

COMPARISON OF WEAK AND STRONG MOMENTS FOR VECTORS WITH INDEPENDENT COORDINATES

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ABSTRACT. We show that for $p \geq 1$, the p -th moment of suprema of linear combinations of independent centered random variables are comparable with the sum of the first moment and the weak p -th moment provided that $2q$ -th and q -th integral moments of these variables are comparable for all $q \geq 2$. The latest condition turns out to be necessary in the i.i.d. case.

1. INTRODUCTION AND MAIN RESULTS

In many problems arising in probability theory and its applications one needs to study variables of the form $S = \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|$, where X_1, \dots, X_n are independent random variables and T is a non-empty subset of \mathbb{R}^n . In particular it is of interest to estimate tails of S . Such estimates are strictly related to bounds for L_p -norms of S (i.e. $\|S\|_p := (\mathbb{E}|S|^p)^{1/p}$) for $p \geq 1$ (see Corollary 1.3 and its proof in Section 4 below). There is a trivial lower estimate:

$$(1.1) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \geq \max \left\{ \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|, \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right\}.$$

It turns out that in some situations this obvious lower bound may be reversed, i.e. there exist numerical constants C_1 and C_2 such that

$$(1.2) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq C_1 \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| + C_2 \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p}.$$

This is for example the case (with $C_1 = 1$), when X_i are normally distributed. This is an easy consequence of the Gaussian concentration (cf. Chapter 3 of [11]). Dilworth and Montgomery-Smith [3] established the inequality (1.2) for X_i being symmetric Bernoulli random variables. This result was generalized in [7] to symmetric variables with logarithmically concave tails and in [10, Theorem 2.3] to symmetric random variables such that $\|X_i\|_q \leq C_q^p \alpha \|X_i\|_p$ for all $q \geq p \geq 2$.

The main result of this paper is the following.

Theorem 1.1. *Let X_1, \dots, X_n be independent mean zero random variables with finite moments such that*

$$(1.3) \quad \|X_i\|_{2p} \leq \alpha \|X_i\|_p \quad \text{for every } p \geq 2 \text{ and } i = 1, \dots, n,$$

where α is a finite positive constant. Then for every $p \geq 1$ and every non-empty set $T \subset \mathbb{R}^n$ we have

$$(1.4) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq C(\alpha) \left[\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right],$$

where $C(\alpha)$ is a constant which depends only on α .

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It turns out that Theorem 1.1 may be reversed in the i.i.d. case.

Theorem 1.2. *Let X_1, X_2, \dots be i.i.d. random variables. Assume that there exists a constant L such that for every $p \geq 1$, every n and every non-empty set $T \subset \mathbb{R}^n$ we have*

$$(1.5) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq L \left[\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right].$$

Then

$$(1.6) \quad \|X_1\|_{2p} \leq \alpha(L) \|X_1\|_p \quad \text{for } p \geq 2,$$

where $\alpha(L)$ is a constant which depends only on $L \geq 1$.

It will be clear from the proof of Theorem 1.2 that it suffices to assume (1.5) for $T = \{\pm e_j : j \in \{1, \dots, n\}\}$ only, where $\{e_1, \dots, e_n\}$ is the canonical basis of \mathbb{R}^n .

The comparison of weak and strong moments (1.4) yields also a deviation inequality for $\sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|$.

Corollary 1.3. *Assume X_1, X_2, \dots satisfy the assumptions of Theorem 1.1. Then for any $u \geq 0$ and any non-empty set T in \mathbb{R}^n ,*

$$(1.7) \quad \mathbb{P} \left(\sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| \geq C_1(\alpha) \left[u + \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| \right] \right) \leq C_2(\alpha) \sup_{t \in T} \mathbb{P} \left(\left| \sum_{i=1}^n t_i X_i \right| \geq u \right),$$

where constants $C_1(\alpha)$ and $C_2(\alpha)$ depend only on the constant α in (1.3).

Another consequence of the main theorem is the following Khintchine-Kahane type inequality.

Corollary 1.4. *Assume X_i , $1 \leq i \leq n$ satisfy the assumptions of Theorem 1.1. Then for any $p \geq q \geq 2$ and any non-empty set T in \mathbb{R}^n we have,*

$$\left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq C_3(\alpha) \left(\frac{p}{q} \right)^{\max\{1/2, \log_2 \alpha\}} \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^q \right)^{1/q}$$

where a constant $C_3(\alpha)$ depend only on the constant α in (1.3).

