thus the right-hand side of the formula of Theorem 5 equals

$$\sup_{\varphi \in \mathscr{H}} \left[\int_\Omega \frac{d\lambda'}{dm} \cdot \varphi\, dm - J(\varphi) \right],$$

where $\mathscr{H} = \mathscr{C}_c \cap \operatorname{dom} I_{f^*}$ (or $\mathscr{C}_0 \cap \operatorname{dom} I_{f^*}$). Since ψ is bounded one has $\mathscr{H} \subset \operatorname{dom} J$, hence $\mathscr{H} \cap \operatorname{dom} J = \mathscr{H}$ and $\varphi_0 \in \mathscr{H} \cap \operatorname{dom} J$. Moreover $\mathscr{H}$ is PCU-stable. We can apply Theorem 2 with $\mathscr{Y} = [L^1_{\mathrm{loc}}]^d$ (or $[L^1]^d$), $\mathscr{X} = \mathscr{C}_c$ or $\mathscr{C}_0$. Thus

$$\sup_{\varphi \in \mathscr{H}} \left[\int_\Omega \frac{d\lambda'}{dm} \cdot \varphi\, dm - J(\varphi) \right] = \int_\Omega k\left(\cdot, \frac{d\lambda'}{dm}\right) dm$$

with $k(x, \cdot) = [j^*(x, \cdot) \nabla \delta^*(\cdot \mid \Gamma(x))]^{**}$ and $\Gamma = \operatorname{ess\,sup}_{u \in \mathscr{H}} \{u(\cdot)\}$. Again $\Gamma(x) = \operatorname{cl}\{\varphi(x) \mid \varphi \in \mathscr{H}\}$ and one can end the proof as in Theorem 4.

(b) Suppose that the supremum is on the whole space $\mathscr{C}_c$ (or $\mathscr{C}_0$) and that ψ is continuous. Proceeding as in (a), but with $\mathscr{H} = \mathscr{C}_c$ or $\mathscr{C}_0$, the difficulty is to check that, denoting $\Gamma = \operatorname{ess\,sup}_{u \in \mathscr{H} \cap \operatorname{dom} J} \{u(\cdot)\} = \operatorname{cl}\{\varphi(x) \mid \varphi \in \mathscr{C}_c \text{ or } \mathscr{C}_0 \text{ and } \int \psi f^*(\cdot, \varphi)\, d\mu < \infty\}$, one has $\psi(x)\, \delta^*(z \mid \Gamma(x)) = \psi(x)\, h(x, z)$. We may suppose $\psi(x) > 0$. There exists a compact neighborhood K of x such that $\inf_K \psi = \delta > 0$. The remainder is routine.

4. Some Properties of h and g

Throughout this section the duality pair is either $(\mathscr{M}, \mathscr{C}_c)$ or $(\mathscr{M}^b, \mathscr{C}_0)$. Hypotheses (H1) and (H2) are assumed, so

$$\begin{aligned} h(x, z) &= \sup\{z \cdot \varphi(x) \mid \varphi \in \mathscr{C}_c \cap \operatorname{dom} I_{f^*}\} \\ &= \sup\{z \cdot \varphi(x) \mid \varphi \in \mathscr{C}_0 \cap \operatorname{dom} I_{f^*}\} \end{aligned}$$

(see the proof of Theorem 4′).

We will sometimes use in place of (H1) the stronger

(H1)″ $\exists \lambda_0 \in]0, \infty[$, $\exists a \in L^1$ such that a.e., $\forall z$, $f(x, z) \geqslant \lambda_0 |z| - a(x)$.

(Remark that (H1)″ $\Rightarrow$ (H1) with $\varphi_0 = 0$.)

Recall that the recession or asymptotic function $f_\infty(x, \cdot)$ of the convex l.s.c. proper function $f(x, \cdot)$ satisfies

$$\forall z_0 \in \operatorname{dom} f(x, \cdot), \qquad f_\infty(x, z) = \lim_{r \to \infty} \frac{f(x, z_0 + rz)}{r}$$

and $f_\infty(x, z) = \delta^*(z \mid \operatorname{dom} f^*(x, \cdot))$ (Rockafellar [34, Theorem 8.5, p. 66, and Theorem 13.3, p. 116]).

