SINGULAR CONTACT VARIETIES

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Abstract. In this note we propose the generalisation of a notion of a holomorphic contact structure on a manifold (smooth variety) to normal varieties with canonical singularities, give some examples and prove basic properties of such objects.

1. Introduction

1.1. **Background.** The study of holomorphic symplectic manifolds is a classical topic (...). The odd-dimensional counterpart, (holomorphic) contact manifolds came into light when LeBrun and Salamon showed in [LS94] the bijection (the twistor construction) between a subclass of them and quaternion-Kähler manifolds with positive scalar curvature. Conjecturally, the only examples of the latter are some symmetric spaces and one can use the twistor correspondence to translate the conjecture to the complex algebraic category, where partial progress have been achieved, see for example [Bea98], [KPSW00], [BWW18] and [ORCW21], however the problem is still wide open.
In the meantime, the seminal paper of Beauville ([Bea00]) marked the birth of a notion of a (complex) symplectic singularity. Such objects are still actively researched, both from the local and global (i.e. varieties with symplectic singularities) point of view. There is a classical, although not very recent survey by Fu [Fu06]. On the other hand, according to author's knowledge there is only one work by Campana and Flenner [CF02] concerning complex contact singularities.
Therefore, the goal of our note is to restart the study of this undeveloped field, however from a different angle. Since our interest in contact structures comes from the conjecture of LeBrun and Salamon, we will adopt more global point of view than [CF02].

1.2. **Notation and conventions.** All objects considered in this note are complex algebraic varieties and by symplectic (contact) structure we always mean a holomorphic one. Likewise, a *manifold* for us is a smooth algebraic variety. For a line bundle L we denote by $L^\bullet$ its total space with the image of the zero section removed, it is clearly a $\mathbb{C}^*$–principal bundle.

2. Definition of contact variety and symplectisation

Definition 2.1. Contact variety is a normal algebraic variety X over $\mathbb{C}$ of odd dimension $2n+1$ with canonical singularities and a globally defined line bundle L such that on the smooth locus X_{reg} we have an exact sequence of vector bundles:

$$0 \to F \to TX_{reg} \xrightarrow{\vartheta} L_{|X_{reg}} \to 0$$

which defines contact structure on X_{reg}, i.e. $d\vartheta : \bigwedge^2 F \to L_{|X_{reg}}$ is nowhere degenerate. Equivalently one can demand that $\vartheta \wedge (d\vartheta)^{\wedge n}$ as an element of $H^0(X_{reg}, \Omega^{2n+1}_{|X_{reg}} \otimes L^{n+1}_{|X_{reg}})$ has no zeroes. We will sometimes call L from the definition *contact line bundle.*

It is clear that the singularities of such varieties are contact in the sense of [CF02]. We will note some easy consequences of the given definition, which will be useful later:

Proposition 2.2. *$-K_X$ is a Cartier divisor and we have $\mathcal{O}(-K_X) = L^{\otimes n+1}$ in $\mathrm{Pic}(X)$. Therefore singularities of X are rational Gorenstein.*

Proof. $\vartheta \wedge (d\vartheta)^{\wedge n}$ gives equality of $(n+1)L$ and $-K_X$ in class group of X_{reg}. Since X is normal, we can take unique closures of both divisors and prolong considered equality to whole X and because L is in fact a Cartier divisor, canonical class also is. This means that X is quasi-Gorenstein. We know that singularities of X are canonical and those are rational in characteristic 0 by [KM98, Thm 5.22]. Then from [Fle81, Satz 1.1] it follows that X is Cohen-Macaulay, therefore Gorenstein. □

The slogan "contact geometry is an odd-dimensional counterpart of symplectic geometry" can be made precise in the smooth case by the standard construction of symplectisation, described for example in [Buc09, Thm. E.6]. In essence, for each contact manifold with contact line bundle L, the space $L^\bullet$ is a symplectic manifold. To have an analogous statement in the singular case recall the Namikawa's characterisation of symplecticity:

Theorem 2.3 ([Nam01, Theorem 6])**.** *A normal variety is symplectic if and only if it has rational Gorenstein singularities and its smooth part admits a holomorphic symplectic form.*

It allows us to prove the following:

Theorem 2.4. *For a contact variety X with a contact line bundle L, the space $L^\bullet$ is a symplectic variety.*

Proof. Since $L^\bullet$ is locally trivial $\mathbb{C}^*$-bundle over X, it has rational Gorenstein singularities if and only if X has. On the smooth part we can define holomorphic symplectic form by the standard construction ([Buc09, Section C.5]) □

Theorem 2.5. *Let Y be a symplectic variety equipped with free action of $\mathbb{C}^*$ such that the symplectic form ω is homogeneous of weight 1. Then the quotient X is a contact variety.*

