Monads, Comonads, and transducers

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University College London

Highlights '23

Founded by EPSRC project *“Resources in Computation”*

Monads

MA	$MMA \longrightarrow MA$	$A \longrightarrow MA$
Data structure	Flattening operation	Singleton operation

Monads

$$MA$$

Data structure

$$MMA \longrightarrow MA$$

Flattening operation

$$A \longrightarrow MA$$

Singleton operation

For example

$$MA = A^*$$

Monads

MA

Data structure

$MMA \longrightarrow MA$

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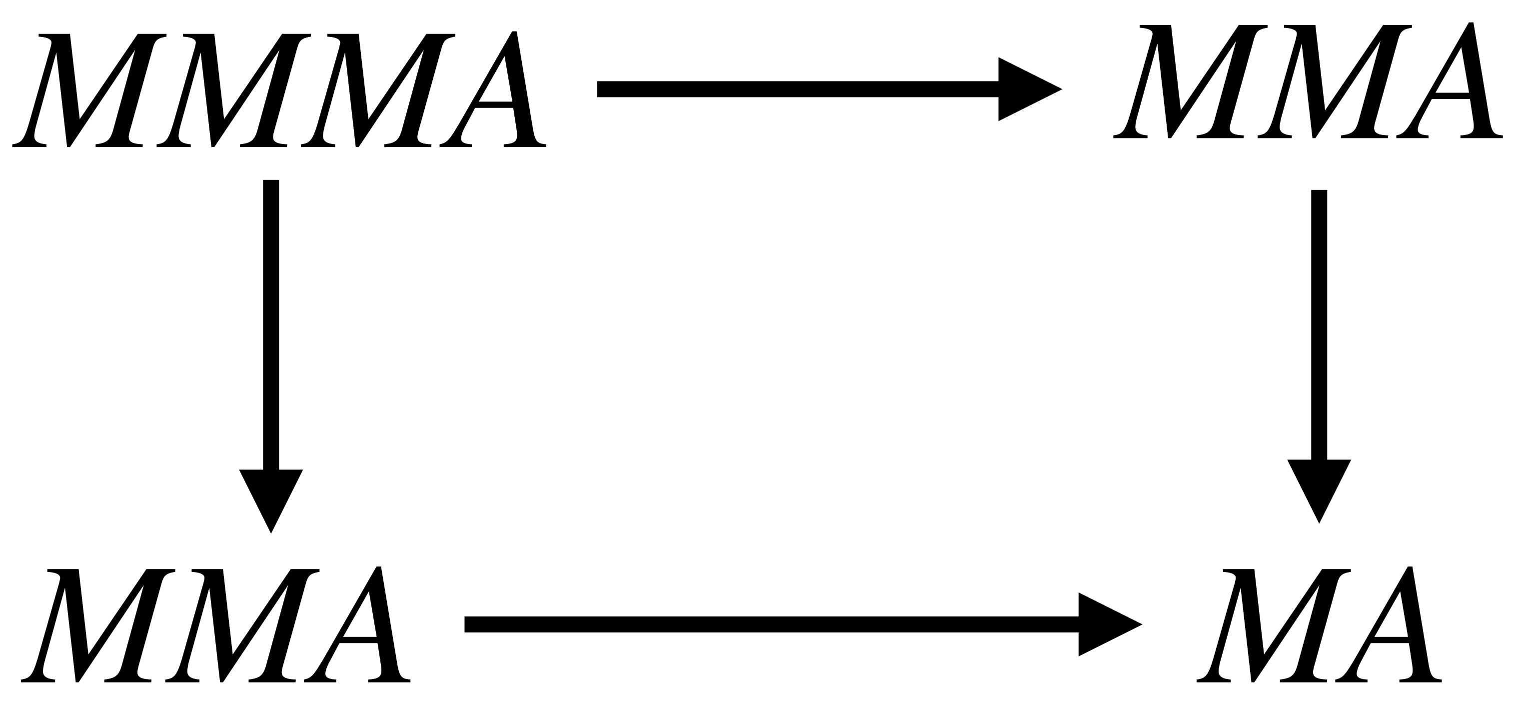
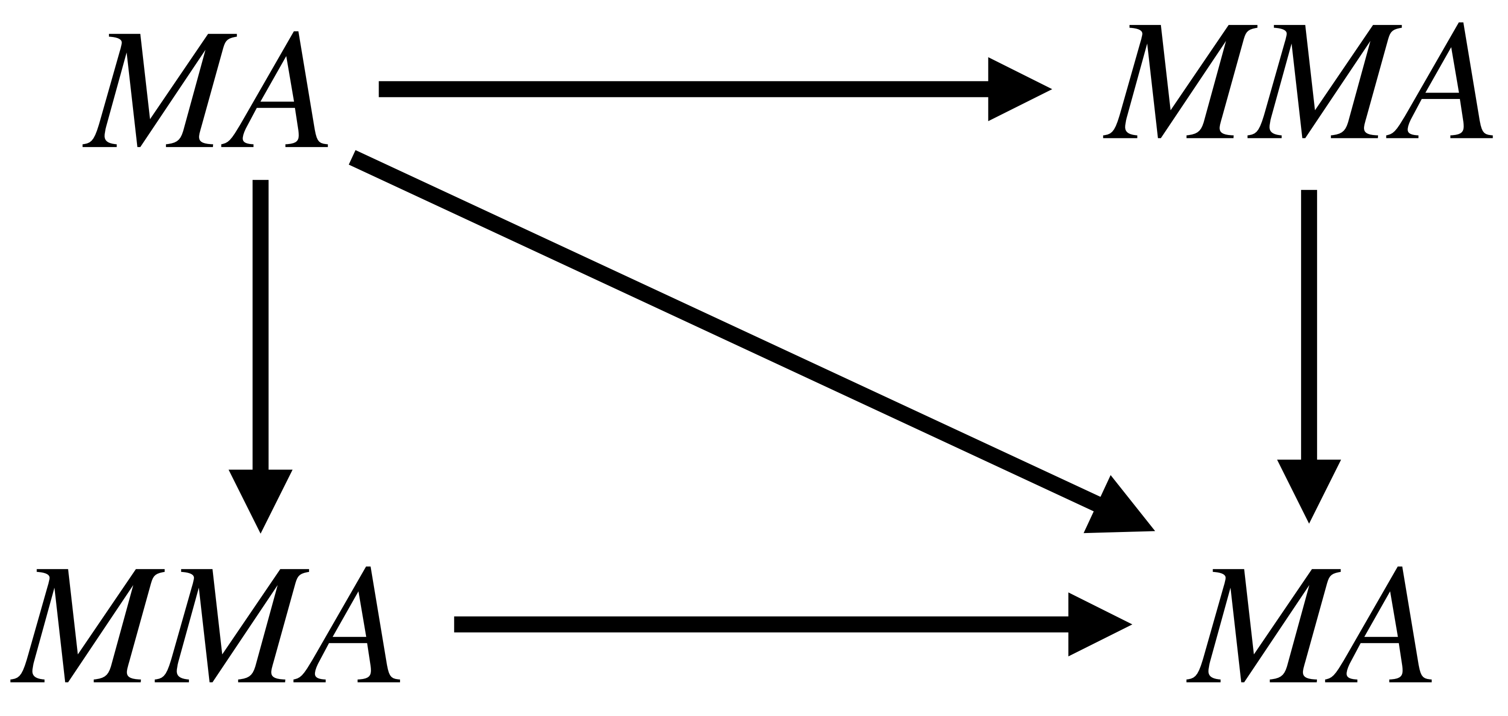
For example