PROPOSITION 6. *Let*

$$E(x)=\Big\{z\in\mathbb{R}^d \mid \exists V \text{ open}, V\ni x, \exists\varphi \text{ continuous on } V \text{ such that}$$
$$\varphi(x)=z \text{ and } \int_V f^*(\cdot,\varphi)\,d\mu<\infty\Big\}$$
$$E_1(x)=\Big\{z\in\mathbb{R}^d \mid \exists V \text{ open}, V\ni x \text{ such that } \int_V f^*(\cdot,z)\,d\mu<\infty\Big\}.$$

Then

(1) $\forall(x,z)$, $h(x,z)=\delta^*(z\mid E(x))$,

(2) *if* $x\in\Omega\setminus\operatorname{supp}\mu$, $E(x)=E_1(x)=\mathbb{R}^d$ *and* $h(x,\cdot)=\delta(\cdot\mid\{0\})$,

(3) *under* (H1)″, $\forall x$, $E_1(x)\subset E(x)\subset\overline{E_1(x)}$.

EXAMPLE. Without (H1)″, (3) may be false. Let $\Omega=]-\pi,\pi[$, μ the Lebesgue measure, $d=2$,

$$D_x=\{\lambda(\cos x,\sin x)\mid\lambda\in\mathbb{R}\},\qquad f(x,\cdot)=\delta(\cdot\mid D_x).$$

Then $f^*(x,\cdot)=\delta(\cdot\mid D_x^\perp)$ and $E_1(0)=\{(0,0)\}$, $E(0)=\{0\}\times\mathbb{R}$.

Proof. (1) This is proved in Valadier [42, Proposition 7, p. 22] and is known since Olech [28].

(2) If $x\notin\operatorname{supp}\mu$, $V=\Omega\setminus\operatorname{supp}\mu$ is an open neighborhood of x and $\int_V f^*(x,z)\,\mu(dx)=0$ for any z. So $E(x)=E_1(x)=\mathbb{R}^d$ and $h(x,\cdot)=\delta(\cdot\mid\{0\})$.

(3) The inclusion $E_1(x)\subset E(x)$ is obvious. Let $z\in E(x)$. Let V and φ corresponding to z. We may, changing V in a smaller neighborhood, suppose φ bounded. For any $\varepsilon>0$, let $V_\varepsilon=\{y\in V\mid |\varphi(y)-z|<\varepsilon\}$ and W_ε a compact neighborhood of x contained in V_ε. There exists $\theta_\varepsilon\colon V\to[0,1]$ continuous such that $\theta_\varepsilon(x)=1$ on W_ε and $\operatorname{supp}\theta_\varepsilon\subset V_\varepsilon$. Define

$$\varphi_\varepsilon=\theta_\varepsilon z+(1-\theta_\varepsilon)\varphi.$$

Then $\varphi_\varepsilon(x)=z$ and $\sup_{y\in V}|\varphi_\varepsilon(y)-\varphi(y)|\leqslant\varepsilon$.

By (H1)″ the functional I on $L^\infty(V,\mu)$, defined by $I(v)=\int_V f^*(\cdot,v)\,d\mu$, is bounded on a (norm) neighborhood of 0, so it is continuous on int(dom I), which contains $[0,\varphi[$. Hence if $r\in[0,1[$

$$\lim_{\varepsilon\to0}\int_V f^*(\cdot,r\varphi_\varepsilon)\,d\mu=\int_V f^*(\cdot,r\varphi)\,d\mu.$$

So for ε sufficiently small, $f^*(\cdot,r\varphi_\varepsilon)\in L^1$, hence $\int_{\operatorname{int}(W_\varepsilon)}f^*(\cdot,rz)\,d\mu<\infty$ and $rz\in E_1(x)$. Finally, $z\in\overline{E_1(x)}$.

PROPOSITION 7. (1) *One has μ-a.e.*

$$g(x, \cdot) \leqslant f(x, \cdot)$$
$$h(x, \cdot) = g_\infty(x, \cdot) \leqslant f_\infty(x, \cdot),$$

(2) *I_f is $\sigma(L^1, \mathscr{C}_0)$ (resp. $\sigma(L^1_{\mathrm{loc}}, \mathscr{C}_c)$) l.s.c. iff μ-a.e. $h(x, \cdot) = f_\infty(x, \cdot)$ (equivalently $h(x, \cdot) \geqslant f_\infty(x, \cdot)$).*

(3) *If Ω' is an open subset of Ω and if $x \mapsto \operatorname{epi} f^*(x, \cdot)$ is l.s.c. on Ω', then $\forall x \in \Omega'$, $f_\infty(x, \cdot) \leqslant h(x, \cdot)$. As a consequence if $\mu(\Omega \setminus \Omega') = 0$, I_f is l.s.c.*

EXAMPLE. Let $\Omega = \mathbb{R}$, μ the Lebesgue measure, $d = 1$, K a compact subset of $\mathbb{R}$ with $\operatorname{int}(K) = \varnothing$ and $\mu(K) > 0$, and

$$f(x, z) = \begin{cases} |z| & \text{if } \ x \in K \\ 0 & \text{otherwise,} \end{cases}$$

Then $I_{f^*}(\varphi) = \delta(\varphi \mid \{0\})$, so $\bar{I}_f = 0 \neq I_f$.