Proof. Since we have a free $\mathbb{C}^*$ action, Y locally looks like $U \times \mathbb{C}^*$ (also near the singularities). Therefore, if singularities of Y (U) are rational Gorenstein, so are singularities of X (and those are canonical). On the smooth locus of X there exists a twisted 1-form by the standard argument ([Buc09, Proposition C.16]). □

It should be clear now that we defined contact variety in such a way to keep the correspondence (symplectisation) in the singular setting. Thanks to it we obtain a natural way to study contact varieties - we can check which properties of symplectic varieties behave well when they are equipped with a free $\mathbb{C}^*$ action and in this way we obtain an analogous property for contact varieties. We will see one such case in the next section.

3. Kaledin's stratification for contact varieties

We have the following analogue of Kaledin's result ([Kal06]):

Theorem 3.1. *Let X be a contact variety. Then we have a canonical stratification $X = X_0 \supset X_1 \supset X_2 \supset ...$ such that:*

1. *X_{i+1} is the singular part of X_i*
2. *the normalisation of each irreducible component of X_i is a contact variety.*

The statement first appeared without a proof in [MnOSC$^+$15, Prop. 5.9].

Proof. Consider the symplectisation of X - $L^\bullet$. By [Kal06, Thm 2.3] it is stratified and the normalisation of each irreducible component of stratum is a symplectic variety. It is clear that $\mathbb{C}^*$ action has to preserve the stratification. On the strata we still have a free $\mathbb{C}^*$ action, and the induced action on the normalisation of each component is still free. Therefore we only need to check that the induced symplectic forms on the strata are homogeneous of weight 1. On the smooth locus, the symplectic form ω defines Poisson bivector $\Theta \in \bigwedge^2 TL^\bullet_{reg}$ which in turn defines the Poisson bracket of functions $\{f, g\} \in \mathcal{O}_{L^\bullet_{reg}}$. Since $L^\bullet$ is normal, $\{f, g\}$ can be extended to the whole variety and its restriction to a stratum is again a Poisson bracket. Going the other way, we obtain a symplectic form on the smooth part of the stratum and we need to check that it is homogeneous of weight 1. Both Θ and $\{,\}$ have weight -1 on $L^\bullet_{reg}$, therefore so does the extension of the bracket over the (components of the) singular locus and its normalisation(s). It follows that the induced symplectic forms have weights 1, as demanded.

□

One immediate consequence of this result is that codimensions of the strata are even. In particular, it provides another proof of the main result of [CF02] that there are no isolated contact singularities.

4. Quotients of contact manifolds

We will begin by considering some simple examples and non-examples, namely quotients of $\mathbb{P}^3$ by finite subgroups of contactomorphisms. Start with associated affine space $\mathbb{C}^4$ equipped with symplectic form $\omega = dx_0 \wedge dx_2 + dx_1 \wedge dx_3$. For a point in $\mathbb{C}^4 \ni (a_0, a_1, a_2, a_3)$ the ω-perpendicular space at this point is given by $a_0x_2 + a_1x_3 - a_2x_0 - a_3x_1 = 0$. Twisted 1-form with the kernel given by the above equation can be written as $\vartheta = x_2dx_0 + x_3dx_1 - x_0dx_2 - x_1dx_3$.

Example 4.1. Consider the action of $\mathbb{Z}_2$ on $\mathbb{C}^4$, where the generator A acts via $A \cdot (x_0, x_1, x_2, x_3) = (x_0, x_1, -x_2, -x_3)$. This is not a symplectomorphism, since it does not preserve the symplectic form, but it preserves the perpendicular space at each point, so it is a contactomorphism of the associated projective space. However $A \cdot \vartheta = -\vartheta$.

Example 4.2. Again consider the action of of $\mathbb{Z}_2$ on $\mathbb{C}^4$, but this time the generator B acts via $B \cdot (x_0, x_1, x_2, x_3) = (x_0, -x_1, x_2, -x_3)$. It is a symplectomorphism descending to a contactomorphism on $\mathbb{P}^3$ and moreover it preserves the contact form, $B \cdot \vartheta = \vartheta$.

Example 4.3. Now consider the action of $\mathbb{Z}_4$ on $\mathbb{C}^4$, where the generator C acts via $C \cdot (x_0, x_1, x_2, x_3) = (ix_0, ix_1, -ix_2, -ix_3)$. It is again symplectomorphism which descends to a contactomorphism, but it does not preserve the contact form, $C \cdot \vartheta = -\vartheta$.