$MA = A^*$

Together with coherence axioms





Comonads

MA

Data structure

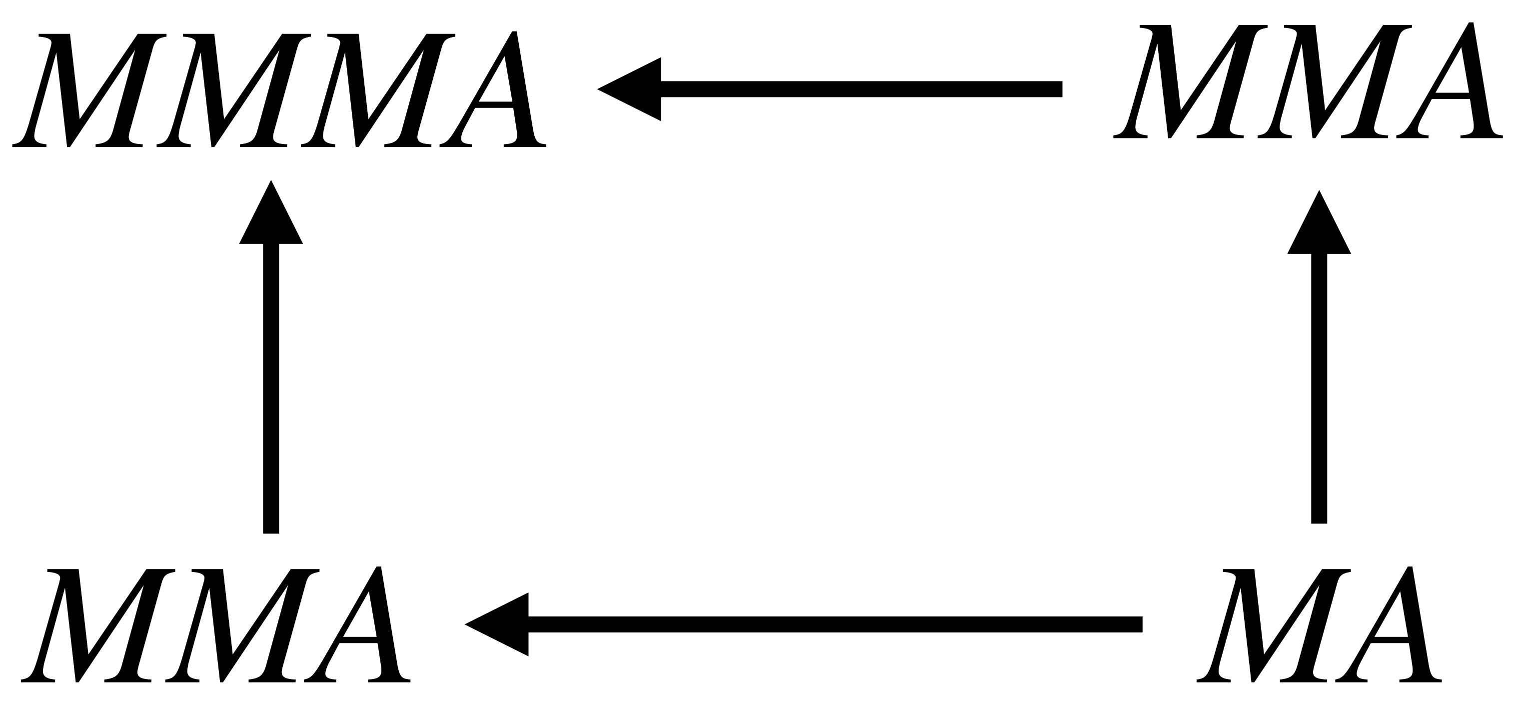
$MMA \longleftarrow MA$

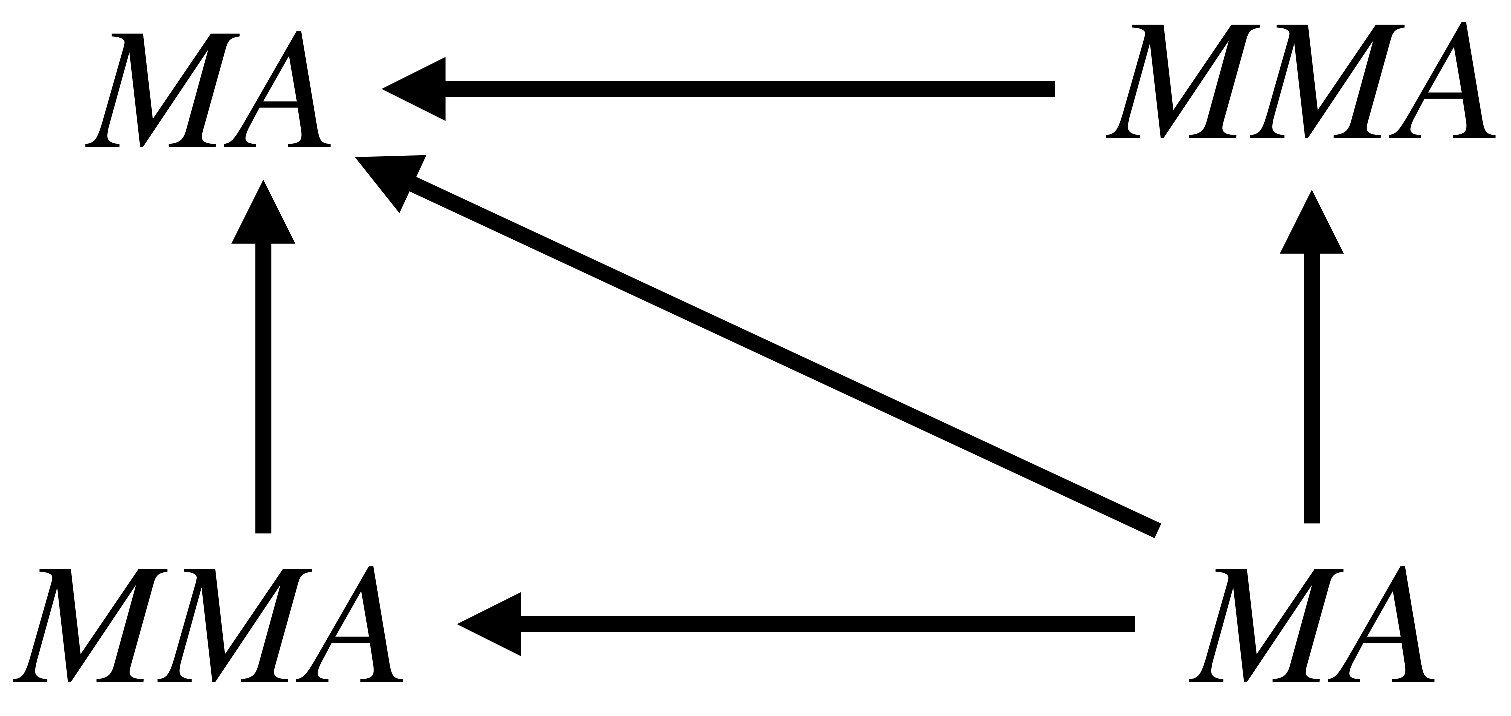
Expanding operation

$A \longleftarrow MA$

Extracting operation

Together with coherence axioms





Slogan:

Monads = Languages

Monads + Comonads = Transducers

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Regular
=
Recognisable by finite algebras

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M. Bojańczyk. 2015.
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Monads = Languages

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This talk.

Monad and comonad

$$MA = A^+$$

$$MMA \to MA$$

$$A \to MA$$

$$MA \to MMA$$

$$MA \to A$$

Monad and comonad

$$MA = A^+$$

$MMA \to MA$	Flatten	$[[1, 2, 3], [4, 5], [6, 7]] \mapsto [1,2,3,4,5,7]$
$A \to MA$	Singleton	$7 \mapsto [7]$

$$MA \to MMA$$

