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A NEW PUMPING METHOD FOR 2-VASS

2-VASS model

A **2-VASS** stands for a 2-D Vector Addition System with States. It is a state graph with $\mathbb{Z}^2$ vectors on its edges.

Def.: a **2-VASS** V is a pair (Q, T), where $T \subseteq Q \times \mathbb{Z}^2 \times Q$.

Def.: a **run** π in 2-VASS is any walk in the state graph such that its all prefix sums are in $\mathbb{N}^2$.

In this paper, we aimed to get a better intuition on how VASS works under the hood. Although it is a popular model, there is still much unknown about it. We showed that any 2-VASS run belongs to one of two classes that have a nice geometric interpretation.

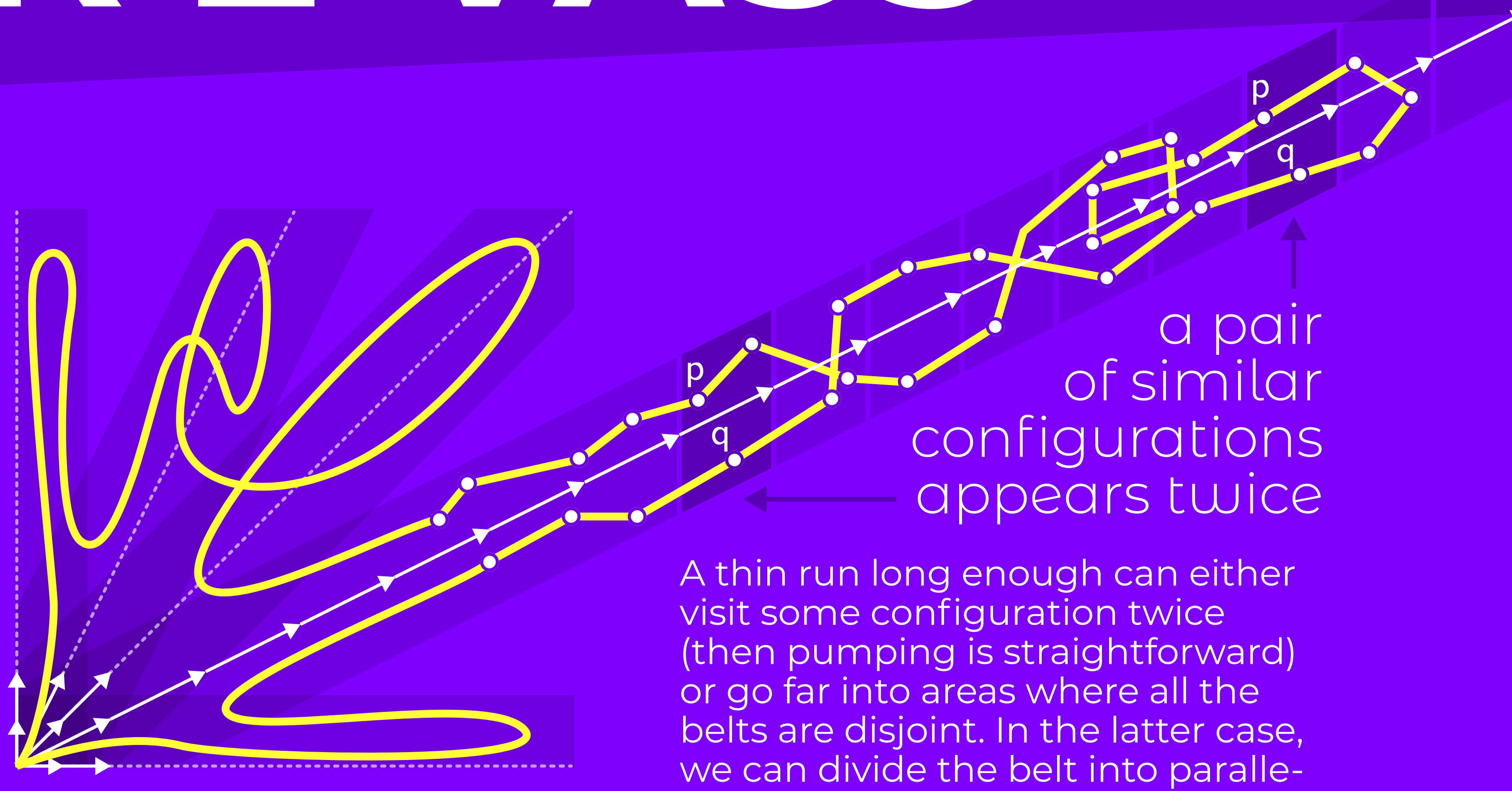
thin runs

Intuitively, a thin run behaves almost like 1-D object. Its definition uses a constant M that depends only on VASS V.

Consider the set $[M]^2$ of all vectors with natural coefficients $\leq$ M. It defines a set of half-lines starting in the origin of the coordinate system.