Proof. Parts (1) and (2) have been proved in Valadier [41, 42]. For a somewhat more direct proof see Bouchitté [5, 7].

(3) Let $x_0 \in \Omega'$. If $z_0 \in \operatorname{dom} f^*(x_0, \cdot)$, by the Michael theorem [26] there exists a continuous selector (φ, ψ) of $x \mapsto \operatorname{epi} f^*(x, \cdot)$ such that $(\varphi(x_0), \psi(x_0)) = (z_0, f^*(x_0, z_0))$. Let K be a compact neighborhood of x contained in Ω'. Then

$$\int_{\operatorname{int} K} f^*(\cdot, \varphi)\, d\mu \leqslant \int_K \psi\, d\mu < \infty.$$

Hence $z_0 \in E(x_0)$. Therefore $f_\infty(x_0, \cdot) \leqslant h(x_0, \cdot)$. The last assertion follows from (2).

THEOREM 8. (1) *Under one of the hypotheses*

(H3) *$\forall z$, $f^*(\cdot, z)$ is u.c.s. on Ω,*

(H4) *f is l.s.c. on $\Omega \times \mathbb{R}^d$ and $f(\cdot, 0)$ is locally bounded,*

one has $\forall x \in \Omega$, $f_\infty(x, \cdot) \leqslant h(x, \cdot)$ (hence I_f is l.s.c.).

(2) *Under* (H3) *or* (H4) *and moreover*

(H5) *$\forall z$, $f_\infty(\cdot, z)$ is u.c.s.,*

one has

$$h(x, z) = \begin{cases} f_\infty(x, z) & \textit{if } \ x \in \operatorname{supp} \mu \\ \delta(z \mid \{0\}) & \textit{if } \ x \in \Omega \setminus \operatorname{supp} \mu. \end{cases}$$

Remarks and Comments. (1) For I_f being $\sigma(L^1, \mathscr{C}_0)$ l.s.c. it is sufficient to have (H3) or (H4) on an open set Ω' such that $\mu(\Omega\backslash\Omega')=0$, for example (as said in [24]) if

$$f(x, z)=\begin{cases} f_1(z) & \text{if } x\in\Omega_1 \\ f_2(z) & \text{if } x\in\Omega_2, \end{cases}$$

where Ω_1 and Ω_2 are disjoint open subsets such that

$$\mu(\Omega\backslash(\Omega_1\cup\Omega_2))=0.$$

(2) If $f(x, \cdot)$ does not depend on x, (H3) and (H5) are obviously satisfied. If moreover $\operatorname{supp}\mu=\Omega$, the formula of Theorem 5 becomes, $\forall\psi$ Borel bounded positive function,

$$\sup\left\{\int \psi\varphi\cdot d\lambda-\int \psi f^*(\cdot, \varphi)\, d\mu \mid \varphi\in\mathscr{C}_0\cap \operatorname{dom} I_{f^*}\right\}$$
$$=\int \psi f\left(\frac{d\lambda_a}{d\mu}\right) d\mu+\int \psi f_\infty\left(\frac{d\lambda_s}{d\,|\lambda_s|}\right) d|\lambda_s|.$$

This is the starting formula (for ψ continuous) of Temam [37] and Demengel and Temam [19].

(3) In case f is l.s.c. on whole the space $\Omega\times\mathbb{R}^d$, hypothesis (H4), Giaquinta, Modica, and Soucek [21], and Dal Maso [16] obtain, thanks to a result of Reschetniak [30] about sublinear functions of measures, that the functional

$$G\left|\begin{array}{l} \lambda\mapsto\int f\left(\cdot, \dfrac{d\lambda_a}{d\mu}\right) d\mu+\int f_\infty\left(\cdot, \dfrac{d\lambda_s}{d\,|\lambda_s|}\right) d\,|\lambda_s| \\ {[\mathscr{M}^{\mathrm{b}}]^d}\to\,]-\infty, \infty] \end{array}\right.$$

is $\sigma(\mathscr{M}^{\mathrm{b}}, \mathscr{C}_0)$ l.s.c. As a consequence I_f is $\sigma(L^1, \mathscr{C}_0)$ l.s.c., hence $g(x, \cdot)=f(x, \cdot)$ μ-a.e. But it can happen that $G\neq\bar{F}_1$. Indeed consider the following example suggested in [16, 4.4, p. 414].

Example. Let $\Omega=\mathbb{R}$, $\mu=dx$, $d=1$.

$$f(x, z)=\begin{cases} |z| & \text{if } |z|\,|x|^{1/2}\leqslant 1 \\ 2\,|z|-|x|^{-1/2} & \text{if } |z|\,|x|^{1/2}\geqslant 1. \end{cases}$$

Then f is continuous on $\Omega\times\mathbb{R}$, (H4) is satisfied, but (H5) does not hold.