The study of examples leads us to the following:

Theorem 4.4. *Let X be a contact manifold with contact distibution F, contact line bundle L and twisted form ϑ. Let $G \subset \mathrm{Aut}(X, F)$ be a finite subgroup of contactomorphisms. Then the quotient is a contact variety if and only if for all $x \in X$ the stabilizer G_x preserves the contact form ϑ restricted to x.*

To prove this theorem we recall a useful lemma:

Lemma 4.5. *[* [DN89, Th. 2.3]*] Let X be a variety equipped with an action of reductive algebraic group G such that there exists a good quotient. Let E be a G-vector bundle on X. Then E descends to quotient variety if and only if for all $x \in X$ such that the orbit $G \cdot x$ is closed the stabilizer G_x acts trivially on the fiber E_x.*

Proof. If the quotient is a contact variety then we have a contact line bundle $\tilde{L}$ on it, and this bundle must be descended from L. Therefore by 4.5 $\forall_{x \in X} G_x$ acts trivially on L_x. It follows that $G_x \cdot \vartheta_{|x} = \vartheta|x$.

Now lets suppose that for all points x the stabilizer subgroup G_x preserves $\vartheta_{|x}$. Then G_x acts trivially on the fiber L_x, so again by 4.5 we know that the line bundle L descends to the quotient. Away from the fixed points of the action we still have the contact exact sequence, and quotient singularities are canonical. □

5. Resolution of singularities

A natural question arising in the study of structures on singular varieties is: does considered structure behave well with respect to resolution of singularities? In our case: is the resolution of contact variety a contact manifold in the usual sense? In general the answer is no, although for special (crepant) resolutions, it is affirmative.

Example 5.1. Take $\mathbb{P}^{2n+1}$ equipped with standard contact form. Let $G = \mathbb{Z}_2^{\times n}$ and $H = \mathbb{Z}_2$ be finite subgroups of contactomorphisms. Then quotient of $\mathbb{P}^{2n+1}$ by G or by H is a (toric) contact variety in the sense of our definition but only the former has crepant resolution, which is isomorphic to $\mathbb{P}(T^*(\mathbb{P}^1 \times ... \times \mathbb{P}^1))$, which is well-known to be a contact manifold.

Theorem 5.2. *If $X' \xrightarrow{f} X$ is a crepant morphism (i.e. $K_{X'} \approx f^*(K_X)$) with $\mathrm{Exc}(f) \subset \mathrm{Sing}(X)$ and X is a contact variety, then X' is again a contact variety with structure F' compatible with projection to X, i.e. $f_*F' = F$ on the smooth locus of X and a contact line bundle f^*L. In particular, a terminalization of a contact variety preserves the contact structure and a crepant resolution of singularities produces a usual contact manifold.*

Proof. To prove the theorem we need to define ϑ' on the smooth locus of X' and check whether it satisfies the nondegeneracy condition. To this end, let $\widetilde{X} \xrightarrow{g} X'$ be a resolution of singularities, which is also (via composition) a resolution of singularities for X. By [GKKP11, Thm 1.4] the sheaf $(fg)_*\Omega^1_{\widetilde{X}}$ is reflexive. Then by projection formula it follows that $(fg)_*(\Omega^1_{\widetilde{X}} \otimes (fg)^*L)$ is also reflexive. ϑ is a section of the last sheaf, so by reflexivity it extends to a section of $\Omega^1_{\widetilde{X}} \otimes (fg)^*L$, since the singular locus of X has codimension ≥ 2. Finally since there is an isomorphism between the smooth locus of X' and $\widetilde{X} \setminus \mathrm{Exc}(g)$, we can define ϑ' on smooth locus of X'.
Now observe that $\vartheta \wedge (d\vartheta)^{\wedge n}$ can be extended to nowhere vanishing section of $K_X \otimes L^{n+1}$. Since f is crepant, the pullback of this section by f is a nowhere vanishing section of $K_{X'} \otimes f^*L^{n+1}$. It agrees with $\vartheta' \wedge (d\vartheta')^{\wedge n}$ on the intersection of X'_{reg} and any open $U \subset X'$ trivializing f^*L.
To finish the proof, observe that ϑ' is surjective on X'_{reg}: if it were not the case for some $x \in X'$ then in some neighbourhood of x trivializing f^*L we would have $\vartheta \wedge (d\vartheta)^{\wedge n}(x) = 0$, which is absurd. □

Theorem and proof above motivate the following definition:

Definition 5.3. Let $f : \widetilde{X} \to X$ be a resolution of singularities for contact variety X. We say that it is contact iff $\widetilde{X}$ is a contact manifold with compatible contact structure.

In light of this definition, one may wonder whether there exist non-crepant but contact resolutions.