Def. we call a run π **thin** iff each point it visits lies in the distance at most M from some of the half-lines.

Example In the picture on the right M was set to 3. The yellow run (states were omitted) has to stay close to the five dashed half-lines determined by the white vectors.



Belts go in directions of all 2-D vectors from $[M]^2$ and have width M.

A thin run long enough can either visit some configuration twice (then pumping is straightforward) or go far into areas where all the belts are disjoint. In the latter case, we can divide the belt into parallelograms using its directional vector. A repeated pair of states marks a part of the run that can be pumped.

pumping thin runs

main theorem

FOR ANY RUN π

1-D LIKE

if π is not **thin**

EASILY PUMPABLE

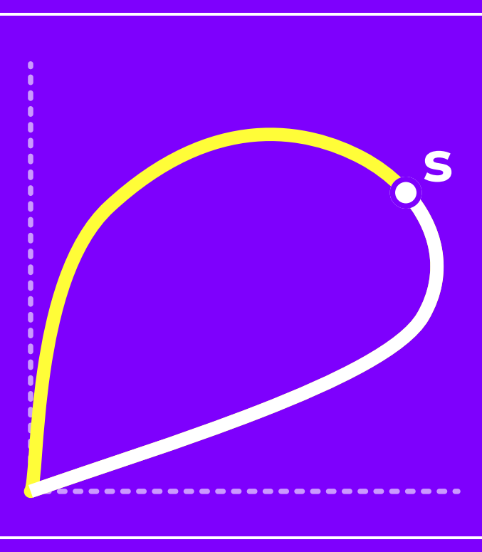
2-D LIKE

then it is **thick**

ADMITS OUR PUMPING

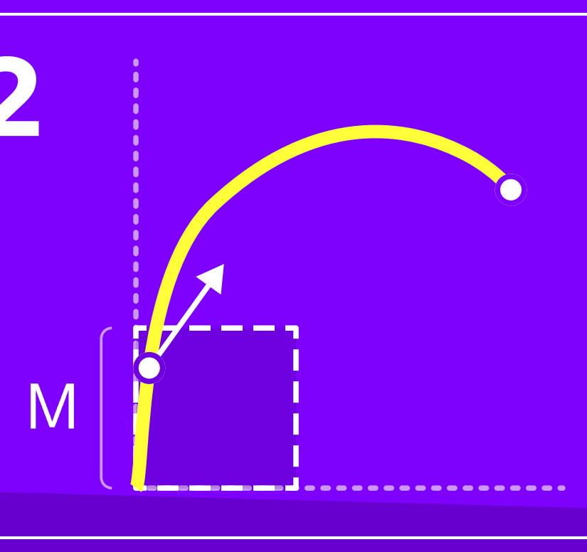
thick run

The definition of a thick run is a bit more involved. In this poster, we omit some details and present its pictorial form.