We postpone proofs of the above results and first present a number of remarks and open questions.

Remark 1.5. Exponent $\max\{1/2, \log_2 \alpha\}$ in Corollary 1.4 is optimal.

Indeed, since $\|g\|_p \sim \sqrt{p/e}$ as $p \rightarrow \infty$ one cannot go below $1/2$ by the central limit theorem.

To see that $\log_2 \alpha$ term cannot be improved it is enough to consider $\alpha > \sqrt{2}$. Let $r = 1/\log_2 \alpha \in (0, 2)$ and let X be a symmetric random variable given by $\mathbb{P}(|X| \geq t) = e^{-t^r}$ (with $2 > r > 0$), i.e. $X = |\mathcal{E}|^{1/r} \text{sgn } \mathcal{E}$, where \mathcal{E} has the symmetric exponential distribution. By Stirling's formula $\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} e^{f(x)}$ with $f(x) \in (0, 1/12)$ for $x \geq 1$, so for $p \geq 2$,

$$\frac{\|X\|_{2p}}{\|X\|_p} = \frac{\Gamma\left(\frac{2p}{r} + 1\right)^{1/(2p)}}{\Gamma\left(\frac{p}{r} + 1\right)^{1/p}} \leq 2^{1/r} \left(\frac{r}{\pi p}\right)^{1/(4p)} e^{1/(24p)} \leq 2^{1/r} = \alpha.$$

Moreover, $\|X\|_p \sim \left(\frac{p}{er}\right)^{1/r}$ for $p \rightarrow \infty$, so the assertion of Corollary 1.4 cannot hold with any exponent better than $\log_2 \alpha$.

Remark 1.6. If the variables X_i are symmetric then the term $\mathbb{E} \sup_{t \in T} |\sum_{i=1}^n t_i X_i|$ in (1.4) may be replaced by $\mathbb{E} \sup_{t \in T} |\sum_{i=1}^n t_i X_i|$.

Proof. Let s be any point in T . Then $T \subset T - T + s$, so by the triangle inequality

$$\left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq \left(\mathbb{E} \sup_{t \in T-T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} + \left(\mathbb{E} \left| \sum_{i=1}^n s_i X_i \right|^p \right)^{1/p}.$$

Estimate (1.4) applied to the set $T - T$ yields

$$\left(\mathbb{E} \sup_{t \in T-T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq C(\alpha) \left[\mathbb{E} \sup_{t \in T-T} \left| \sum_{i=1}^n t_i X_i \right| + \sup_{t \in T-T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right].$$

The set $T - T$ is symmetric, so

$$\mathbb{E} \sup_{t \in T-T} \left| \sum_{i=1}^n t_i X_i \right| = \mathbb{E} \sup_{t \in T-T} \sum_{i=1}^n t_i X_i \leq 2 \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i,$$

where the last estimate follows, since $(X_i)_{i=1}^n$ and $(-X_i)_{i=1}^n$ are equally distributed. Moreover,

$$\sup_{t \in T-T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq 2 \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p},$$

what finishes the proof of the remark. \square

Remark 1.7. If the variables X_i are not centered then (1.4) holds provided that the assumption (1.3) is replaced by

$$\|X_i - \mathbb{E}X_i\|_{2p} \leq \alpha \|X_i - \mathbb{E}X_i\|_p \quad \text{for } p \geq 2 \text{ and } i = 1, \dots, n.$$

Proof. We have

$$\left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i (X_i - \mathbb{E}X_i) \right|^p \right)^{1/p} + \sup_{t \in T} \left| \sum_{i=1}^n t_i \mathbb{E}X_i \right|.$$

Theorem 1.1 applied to centered variables $X_i - \mathbb{E}X_i$, $i = 1, \dots, n$, yields

$$\begin{aligned} & \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i (X_i - \mathbb{E}X_i) \right|^p \right)^{1/p} \\ & \leq C(\alpha) \left[\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i (X_i - \mathbb{E}X_i) \right| + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i (X_i - \mathbb{E}X_i) \right|^p \right)^{1/p} \right]. \end{aligned}$$