One can check that $\forall x$, $h(x, z)=2\,|z|$ and $f_\infty(0, z)=|z|$. Thus $G(\delta_0)=1$ and $\bar{F}_1(\delta_0)=2$.

(4) In [1, 16] (where the more difficult problem of a functional depending on the gradient is studied), a sufficient condition ensuring $\bar{F}=G$ is set. This condition implies that f is continuous in x and has linear growth in z; more precisely,

$$\forall \varepsilon>0, \exists \delta>0, |x_1-x_2|<\delta \Rightarrow \forall z, |f(x_1, z)-f(x_2, x)| \leqslant \varepsilon(1+|z|).$$

This hypothesis is far more stringent that the one of (2) of Theorem 8. Indeed (H3) or (H4) supplemented with (H5) does not imply the continuity of $f(\cdot, z)$ but only the continuity of $f_\infty(\cdot, z)$ (remark that f being l.s.c., $f_\infty(\cdot, z)$ is l.s.c. too).

Proof of Theorem 8. (1) By Proposition 7 it is sufficient to prove that the multifunction $Q: x \mapsto \operatorname{epi} f^*(x, \cdot)$ is l.s.c.

(a) Under (H3). Let U be an open subset of $\mathbb{R}^d \times \mathbb{R}$. Then

$$\begin{aligned}\{x \in \Omega \mid Q(x) \cap U \neq \varnothing\} &= \{x \mid \exists (z, r) \in U \text{ such that } f^*(x, z) \leqslant r\} \\ &= \bigcup_{(z,r)\in U} \{x \mid f^*(x, z)<r\}\end{aligned}$$

(the change from $\leqslant$ to $<$ is easy) which is open.

(b) Under (H4). Recall that, for $(z, t) \in \mathbb{R}^d \times \mathbb{R}$,

$$\tilde{f}(x, z, t)=\delta^*((z, t) \mid Q(x))=\begin{cases}-tf(x, z/-t) & \text{if } t<0 \\ f_\infty(x, z) & \text{if } t=0 \\ +\infty & \text{if } t>0.\end{cases}$$

From Lemma A2 it is sufficient to prove that $\tilde{f}$ is l.s.c. This is a consequence of Dal Maso [16].

(2) Under (H5)

$$\begin{aligned}V &= \{x \in \Omega \mid \exists z \in \mathbb{R}^d \text{ such that } f_\infty(x, z)<h(x, z)\} \\ &= \bigcup_z \{x \mid f_\infty(x, z)<h(x, z)\}\end{aligned}$$

is open (h defined in Theorem 4 is l.s.c.). From Proposition 7(1) a.e. $f_\infty(x, \cdot) \geqslant h(x, \cdot)$, so V is negligible, hence $V \cap \operatorname{supp} \mu=\varnothing$.

If $x \in \operatorname{supp} \mu$, $x \notin V$ and then using (1), $f_\infty(x, \cdot)=h(x, \cdot)$. If $x \notin \operatorname{supp} \mu$, the result follows from Proposition 6(2).

5. Examples

The proofs of the results stated in Examples 1 to 4 are left to the reader. For details see Bouchitté [5, 7].

EXAMPLE 1. Let Ω be an open subset of $\mathbb{R}^N$ and Σ be an $N-1$-dimensional hypersurface contained in Ω. Let μ denote the measure $dx + H^{N-1}(\Sigma \cap \cdot)$, where H^{N-1} is the $N-1$-dimensional Hausdorff measure (thus $H^{N-1}(\Sigma \cap \cdot)$ is the area measure of Σ). We suppose that Σ is regular, that is, μ is finite on compact sets and $\Omega \backslash \Sigma$ is dense in Ω.

Let

$$f(x, z) = \begin{cases} |z| & \text{if } x \in \Omega \backslash \Sigma \\ \frac{1}{2}|z|^2 & \text{if } x \in \Sigma, \end{cases}$$

Let

$$\beta(z) = \begin{cases} \frac{1}{2}|z|^2 & \text{if } |z| \leqslant 1 \\ |z| - \frac{1}{2} & \text{if } |z| \geqslant 1. \end{cases}$$

Remark that $\beta = \frac{1}{2}|\cdot|^2 \nabla |\cdot|$. Then, if $\lambda_a + \lambda_s$ is the μ-decomposition of λ,

$$\bar{F}(\lambda) = \int_{\Omega \backslash \Sigma} d\,|\lambda_a| + \int_\Sigma \beta\left(\frac{d\lambda_a}{d\mu}(x)\right) dH^{N-1}(x) + |\lambda_s|\,(\Omega).$$