1


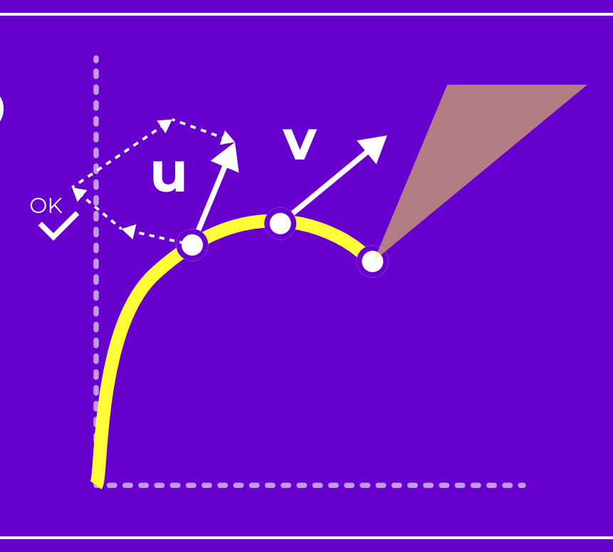
two symmetrical parts

First of all, thick run splits into two parts, both of which have to meet criteria 2-4. For symmetry, we consider the second part in a reversed direction, such that they both start in point (0, 0) and end in some midpoint **s**.

2


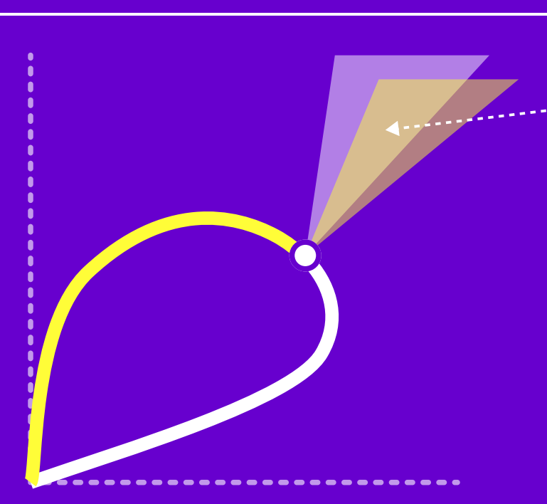
positive 'pump' close to origin

Close to (0, 0), the thick run has to visit a point in which one may insert some cycle with a strictly* positive effect. Intuitively, this will let us not to worry about going to negative coordinates in the later part of the run: adding the positive vector will make up for that.

3


two vectors spanning a cone

Additionally, the run has to visit two more points in which some loops with effects **u** and **v** could be pasted. The difference is that now we allow loops going bellow 0 thanks to **2**. Positive linear combinations of **u** and **v** form a cone, in which **s** can be freely moved.

4


nonempty intersection of cones

The final requirement is that the two cones coming from both parts of the run have to have a nontrivial intersection. This is the key property that ensures there is a way to pump both parts such that they always agree at the midpoint **s**.

pumping thick runs

The definition of a thick run is designed to allow 2-D pumping. A total of 6 vectors appears in it. We have shown there is always a way to insert some multiplicity of those vectors while preserving the validity of the run.

Proof sketch To prove that, express the following two requirements as a system of linear inequalities:

- ends of both parts of the run meet at the same point,
- cycles used to 'produce' vectors **u** and **v** do not go bellow 0.