To conclude it is enough to observe that

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i (X_i - \mathbb{E}X_i) \right| & \leq \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| + \sup_{t \in T} \left| \sum_{i=1}^n t_i \mathbb{E}X_i \right|, \\ \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i (X_i - \mathbb{E}X_i) \right|^p \right)^{1/p} & \leq \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} + \sup_{t \in T} \left| \sum_{i=1}^n t_i \mathbb{E}X_i \right|, \end{aligned}$$

and

$$\sup_{t \in T} \left| \sum_{i=1}^n t_i \mathbb{E}X_i \right| \leq \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|.$$

\square

Open questions. For Gaussian random vectors (1.2) holds with $C_1 = 1$. This is also the case for X_i symmetric, independent with log-concave distributions [6, Remark 3.16 and Corollary 2.19]. However, we do not know the general conditions for the distributions of X_i which are sufficient for (1.2) to hold with $C_1 = 1$.

It is of interest to study the comparison of weak and strong moments for random vectors $X = (X_1, \dots, X_n)$ with dependent coordinates. A natural and important class to investigate in this context are vectors with log-concave distributions (cf. [2] for an up to date survey of properties of such vectors). Paouris [12] showed that (1.2) holds for log-concave vectors and sets T being balls in Euclidean spaces (see also [1]). This was generalized in [9] to balls in L_r -spaces with $1 \leq r < \infty$. Unfortunately there are very few classes of log-concave vectors such that (1.2) is known to be satisfied for all sets T – this includes vectors uniformly distributed on l_r^n -balls (with $1 \leq r \leq \infty$). [6, Remark 3.16 and Theorem 5.27], or more generally vectors with densities of the form $\exp(-\varphi(\|x\|_r))$, where $\varphi: [0, \infty) \rightarrow (-\infty, \infty)$ is non-decreasing and convex, and $1 \leq r \leq \infty$ [8, Proposition 6.5].

The organization of this paper is as follows. In Section 2 we prove Theorem 1.1 for unconditional sets T only. Using this result we generalize it to the case of an arbitrary T in Section 3. In Section 4 we prove Corollaries 1.3 and 1.4. Finally, in Section 5 we present the proof of Theorem 1.2.

Throughout this paper by a letter C we denote universal constants and by $C(\alpha)$ constants depending only on the parameter α . The values of the constants C , $C(\alpha)$ may differ at each occurrence. We will also frequently work with a Bernoulli sequence ε_i of i.i.d. symmetric random variables taking values ± 1 . We assume that variables ε_i are independent of other random variables.

2. THE CASE OF UNCONDITIONAL SETS

In this section we show that Theorem 1.1 holds under additional assumptions that the set T is unconditional and the variables X_i are symmetric. Recall that a set T in \mathbb{R}^n is called *unconditional* if it is symmetric with respect to the coordinate axes, i.e. $(\eta_i t_i)_{i=1}^n \in T$ for any $t = (t_i)_{i=1}^n \in T$ and any choice of signs $\eta_1, \dots, \eta_n \in \{-1, 1\}$.

Proposition 2.1. *Let $r \in (0, 1)$ and $L \geq 1$. Assume that variables Y_1, \dots, Y_n are independent and symmetric and*

$$(2.1) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i Y_i \right|^p \right)^{1/p} \leq L \left[\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i Y_i + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i Y_i \right|^p \right)^{1/p} \right]$$

for all $p \geq 1$ and all unconditional sets T . Then variables $X_i := |Y_i|^{1/r} \operatorname{sgn} Y_i$ satisfy

$$(2.2) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq (2L)^{1/r} \left[\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right]$$

for all $p \geq 1$ and all unconditional sets $T \subset \mathbb{R}^n$.

Proof. Definition of X_i and unconditionality of T yield

$$\sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| = \sup_{t \in T} \left| \sum_{i=1}^n t_i |Y_i|^{1/r} \operatorname{sgn} Y_i \right| = \sup_{t \in T} \sum_{i=1}^n |t_i| |Y_i|^{1/r}.$$

Let $s = (1 - r)^{-1}$ and let B_s^n denote the unit ball of ℓ_s^n . Then $1/s + r = 1$ and by Hölder's duality we have

$$\sup_{t \in T} \left| \sum_{i=1}^n |t_i| |Y_i|^{1/r} \right|^r = \sup_{t \in T} \sup_{u \in B_s^n} \sum_{i=1}^n u_i |t_i|^r Y_i = \sup_{t \in T} \sum_{i=1}^n t_i Y_i,$$

where

$$T_r := \{(u_i |t_i|^r)_{i=1}^n : t \in T, u \in B_s^n\}$$

is unconditional in \mathbb{R}^n . Therefore (2.1) applied with p/r and T_r instead of p and T yields

$$\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \leq L^{p/r} \left[\mathbb{E} \sup_{t \in T_r} t_i Y_i + \sup_{t \in T_r} \left(\mathbb{E} \left| \sum_{i=1}^n t_i Y_i \right|^{p/r} \right)^{r/p} \right]^{p/r}.$$