EXAMPLE 2. Let Ω be an open subset of $\mathbb{R}^N$, μ the Lebesgue measure, $a: \Omega \to [0, \infty[$ a locally integrable function, and $f(x, z) = a(x)\,|z|$. Then, if

$$\tilde{a}(x) = \overline{\lim_{\delta \to 0_+}}\,[\mu(B(x, \delta))]^{-1} \int_{B(x,\delta)} a(y)\, dy$$

and $\hat{a}$ is the l.s.c. hull of $\tilde{a}$,

$$\bar{F}(\lambda) = \int_\Omega \hat{a}\, d\,|\lambda|.$$

Remark. As soon as $f^*(x, \cdot)$ is an indicator, $\bar{F}(\lambda) = \delta^*(\lambda \,|\, \Phi)$, where Φ is the set of $\mathscr{C}_c$-selectors of a l.s.c. multifunction Γ. For the existence of Γ see Valadier [44]. In Examples 2 and 4 below, it is possible to "calculate" Γ.

EXAMPLE 3. Let Ω be an open subset of $\mathbb{R}^N$, μ the Lebesgue measure, $a: \Omega \to [0, \infty[$ a measurable function, and $f(x, z) = \frac{1}{2}a(x)\,|z|^2$. Then, if Ω' is the greatest open subset on which $1/a$ is locally integrable (with the convention $1/0 = +\infty$), one has

$$\bar{F}(\lambda) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega'} a(x) \left|\frac{d\lambda_a}{dx}\right|^2 dx & \text{if } |\lambda_s|\,(\Omega') = 0 \\ +\infty & \text{if } |\lambda_s|\,(\Omega') > 0. \end{cases}$$

EXAMPLE 4. Let Ω be an open subset of $\mathbb{R}^N$, μ the Lebesgue measure, and $A:\Omega\to\mathbb{R}^d$ a measurable function such that $|A(x)|=1$ a.e. Let $f(x,z)=[A(x)\cdot z]^+$. If $\tilde{A}$ is defined as in Example 2 but coordinate-wise, that is,

$$\forall i\in\{1,...,d\},\qquad \tilde{A}_i(x)=\overline{\lim_{\delta\to 0_+}}\,[\mu(B(x,\delta)]^{-1}\int_{B(x,\delta)} A_i(y)\,dy,$$

and if Ω' is the greatest open subset on which $\tilde{A}$ is continuous, then $\bar{F}(\lambda)=\int_{\Omega'}[(d\lambda/d\,|\lambda|)(x)\cdot\tilde{A}(x)]^+\,|\lambda|\,(dx)$.

Remarks. (1) On Ω', $|\tilde{A}(x)|=1$ because $\tilde{A}(x)=A(x)$ a.e.

(2) The existence of Ω' and $\tilde{A}$ can be proved without the $\sim$ operation. Indeed Ω' is the greatest open subset on which A is a.e. equal to a (unique) continuous function. The existence of Ω' follows from the Lindelöf property. One can treat also $f(x,z)=|A(x)\cdot z|$: in this case it is necessary to topologize the unit sphere identifying opposite points.

EXAMPLE 5 (which describes the usual case in plasticity theory). Let Ω be an open bounded subset of $\mathbb{R}^N$ and E the space of symmetric tensors of order 2 ($\dim E=N(N+1)/2$). Recall that E has a euclidean stucture for which the orthogonal of $\mathbb{R}I$ (the one-dimensional subspace of diagonal tensors) is the space E^D of rensors whose traces vanish.

Let B be a closed convex-valued l.s.c. multifunction such that $\forall x$, $0\in B(x)$. We suppose moreover that $\forall\varphi\in\mathscr{C}_0$, $\varphi(x)\in B(x)$ a.e. $\Rightarrow\varphi(x)\in B(x)$ everywhere (remark that this avoids $\Omega=]-1,1[$, $B(x)=[0,1]$ if $x\neq 0$, $B(0)=\{0\}$). There exist many l.s.c. discontinuous multifunctions which satisfy this hypothesis. In practice $B(x)=B^D(x)+\mathbb{R}I$, where $B^D(x)$ is a convex compact subset of E^D containing 0.