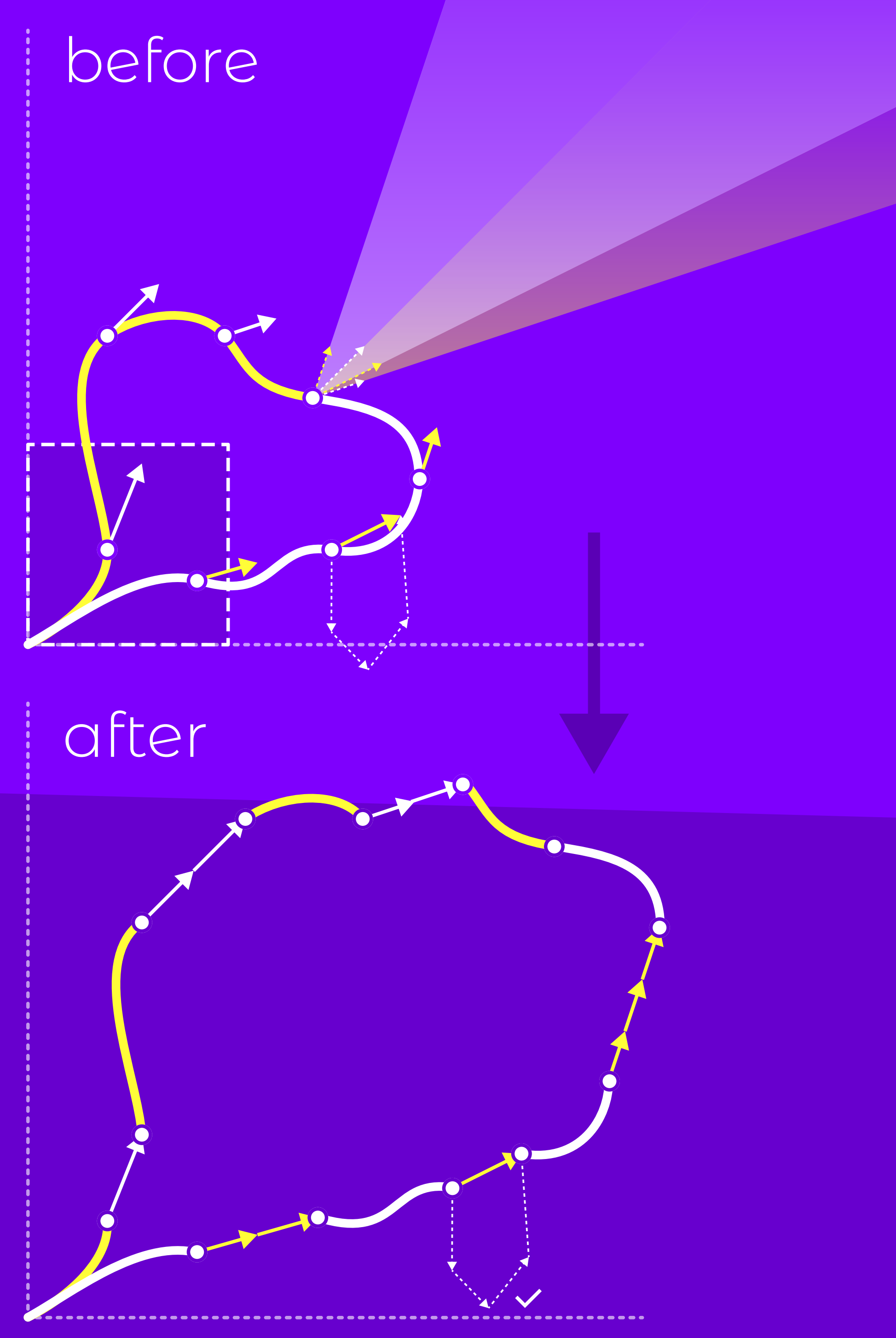
We then show that this system has a small, non-negative integer solution.

motivation

Our original goal was to solve the regular separation problem for 2-VASS languages. Although that question remains open, we believe the pumping method we propose may prove useful in resolving it.

Additionally, using our method, we can prove a short run property: *If there exists a run, there is also a short one of size* $p(|Q| \cdot \|V\|)^{|Q|}$ (where p is some fixed polynomial)

It is an open question if similar pumping schemes exist in 3-D.



Inserting new loops preserves the validity of the run.