An algebraic theory for single-use transducers
over data words

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Single-use register Mealy machines

$$\mathbb{A}$$

Single-use register Mealy machines

$$\mathbb{A} = \{1, 2, 3 \ldots\}$$

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$$\square$$

$$\downarrow$$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$\vdash$

$\boxed{2}$

$\downarrow$

2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Shift all letters one position to the right

$\vdash \; 2$

$\boxed{1}$

$\downarrow$

$2 \;\; 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 1 \;\; 2$

Single-use register Mealy machines

Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$$\vdash \quad 2 \quad 1$$

$$\boxed{3}$$

$$\downarrow$$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

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Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$$\vdash \;\; 2 \;\; 1 \;\; 3$$

$$\boxed{3}$$

$$\downarrow$$

$$2 \;\; 1 \;\; 3 \;\; 3 \;\; 1 \;\; 2 \;\; 3 \;\; 1 \;\; 2$$

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Shift all letters one position to the right

$$\mathbb{A}^* \to (\vdash + \mathbb{A})^*$$

$$\vdash \ 2 \ 1 \ 3 \ 3 \ 1 \ 2 \ 3 \ 1$$

$$\boxed{2}$$

$$\downarrow$$

$$2 \ 1 \ 3 \ 3 \ 1 \ 2 \ 3 \ 1 \ 2$$

Single-use register Mealy machines

Replace all letters with the first letter

$$\mathbb{A}^* \to \mathbb{A}^*$$

$$\square$$

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$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

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Replace all letters with the first letter

$$\mathbb{A}^* \to \mathbb{A}^*$$

2

$\boxed{2}$

$\downarrow$

2 1 3 3 1 2 3 1 2

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Replace all letters with the first letter

$\mathbb{A}^* \to \mathbb{A}^*$

$2 \quad 2$

$\square$

$\downarrow$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

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Replace all letters with the first letter

$$\mathbb{A}^* \to \mathbb{A}^*$$

$$2 \quad 2$$

$$\boxed{?}$$

$$\downarrow$$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

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Replace all letters with the first letter

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$$2 \quad 2$$

$$\boxed{?}$$

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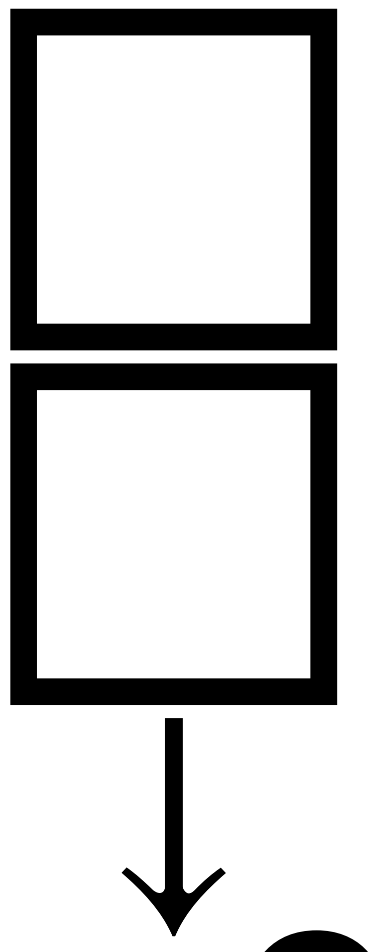
Replace all letters with the first letter

$$\mathbb{A}^* \to \mathbb{A}^*$$



2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Replace all letters with the first letter

$\mathbb{A}^* \to \mathbb{A}^*$

2

$\boxed{2}$

$\boxed{2}$

$\downarrow$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

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Replace all letters with the first letter

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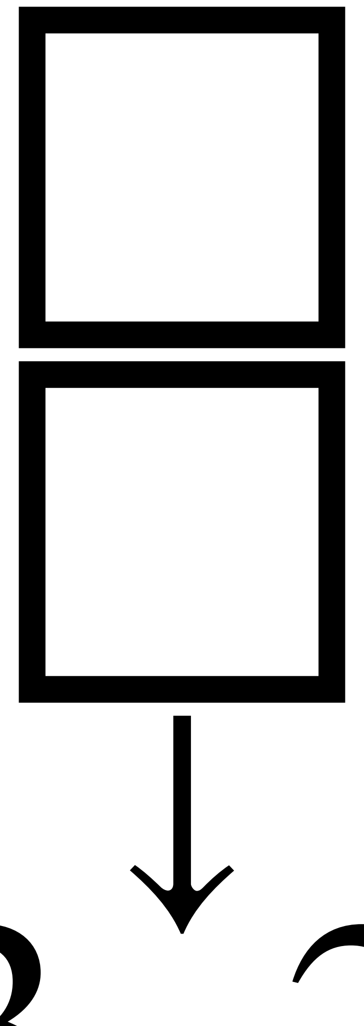
2 2 2



2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Replace all letters with the first letter

$\mathbb{A}^* \to \mathbb{A}^*$

$$2 \quad 2 \quad 2$$

$$?$$

$$\downarrow$$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

Single-use register Mealy machines

Compare every letter with the previous one

$$\mathbb{A}^* \to \{Y, N\}^*$$

$$\square$$

$$\downarrow$$

2 1 3 3 1 2 3 1 2

Single-use register Mealy machines

Compare every letter with the previous one

$$\mathbb{A}^* \to \{Y, N\}^*$$

N

$\boxed{2}$

$\downarrow$

$$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$$

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Compare every letter with the previous one

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$$N$$

$$\boxed{2}$$

$$\downarrow$$

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$N \quad N$

$\boxed{1}$

$\downarrow$

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$$\boxed{3}$$

$$\downarrow$$

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$$\square$$

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$$N \;\; N \;\; N \;\; Y$$

$$\boxed{3}$$

$$\downarrow$$

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$$N \quad N \quad N \quad Y \quad N \quad N \quad N \quad N \quad N$$

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$$\boxed{2}$$

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$$\square$$

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$$\square$$

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Single-use register Mealy machines

Compare every letter with the first letter

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$Y \quad N$

$$\begin{array}{c} ? \\ \downarrow \end{array}$$

$2 \quad 1 \quad 3 \quad 3 \quad 1 \quad 2 \quad 3 \quad 1 \quad 2$

Single-use register Mealy machines

Shift all letters one position to the right

~~Replace all letters with the first letter~~

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Why single-use transducers?

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- Single-use Mealy machines admit Krohn-Rhodes decompositions

Bojanczyk, S. 2020

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  4. Compositions of two-way primes with atoms

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- Single-use Mealy machines admit Krohn-Rhodes decompositions
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    1. Single-use two-way automata
    2. Single-use copyless SSTs
    3. Regular list functions with atoms
    4. Compositions of two-way primes with atoms
- Single-use automata are equivalent to orbit-finite semigroups

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Orbit-finite semigroups

$$\mathbb{A}^2 \qquad\qquad P_{fin}(\mathbb{A})$$

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- Set with one associative operations

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$$\mathbb{A}^2$$

Semigroups and languages

$$S, \quad h : \Sigma \to S, \quad \lambda : S \to \{Y, N\}$$

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Semigroups and languages

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The first and the last letters are equal

Semigroups and transducers

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Shift all letters one position to the right

Semigroups and transducers

Replace all letters with the first letter

Compare every letter with the first letter

Locality restriction

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$$\lambda(xey) = \lambda(\pi(x)ey)$$

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- y is a prefix of e ($e \cdot b = y$, for some b)

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Local orbit-finte semigroup transductions $\simeq$ Single-use Mealy machines

Research directions

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- Rational single-use functions

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Thank you!