Let $\gamma: R\to\mathbb{R}$ be continuous and ψ be a convex normal integrand on $\Omega\times E$ such that $0\leqslant\psi(x,\cdot)\leqslant\gamma(\cdot)$. The useful integrand in plasticity is

$$f(x,\cdot)=[\psi(x,\cdot)+\delta(\cdot\,|\,B(x))]^*.$$

Let $h(x,\cdot)=\delta^*(\cdot\,|\,B(x))$. Then

$$\forall\lambda\in\mathscr{M}^b(\Omega;E),\qquad \bar{F}_1(\lambda)=\int_\Omega f\left(x,\frac{d\lambda_a}{dx}\right)dx+\int_\Omega h(x,d\lambda_s).$$

Remark. When $B(x)=B^D(x)+\mathbb{R}I$,

$$h(x,z)=\begin{cases}\delta^*(z\,|\,B^D(x)) & \text{if}\quad z\in E^D\\ +\infty & \text{otherwise.}\end{cases}$$

Hence $\operatorname{dom} h(x, \cdot) = E^D$ and, if $u \in BD(\Omega)$ and $Du = \frac{1}{2}(u_{ij} + u_{ji})$ satisfies $\bar{F}_1(Du) < \infty$, the singular part of the measure $\operatorname{div} u = \operatorname{tr}(Du)$ vanishes.

Proof. Since $f^*(x, \cdot) = \psi(x, \cdot) + \delta(\cdot \mid B(x))$, one has for $\varphi \in \mathscr{C}_0$

$$I_{f^*}(\varphi) < \infty \Leftrightarrow \varphi(x) \in B(x) \text{ a.e.} \Leftrightarrow \forall x,\ \varphi(x) \in B(x).$$

Thanks to the Michael theorem [26], for any $z \in B(x)$, there exists $\varphi \in \mathscr{C}_0$ with $\varphi(x) = z$ and $\forall y$, $\varphi(y) \in B(y)$. Thus $h(x, \cdot) = \delta^*(\cdot \mid B(x))$ and, since $g^* = f^* + h^*$, $\forall x$, $g(x, \cdot) = f(x, \cdot)$.

Appendix 1

LEMMA A1. *Let $g: \mathbb{R}^d \to]-\infty, \infty]$ be convex l.s.c., $D \subset \operatorname{dom} g$. Suppose $\bar{D}$ convex. Then the l.s.c. hull*

$$\overline{g + \delta(\cdot \mid D)} \quad \textit{of} \quad g + \delta(\cdot \mid D) \textit{ is equal to } g + \delta(\cdot \mid \bar{D}).$$

In particular $\inf_D g = \inf_{\bar{D}} g$.

Proof. Obviously $g + \delta(\cdot \mid \bar{D}) \leqslant \overline{g + \delta(\cdot \mid D)}$. Without loss of generality we may suppose that the affine subspace generated by D is $\mathbb{R}^d$ itself. So $\operatorname{int}(\operatorname{co} D) \neq \varnothing$. Let $x_0 \in \operatorname{int}(\operatorname{co} D)$, one has $x_0 \in \operatorname{int}(\operatorname{dom} g) \cap \bar{D}$.

(a) As g is continuous at x_0,

$$\begin{aligned}\overline{g + \delta(\cdot \mid D)}(x_0) &= \varliminf_{\substack{x \to x_0 \\ x \in D}} g(x) = g(x_0) \\ &= [g + \delta(\cdot \mid \bar{D})](x_0).\end{aligned}$$

(b) Let $x_1 \in \bar{D}$, $x_1 \neq x_0$, and prove $\overline{g + \delta(\cdot \mid D)}(x_1) \leqslant g(x_1)$. Let $x_\lambda = \lambda x_1 + (1 - \lambda) x_0$. When λ runs through $[0, 1[$, x_λ belongs to $\operatorname{int}(\operatorname{dom} g) \cap \bar{D}$, hence, by (a),

$$\overline{g + \delta(\cdot \mid D)}(x_\lambda) = g(x_\lambda).$$

On a one-dimensional interval like $[x_0, x_1]$, a convex function is u.s.c., so when it is l.s.c. it is continuous. Hence

$$\begin{aligned}\overline{g + \delta(\cdot \mid D)}(x_1) &= \varliminf_{\lambda \to 1^-} \overline{g + \delta(\cdot \mid D)}(x_\lambda) \\ &= \varliminf_{\lambda \to 1^-} g(x_\lambda) = g(x_1).\end{aligned}$$

The last formula is easy.

APPENDIX 2

LEMMA A2. *Let Q be a multifunction on a topological space Ω to the convex subsets of $\mathbb{R}^d$. Then Q is l.s.c. on Ω iff $(x, z') \mapsto \delta^*(z' \mid Q(x))$ is l.s.c. on $\Omega \times \mathbb{R}^d$.*

Proof. Let $\varphi(x, z') = \delta^*(z' \mid Q(x))$.