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$MA \to MMA$	Prefixes
$MA \to A$	

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$MA \to A$	Last element	

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$MA \to A$	Last element	$[1,2,3,4] \mapsto 4$

Monads, comonads, and transducers

Given a regular language:

$$L : M\Sigma \to \{\mathtt{Yes}, \mathtt{No}\}$$

We define the following transduction:

$$M\Sigma \xrightarrow{\text{comonad}} MM\Sigma \xrightarrow{ML} M\{\mathtt{Yes}, \mathtt{No}\}$$

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This gives us a class of M-transductions.

Structure vs. power

M	Expressive Power

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Non-empty lists with prefixes	Mealy machines

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Words over countable orders with a maximal/minimal/underlined element.

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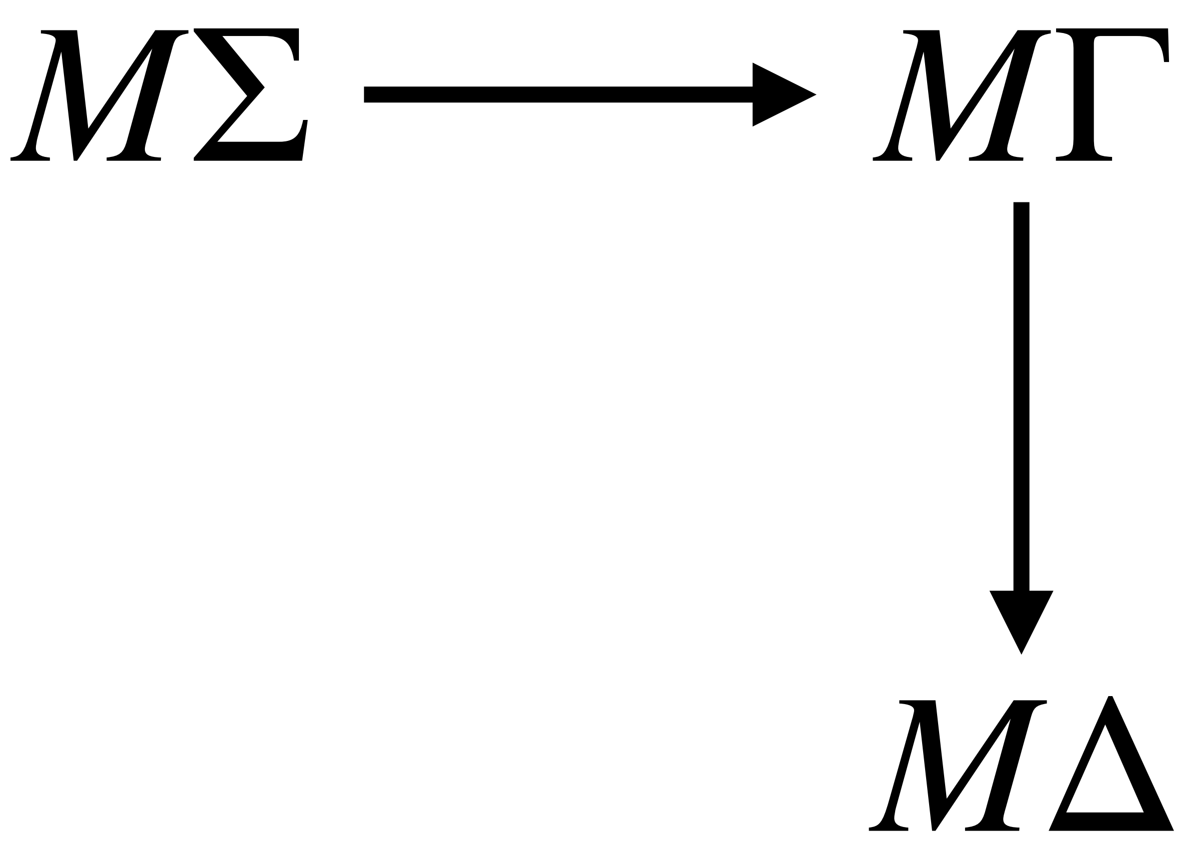

