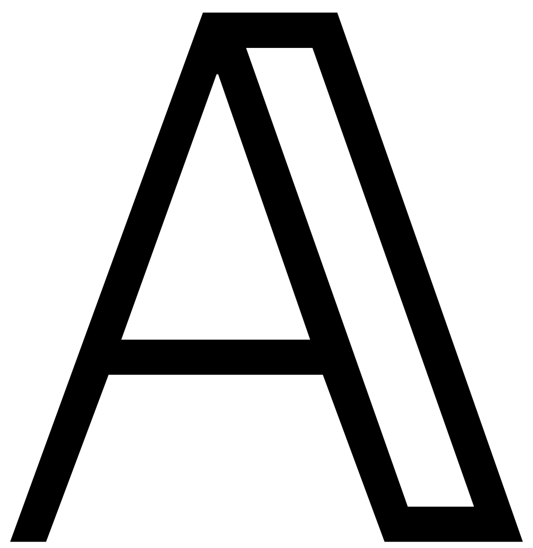
Single-use restriction vs. associativity

Rafał Stefański (University College London)
joint work with Mikołaj Bojańczyk

Cambridge, July 2023

Founded by EPSRC project *“Resources in Computation”*

Data words



Kaminski, Francez, 1994

Data words

$$\mathbb{A} = \{1, 2, 3 \ldots\}$$

Kaminski, Francez, 1994

Register automata

The first letter appears again

2 1 3 3 1 2 3 1 2

Register automata

The first letter appears again

q_{init}

$\square$

$\downarrow$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

Kaminski, Francez, 1994

Register automata

The first letter appears again

q_{check}

$\boxed{2}$

$\downarrow$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

Kaminski, Francez, 1994

Register automata

The first letter appears again

q_{check}

$\boxed{2}$

$\downarrow$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

Kaminski, Francez, 1994

Register automata

The first letter appears again

q_{check}

$\boxed{2}$

$\downarrow$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

Kaminski, Francez, 1994

Register automata

The first letter appears again

q_{found}

$\boxed{2}$

$\downarrow$

$$2\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 1\ \ 2$$

Kaminski, Francez, 1994

Register automata

The first letter appears again

q_{found}

$\boxed{2}$

$\downarrow$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

Kaminski, Francez, 1994

Register automata

The first letter appears again

There are at most 3 different letters in the word

The first and the last letters are equal

No two consecutive letters are equal

Kaminski, Francez, 1994

Semigroups with atoms (nominal semigroups)

- Set with one associative operations
- Each element can store a finite number of atoms
- The operation commutes with atom renaming:

$$\pi(x \cdot y) = \pi(x) \cdot \pi(y)$$

Bojańczyk, 2012

Semigroups with atoms (nominal semigroups)

$$P_{fin}(\mathbb{A})$$

$$x \cdot y = x \cup y$$

Bojańczyk, 2012

Semigroups with atoms (nominal semigroups)

$$\mathbb{A}^2$$

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1, y_2)$$

Bojańczyk, 2012

Orbit-finite semigroups

There are only finitely many elements up to atom renaming

$$\mathbb{A}^2$$

$$\sout{P_{fin}(\mathbb{A})}$$

Bojańczyk, 2012

Semigroups and languages

$$S, \quad h : \Sigma \to S, \quad \lambda : S \to \{Y, N\}$$

$$\Sigma^* \xrightarrow{h^*} S^* \xrightarrow{\text{mult}} S \xrightarrow{\lambda} \{Y, N\}$$

Semigroups and languages

There are at most 3 different letters in the word

$$\binom{\mathbb{A}}{\leq 3} + \bot$$

$$x \cdot y = \begin{cases} x \cup y & \text{if } |x \cup y| \leq 3 \\ \bot & \text{otherwise} \end{cases}$$