(1) Suppose Q is l.s.c. Let $(x_0, z'_0) \in \Omega \times \mathbb{R}^d$ and $r \in \mathbb{R}$, $r < \varphi(x_0, z'_0)$. The set $W = \{(z, z') \in (\mathbb{R}^d)^2 \mid z \cdot z' > r\}$ is open. There exists $z_0 \in Q(x_0)$ such that $(z_0, z'_0) \in W$. There exists U an open neighborhood of z_0 and U' an open neighborhood of z'_0 such that $U \times U'$ is contained in W. As Q is l.s.c. and $z_0 \in Q(x_0) \cap U$, there exists a neighborhood V of x_0 such that $\forall x \in V$, $Q(x) \cap U \neq \varnothing$. Hence

$$(x, z') \in V \times U' \Rightarrow \varphi(x, z') \geqslant z_x \cdot z' \qquad (\text{where } z_x \in Q(x) \cap U)$$
$$> r.$$

Thus φ is l.s.c. at (x_0, z'_0).

(2) Suppose φ is l.s.c. and Q is not l.s.c. at x_0. Let U be an open subset of $\mathbb{R}^d$ such that $Q(x_0) \cap U \neq \varnothing$. We may suppose U convex and $0 \in Q(x_0) \cap U$. Thus $\varphi(x_0, \cdot) \geqslant 0$. There exists a generalized sequence (y_α) such that $y_\alpha \to x_0$ and $Q(y_\alpha) \cap U = \varnothing$. By the Hahn–Banach theorem $\exists z'_\alpha$ and $r \in \mathbb{R}$ such that

$$\varphi(y_\alpha, z'_\alpha) \leqslant r \leqslant \inf_{z \in U} z \cdot z'_\alpha.$$

We may suppose $r = -1$. Thus $z'_\alpha \in \{z' \mid \forall z \in U,\ z \cdot z' \geqslant -1\}$, which is an equicontinuous set (here a bounded subset of $\mathbb{R}^d$). Let z' be a cluster point of the generalized sequence (z'_α). By the lower semi-continuity of φ, $\varphi(x_0, z') \leqslant -1$, which is a contradiction.

Remark. This improves in one direction II.21 of Castaing and Valadier [15].

REFERENCES

1. G. ANZELOTTI, "The Euler Equation for Functionals with Linear Growth," University of Trento, 1983.
2. H. ATTOUCH, G. BOUCHITTÉ, AND M. MABROUK, Formulations variationnelles pour des équations elliptiques semi-linéaires avec second membre mesure, *C. R. Acad. Sci. Paris* **306** (1988), 161–164.
3. H. ATTOUCH AND C. PICARD, Problèmes variationnels et théorie du potentiel non linéaire, *Ann. Fac. Sci. Toulouse Math.* **1** (1979), 89–136.
4. G. BOUCHITTÉ, "Convergence et relaxation de fonctionnelles du calcul des variations à croissance linéaire. Application à l'homogénéisation en plasticité," Publications AVMAC, n° 10, Université de Perpignan, 1985; *Ann. Fac. Sci. Toulouse Math., Sér. 5-VIII* (1986–87), 7–36.