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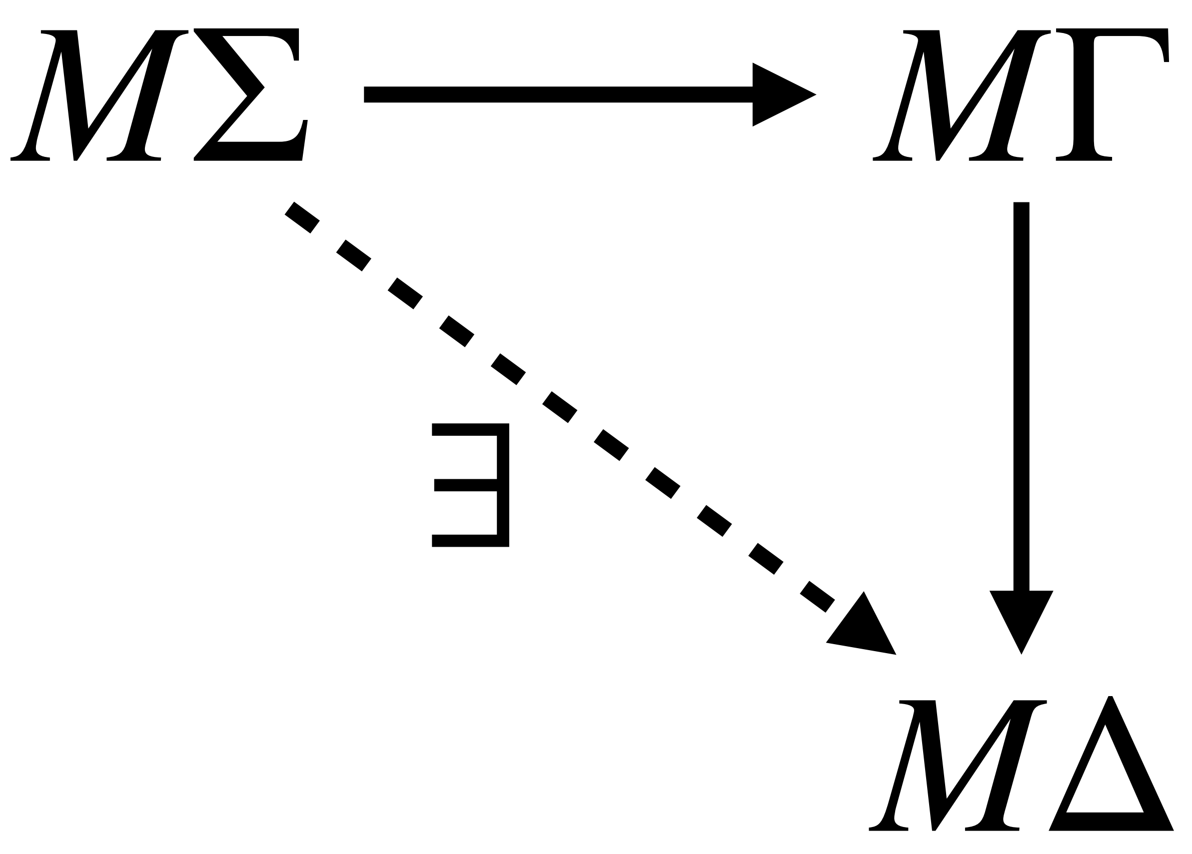