Bojańczyk, 2012

Semigroups and languages

The first letter appears again

Bojańczyk, 2012

Semigroups and languages

~~The first letter appears again~~

The semigroup would have to remember every letter from the word

$$P_{fin}(\mathbb{A})$$

Bojańczyk, 2012

Orbit-finite semigroups

~~The first letter appears again~~

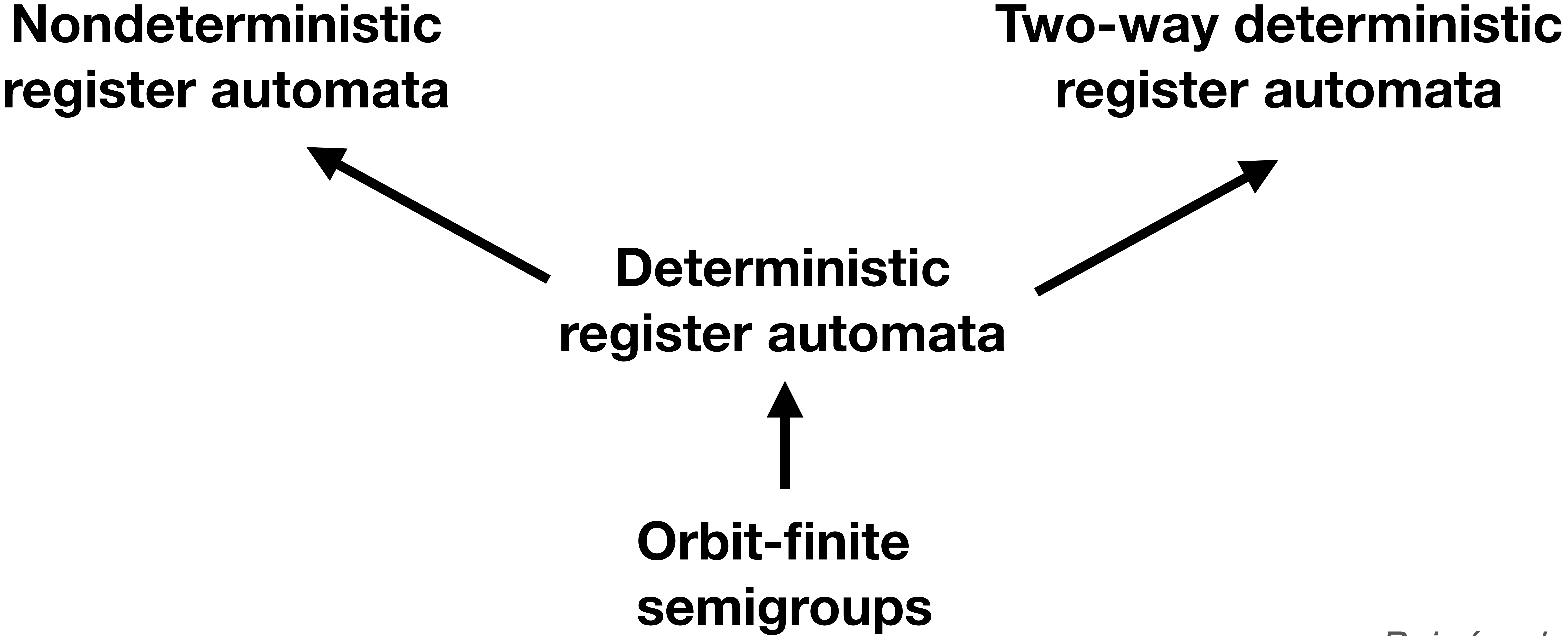
There are at most 3 different letters in the word