5. G. BOUCHITTÉ, "Représentation intégrale de fonctionnelles convexes sur un espace de mesures, I," Publications AVAMAC, n° 2, Université de Perpignan, 1986.
6. G. BOUCHITTÉ, "Preprésentation intégrale de fonctionnelles convexes sur un espace de mesures, II," Publications AVAMAC, Vol. 2, exposé n° 3, Université de Perpignan, 1986.
7. G. BOUCHITTÉ, "Calcul des variations en cadre non réflexif. Représentation et relaxation de fonctionnelles intégrales sur un espace de mesures. Applications en plasticité et homogénéisation," Thèse de Doctorat d'Etat, Perpignan, 1987.
8. G. BOUCHITTÉ, Conjuguée et sous-différentiel d'une fonctionnelle intégrale sur un espace de Sobolev, *C.R. Acad. Sci. Paris*, in press.
9. A. BOURASS AND M. VALADIER, "Conditions de croissance associée à l'inclusion des sections (d'après A. Fougères et R. Vaudène), "Publications AVAMAC, n° 3, Université de Perpignan, 1984.
10. N. BOURBAKI, "Integration," Chaps. 1 à 4, Hermann, Paris, 1965.
11. H. BREZIS, Intégrales convexes dans les espaces de Sobolev, *Israel J. Math.* **13** (1972), 9–23.
12. H. BREZIS, "Analyse fonctionnelle: Théorie et applications," Masson, Paris, 1983.
13. H. BREZIS, Some variational problems of the Thomas–Fermi type, *in* "Variational Inequalities" (Cottle, Gianessi, and J. L. Lions, Eds.), pp. 53–73, Wiley, New York, 1980.
14. H. BREZIS, Nonlinear elliptic equations involving measures, *in* "Contributions to Nonlinear Partial Differential Equation" (Bardos, Damlamian, Diaz, and Hernandez, Eds.), Pitman, New York, 1983.
15. C. CASTAING AND M. VALADIER, "Convex Analysis and Measurable Multifunctions," Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin, 1977.
16. G. DAL MASO, Integral representation on $BV(\Omega)$ of Γ-limit of variational integrals, *Manuscripta Math.* **30** (1980), 387–416.
17. E. DE GIORGI, L. AMBROSIO, AND G. BUTTAZZO, "Integral Representation and Relaxation for Functionals Defined on Measures," Scuola Normale Superiore, Pisa, 1986.
18. C. DELLACHERIE AND P. A. MEYER, "Probabilités et potentiel," Chaps. I à IV, Hermann, Paris, 1975.
19. F. DEMENGEL AND R. TEMAM, Convex functions of a measure and applications, *Indiana Univ. Math. J.* **33**(5) (1984), 673–709.
20. A. FOUGÈRES, "Separability of Integrands," Publications AVAMAC, n° 15, Université de Perpignan, 1986.
21. M. GIAQUINTA, G. MODICA, AND J. SOUCEK, Functionals with linear growth in the calculus of variations, *Comm. Math. Univ. Carolina* **20** (1979), 143–171.
22. C. GOFFMAN AND J. SERRIN, Sublinear functions of measures and variational integrals, *Duke Math. J.* **31** (1964), 159–178.
23. M. GRUN-REHOMME, Caractérisation du sous-différentiel d'intégrandes convexes dans les espaces de Sobolev, *J. Math. Pures Appl.* **56** (1977), 149–156.
24. T. HADHRI, Fonction convexe d'une mesure, *C.R. Acad. Sci. Paris* **301** (1985), 687–690.
25. F. HIAI AND H. UMEGAKI, Integrals, conditional expectations, and martingales of multivalued functions, *J. Multivariate Anal.* **7** (1977), 149–182.
26. E. MICHAEL, Continuous selections, I, *Ann. of Math.* **63** (1956), 361–382.
27. J. J. MOREAU, "Fonctionnelles convexes," Collège de France, Paris, 1966–1967.
28. C. OLECH, The characterization of the weak* closure of certain sets of integrable functions, *SIAM J. Control* **12**(2) (1974), 311–318.
29. C. OLECH, A necessary and sufficient condition for lower semi-continuity of certain integral functionals, *in* "Mathematical Structures, Computational Mathematics, Mathematical Modelling," pp. 373–379, Sofia, 1975.
30. Y. G. RESCHETNIAK, Weak convergence of completely additive vector measures on a set, *Sibirsk. Mat. Zh.* **9** (1968), 1386–1394.

31. R. T. ROCKAFELLAR, Integrals which are convex functionals, *Pacific J. Math.* **24** (1968), 525–539.
32. R. T. ROCKAFELLAR, Integrals which are convex functionals, II, *Pacific. J. Math.* **39** (1971), 439–469.
33. R. T. ROCKAFELLAR, Integral functionals, normal integrands and measurable selections, *in* "Nonlinear Operators and the Calculus of Variations," Lecture Notes in Mathematics, Vol. 543, pp. 157–207, Springer-Verlag, Berlin, 1976.
34. R. T. ROCKAFELLAR, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
35. L. SCHWARTZ, "Théorie des distributions," Hermann, Paris, 1957.
36. R. TEMAM, "Problèmes mathématiques en Plasticité," Gauthier–Villars, Paris, 1983.
37. R. TEMAM, Approximation de fonctions convexes sur un espace de mesures et applications, *Canad. Math. Bull.* **25** (4) (1982), 392–413.
38. TRAN CAO NGUYEN, A characterization of some weak semi-continuity of integral functionals, *Stud. Math.* **66** (1) (1979), 81–92.
39. TRAN CAO NGUYEN, Decomposition of the conjugate integral functional on the space of regular measures, *Sém. Anal. Convexe* **16** (1986), exposé n° 2.
40. M. VALADIER, Fermeture étroite et bipolaire vague, *Sém. Anal. Convexe* **7** (1977), exposé n° 6.
41. M. VALADIER, Closedness in the weak topology of the dual pair $L^1, \mathscr{C}$, *J. Math. Anal. Appl.* **69** (1979), 17–34.
42. M. VALADIER, Fonctions et opérateurs sur les mesures. Formules de dualité, *Sém. Anal. Convexe* **16** (1986), exposé n° 3.
43. M. VALADIER, Multi-applications mesurables à valeurs convexes compactes, *J. Math. Pures Appl.* **50** (1971), 265–297.
44. M. VALADIER, Quelques propriétés de l'ensemble des sections continues d'une multifonction s.c.i., *Sém. Anal. Convexe* **16** (1986), exposé n° 7.
45. M. VALADIER, Fonctions et opérateurs sur les mesures, *C.R. Acad. Sci. Paris* **304** (1987), 135–137.