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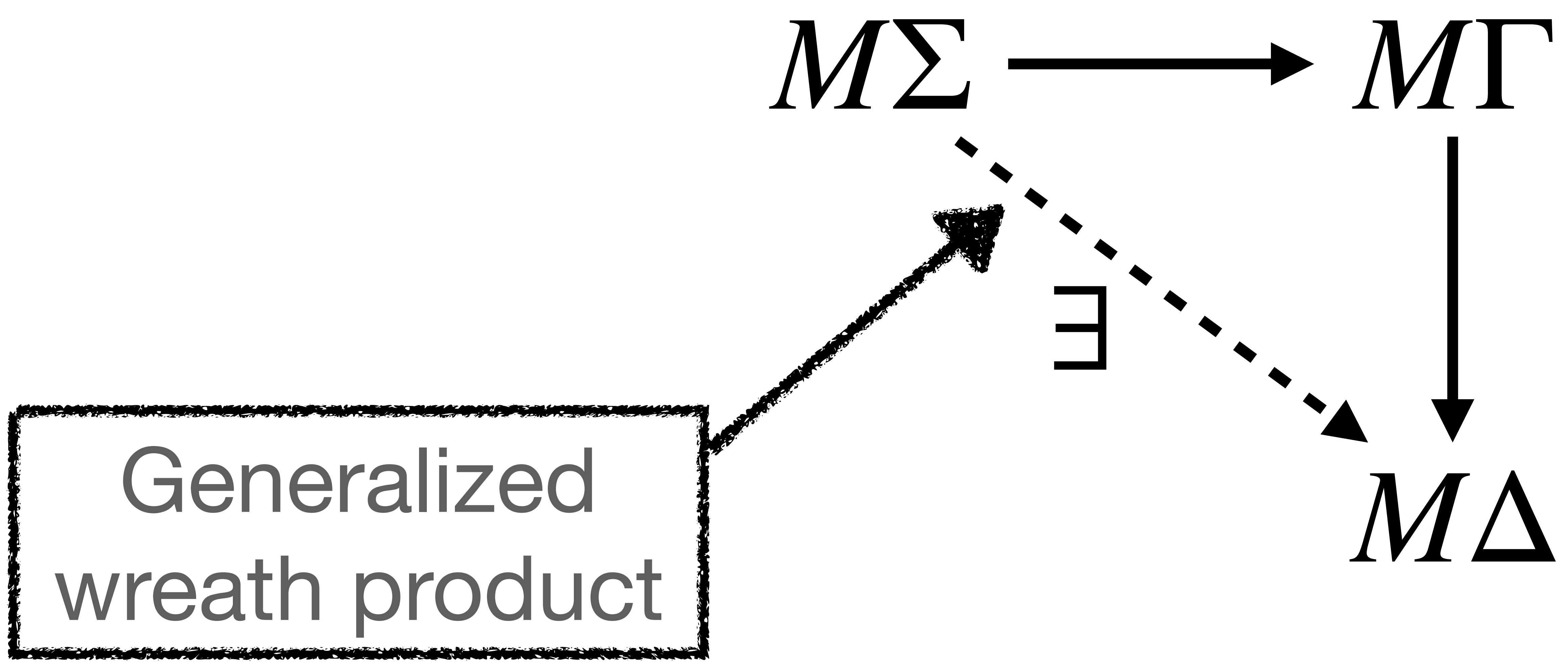