The first and the last letters are equal

No two consecutive letters are equal

Bojańczyk, 2012

Other models



Bojańczyk, 2012

Single-use register automaton

Every read access to a register destroys its contents

Bojańczyk, S. 2020

Single-use register automaton

The first letter appears again

2 1 3 3 1 2 3 1 2

Bojańczyk, S. 2020

Single-use register automaton

The first letter appears again

q_{init}

$\square$

$\downarrow$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

Bojańczyk, S. 2020

Single-use register automaton

The first letter appears again

q_{check}

$\boxed{2}$

$\downarrow$

$$2\ 1\ 3\ 3\ 1\ 2\ 3\ 1\ 2$$

Bojańczyk, S. 2020

Single-use register automata

~~The first letter appears again~~

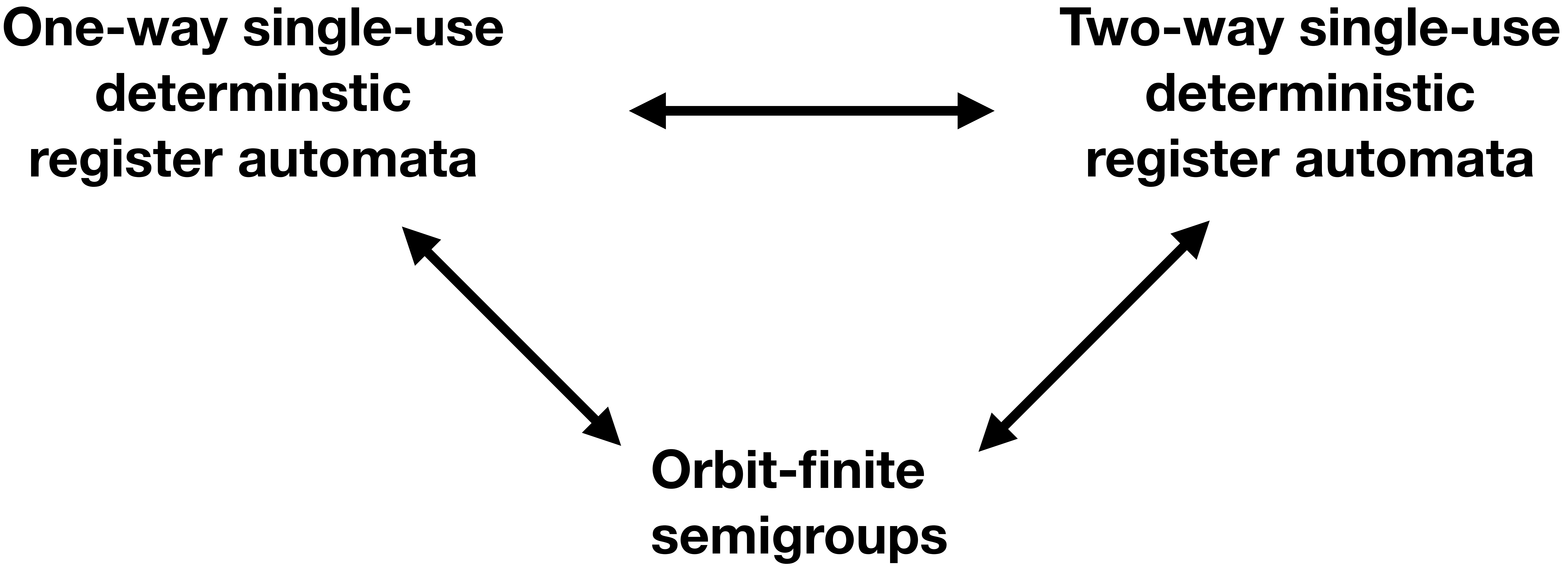
There are at most 3 different letters in the word