Conjecture. If a resolution of singularities as above is contact and obtained contact manifold is projective, then the resolution is crepant.

Possible strategy. Let L be the contact line bundle on X. If $f^*(L)$ is the contact line bundle on $\widetilde{X}$ then we are done, because then $f^*(-K_X) = f^*((n+1)L) = (n+1)f^*(L) = -K_{\widetilde{X}}$. Note that this is the case if the resolution is small.
Since $\widetilde{X}$ is a projective contact manifold, it does not admit birational Mori contractions ([KPSW00][Lemma 2.10]). Note that it still can admit divisorial contractions (like in example above), but they cannot be Mori. From the Mori condition we should be able to somehow rule out the case when the contact line bundle on the resolution is $f^*(L) \otimes \mathcal{O}(-E)$. If one can reason step by step, i.e. by contracting one irreducible divisor at a time then relative nefness and ampleness are the same thing, so it would conclude the proof. □

Unfortunately currently we are unable to conclude, even using additional assumption of projectivity. The problem lies in a fact that in general one cannot assume that resolution algorithm works one smooth blow-up at a time.

6. Projective threefolds

In this section we will use the developed tools to study projective singular contact threefolds.

Remark 6.1. *If X is a contact variety of dimension 3 then it has a crepant resolution of singularities.*

Proof. Take the terminalization of X which by 5.2 is still contact. Its singular locus by 3.1 has dimension 1, however at the same time it must have codimension at least 3 by terminality. Therefore the terminalization is already smooth. □

In the projective case, the only possible manifold resolving X is of the form $\mathbb{P}(T^*S)$ for some smooth surface S. Therefore we can form the following commutative square of morphisms:

$$\begin{array}{ccc} \mathbb{P}(T^*S) & \xrightarrow{f} & X \\ \downarrow p & & \downarrow q \\ S & \overset{g}{\dashrightarrow} & Y \end{array}$$

p is the unique elementary Mori contraction of $\mathbb{P}(T^*S)$ (see [KPSW00] for details) and denote by $[F]$ the class of the fiber of p. Then, since f is the surjective and crepant morphism (resolution of singularities), contraction of $f_*[F]$ is the unique elementary Mori contraction of X, which is denoted by q. Finally g is a contraction (not necessarily Mori) making the diagram commutative.
The singular locus of X consists of smooth, disjoint curves C_i by 3.1 and the exceptional divisors E_i of the resolution are mapped to these curves. They are covered by rational curves and so have negative Kodaira dimension. There are two possibilities:

(1) at least one of E_i is mapped onto S via p,
(2) p maps every E_i to some curve $D_i \subset S$.

In the first case we can observe that $\kappa(S) = -\infty$ as some E_i with $\kappa(E_i) = -\infty$ is mapped onto it by p. Moreover the contraction g cannot be a birational morphism, so Y has to be a point or a curve. If $Y = *$ then since it comes from an elementary Mori contraction of X, it follows that X has $b_2 = 1$ and is Gorenstein-Fano.
Now consider the situation where every E_i is mapped by p to a curve D_i, that is every E_i is a $\mathbb{P}^1$-bundle over D_i. By rigidity lemma no fiber of p gets contracted to a point by f. Therefore every $\mathbb{P}^1$-fiber of E_i is mapped onto C_i, so each of them has to be a projective line and moreover this mapping is isomorphism, since the fibers of f are connected.

Example 6.2 (Basic example in dimension 3)**.** Start with a $\mathbb{C}^4$ with coordinates x_0, x_1, x_2, x_3 and a symplectic form: $dx_0 \wedge dx_2 + dx_1 \wedge dx_3$. The associated projective space $\mathbb{P}^3$ is a contact manifold. Now consider diagonal action of $diag(i, i, -i, -i)$ on C^4, this is a symplectomorphism and it descends to a contactomorphism. The fixed point locus on $\mathbb{P}^3$ consists of two lines: $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$. The quotient is a contact variety X with singular locus being precisely the image of two fixed lines. We can resolve X by two blowups at singular lines. The smooth variety that we obtain is isomorphic to $\mathbb{P}(T^*(\mathbb{P} \times \mathbb{P}))$ and both exceptional divisors are sections of this projective bundle. X is Fano and has $b_2 = 1$. Moreover, the partial resolution (that is X blown up in one of the lines, there is a contactomorphism swapping them) is also a contact variety. The unique Mori contraction maps it to the projective line.

Remark 6.3. *The example above presents all possible toric contact threefolds: they have to be resolved by* $\mathbb{P}(T^*(\mathbb{P}\times\mathbb{P}))$*, and the only crepant* *contractions of this manifold are contractions of two (torus invariant) sections.*

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