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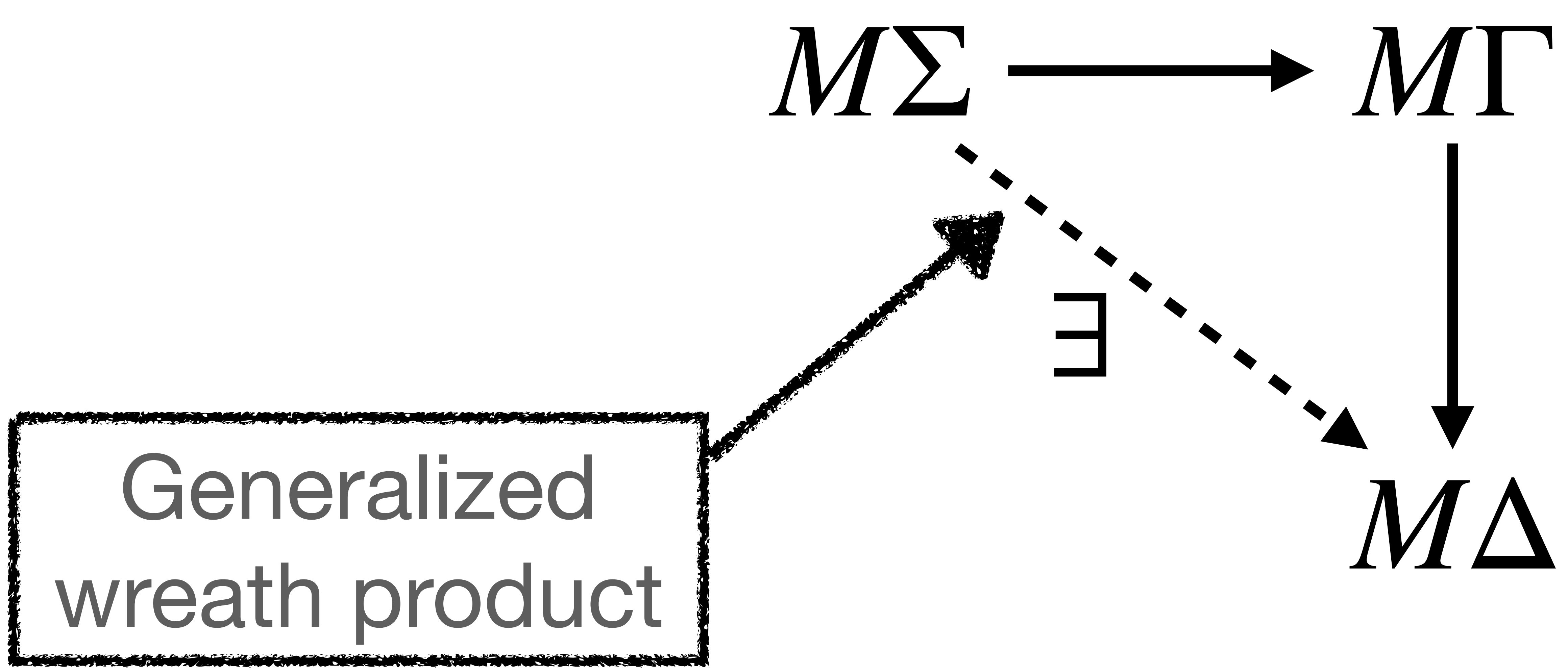

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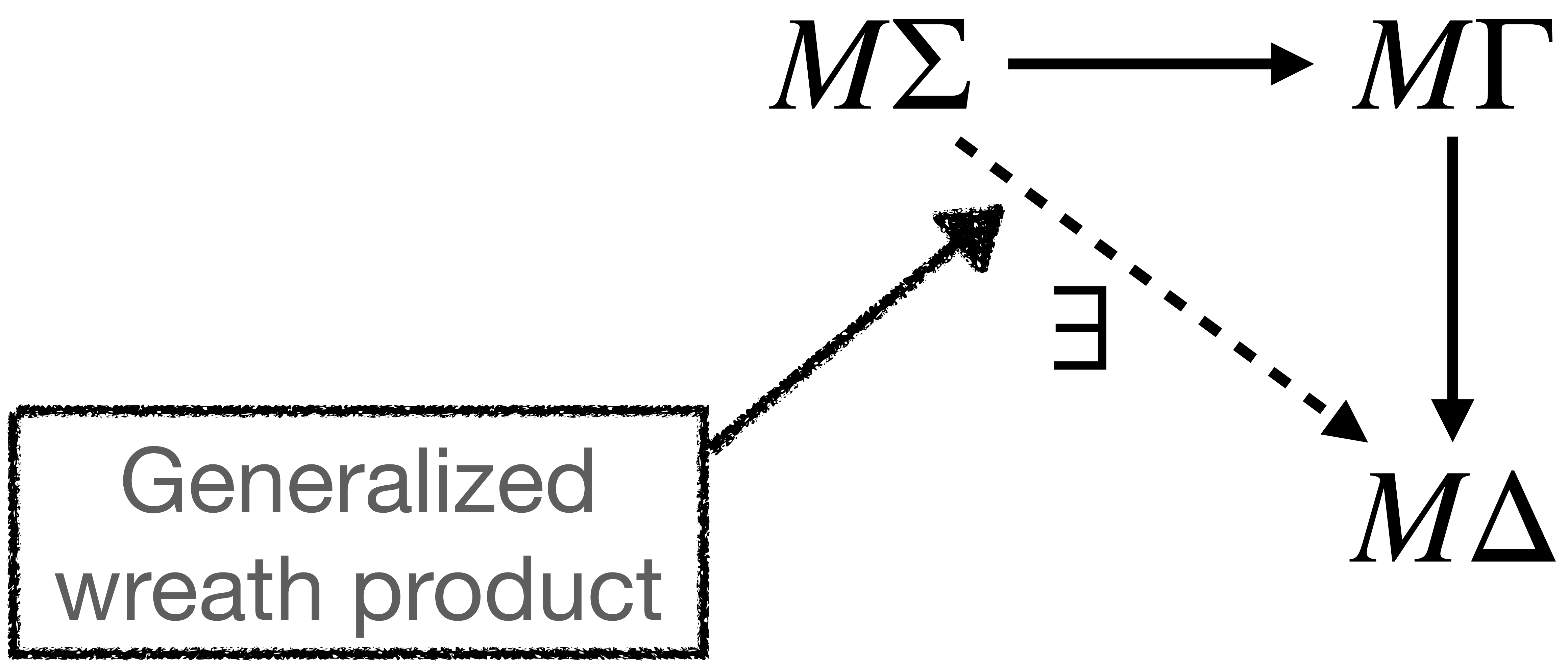