The first and the last letters are equal

No two consecutive letters are equal

Bojańczyk, S. 2020

Single-use models



Bojańczyk, S. 2020

Single-use transducers

Single-use register Mealy machines

Shift all letters one position to the right

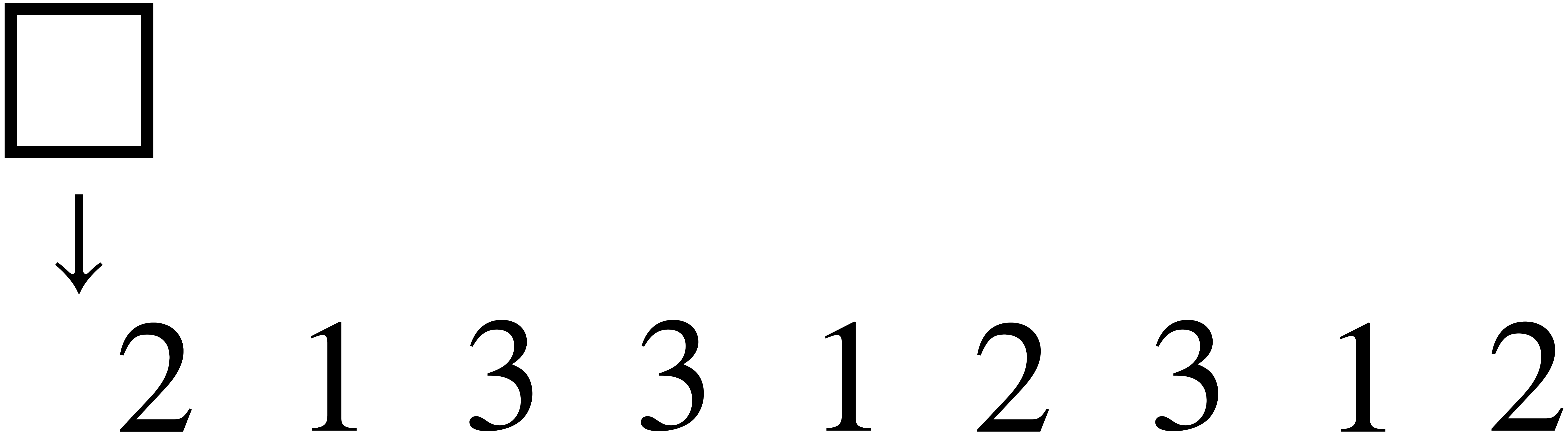
$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$



Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$\vdash$

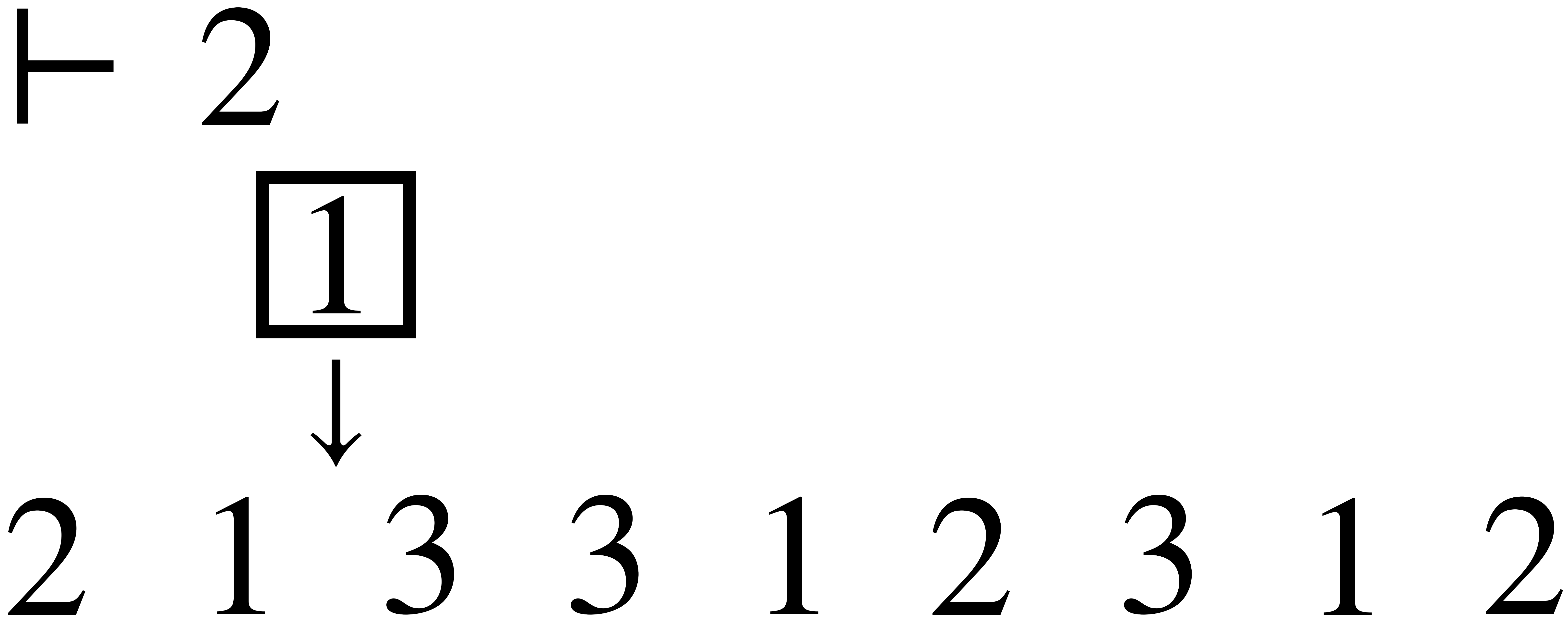
$\boxed{2}$

$\downarrow$

2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Shift all letters one position to the right



Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$$\vdash \; 2 \; 1$$

$$\boxed{3}$$

$$\downarrow$$

$$2 \; 1 \; 3 \; 3 \; 1 \; 2 \; 3 \; 1 \; 2$$

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$\vdash$ 2 1 3

$\boxed{3}$

$\downarrow$

2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$\vdash$ 2 1 3

$\boxed{3}$

$\downarrow$

2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$$\vdash \; 2 \; 1 \; 3 \; 3 \; 1 \; 2 \; 3 \; 1$$

$$\boxed{2}$$

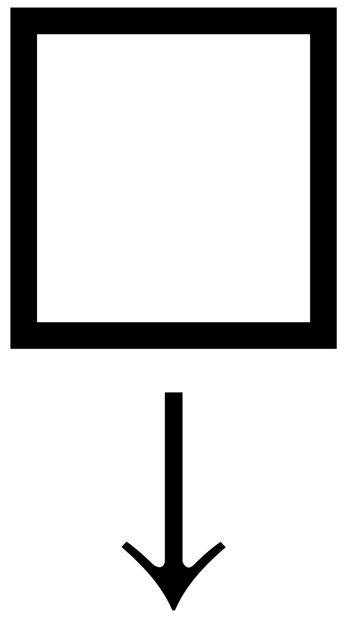
$$\downarrow$$

$$2 \; 1 \; 3 \; 3 \; 1 \; 2 \; 3 \; 1 \; 2$$

Single-use register Mealy machines

Compare every letter with the first letter

$$\mathbb{A}^* \to \{Y, N\}^*$$



2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Compare every letter with the first letter