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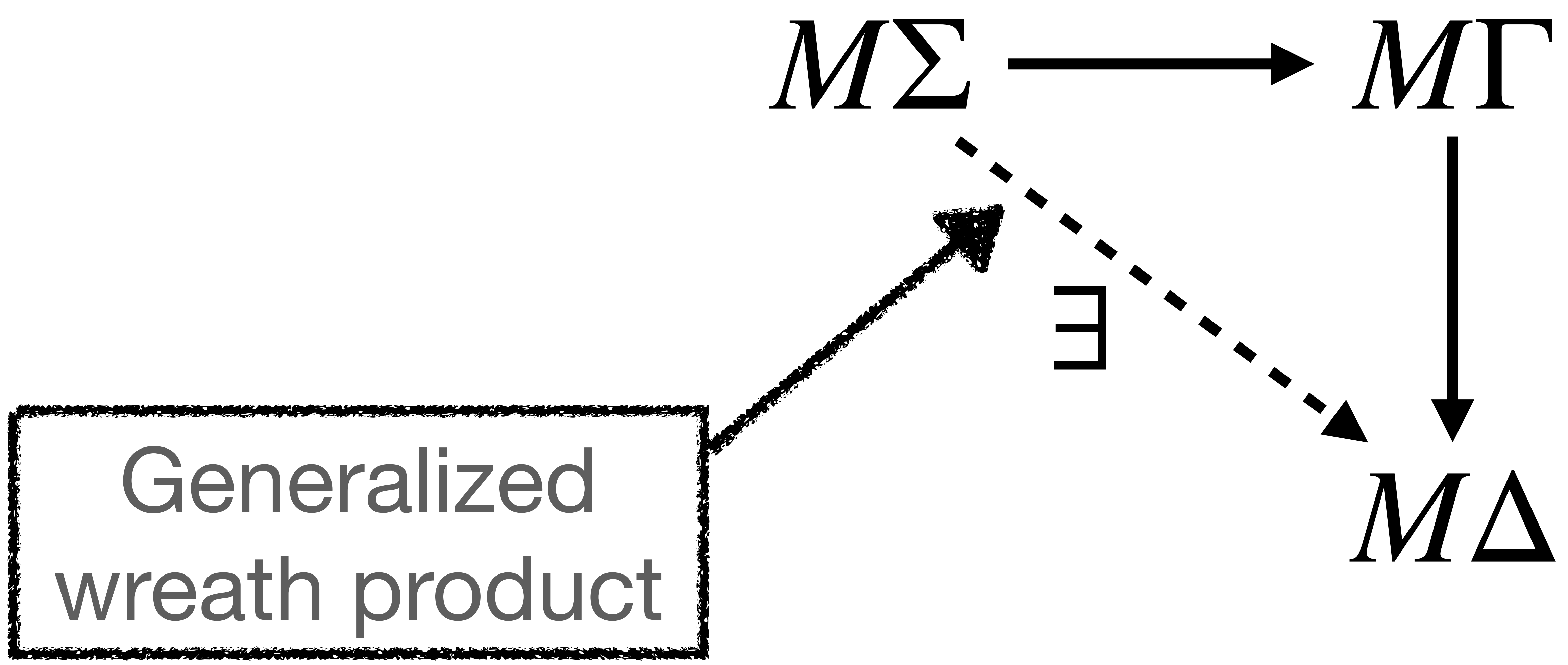

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Thank you!