$$\mathbb{A}^* \to \{Y, N\}^*$$

Y

$\boxed{2}$

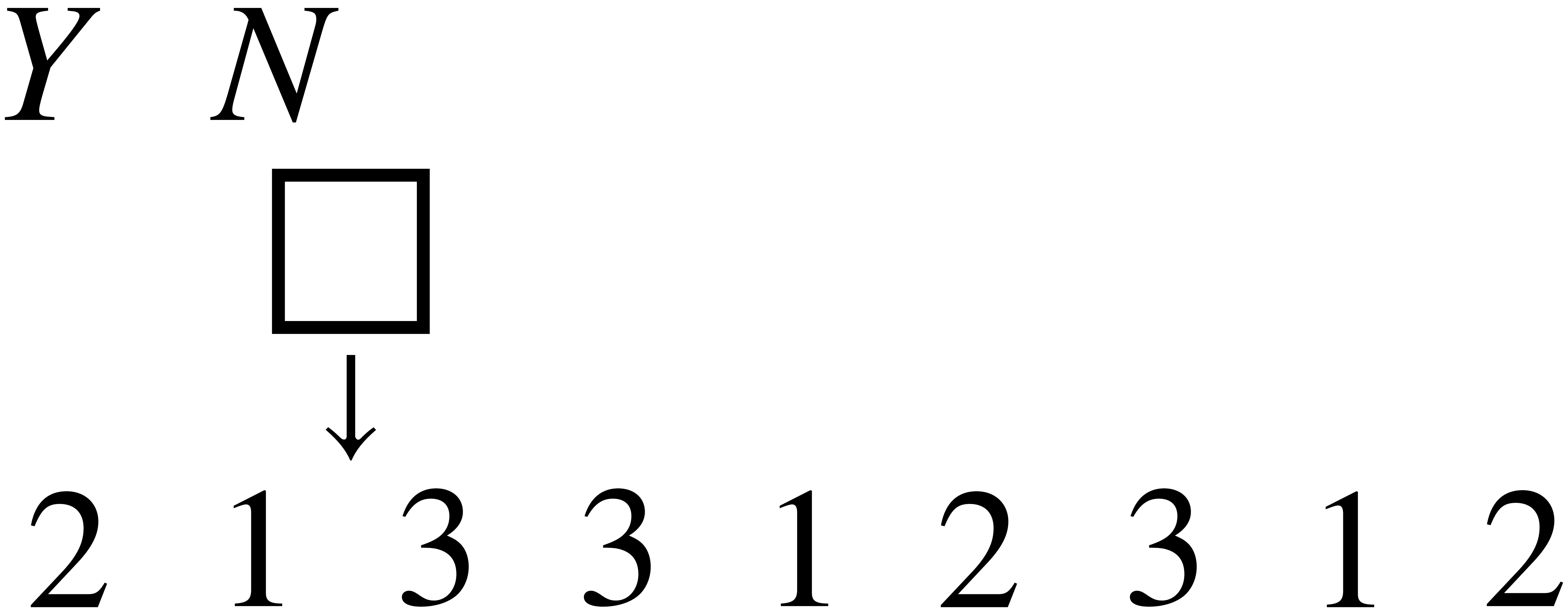
$\downarrow$

2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Compare every letter with the first letter

$$\mathbb{A}^* \to \{Y, N\}^*$$



Single-use register Mealy machines

Compare every letter with the first letter

$$\mathbb{A}^* \to \{Y, N\}^*$$

$$Y \quad N$$

$$?$$

$$\downarrow$$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

Single-use register Mealy machines

Shift all letters one position to the right

~~Replace all letters with the first letter~~

Compare every letter with the previous one

~~Compare every letter with the first letter~~

Why single-use transducers?

- Single-use Mealy machines admit Krohn-Rhodes decompositions
- All of the following models are equivalent:
  1. Single-use two-way automata
  2. Single-use copyless SSTs
  3. Regular list functions with atoms
  4. Compositions of two-way primes with atoms
- Single-use automata are equivalent to orbit-finite semigroups

Bojańczyk, S. 2020

Semigroups and transducers

$$S, \qquad h : \Sigma \to S, \qquad \lambda : S \to \Gamma$$

$$\Sigma^* \xrightarrow{h^*} S^* \xrightarrow{\text{prefixes}} S^* \xrightarrow{\lambda^*} \Gamma^*$$

Semigroups and transducers

Shift all letters one position to the right

$$\mathbb{A}^2 + \mathbb{A}$$

$$(x_1, x_2) \cdot (y_1, y_2) = (y_1, y_2)$$

$$(x_1, x_2) \cdot y = (x_2, y)$$

$$\cdots$$

Semigroups and transducers

Replace all letters with the first letter

Compare every letter with the first letter

$$\mathbb{A}^2$$

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1, y_2)$$

Locality restriction

$$\lambda(xey) = \lambda(\pi(x)ey)$$

As long as:

- π fixes all atoms in e
- e is an idempotent ($e \cdot e = e$)
- y is a prefix of e ($e \cdot b = y$, for some b)

Local orbit-finte semigroup transductions $\simeq$ Single-use Mealy machines

Research directions

- Rational single-use functions
- Krohn-Rhodes decompositions of orbit-finite semigroups
- Atoms with more structure such as $(\mathbb{Q}, \leq)$

Thank you!