We have

$$\mathbb{E} \sup_{t \in T_r} \sum_{i=1}^n t_i Y_i = \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^r \leq \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{r/p}.$$

Moreover,

$$\begin{aligned} \sup_{t \in T_r} \left(\mathbb{E} \left| \sum_{i=1}^n t_i Y_i \right|^{p/r} \right)^{r/p} &\leq \sup_{t \in T} \left(\mathbb{E} \sup_{u \in B_s^n} \left| \sum_{i=1}^n u_i |t_i|^r Y_i \right|^{p/r} \right)^{r/p} \\ &= \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n |t_i| |X_i| \right|^p \right)^{r/p} = \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i |X_i| \right|^p \right)^{r/p}. \end{aligned}$$

Estimates above together with the inequality $(a+b)^{1/r} \leq 2^{1/r-1}(a^{1/r} + b^{1/r})$ yield

$$\left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq \frac{1}{2} (2L)^{1/r} \left[\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i |X_i| \right|^p \right)^{1/p} \right].$$

Hence, in order to prove (2.2) it suffices to show that

$$(2.3) \quad \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i |X_i| \right|^p \right)^{1/p} \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i + 2 \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p}.$$

Let (X'_1, \dots, X'_n) be an independent copy of (X_1, \dots, X_n) . By the triangle inequality for the p -th integral norm and Jensen's inequality we get

$$\begin{aligned} \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i |X_i| \right|^p \right)^{1/p} &\leq \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i (|X_i| - \mathbb{E}|X'_i|) \right|^p \right)^{1/p} + \sup_{t \in T} \left| \mathbb{E} \sum_{i=1}^n t_i |X_i| \right| \\ &\leq \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i (|X_i| - |X'_i|) \right|^p \right)^{1/p} + \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i |X_i| \right| \\ (2.4) \quad &= \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i (|X_i| - |X'_i|) \right|^p \right)^{1/p} + \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i, \end{aligned}$$

where the equation follows by the unconditionality of T .

Let $(\varepsilon_i)_{i=1}^n$ be the Bernoulli sequence, independent of all X_i and X'_i . Then the sequence $(|X_i| - |X'_i|)_{i=1}^n$ has the same distribution as $(\varepsilon_i (|X_i| - |X'_i|))_{i=1}^n$ and for every $t \in \mathbb{R}^n$,

$$\begin{aligned} \left(\mathbb{E} \left| \sum_{i=1}^n t_i (|X_i| - |X'_i|) \right|^p \right)^{1/p} &= \left(\mathbb{E} \left| \sum_{i=1}^n t_i \varepsilon_i (|X_i| - |X'_i|) \right|^p \right)^{1/p} \\ &\leq \left(\mathbb{E} \left| \sum_{i=1}^n t_i \varepsilon_i |X_i| \right|^p \right)^{1/p} + \left(\mathbb{E} \left| \sum_{i=1}^n t_i \varepsilon_i |X'_i| \right|^p \right)^{1/p} \\ (2.5) \quad &= 2 \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p}. \end{aligned}$$

Putting (2.4) and (2.5) together we get (2.3), what completes the proof of (2.2). \square

Corollary 2.2. *Let X_1, \dots, X_n be independent symmetric random variables with finite moments such that*

$$(2.6) \quad \|X_i\|_{2p} \leq \alpha \|X_i\|_p \quad \text{for } p \geq 2 \text{ and } i = 1, \dots, n,$$

where α is a finite positive constant. Then for every $p \geq 1$ and every unconditional set $T \subset \mathbb{R}^n$ we have

$$(2.7) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq C(\alpha) \left[\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right],$$

where $C(\alpha)$ is a constant, which depends only on α .

Proof. Let us first note, that the assumption (2.6) applied k times yields that

$$\|X_i\|_{2^k p} \leq \alpha^k \|X_i\|_p \quad \text{for } p \geq 2.$$

Therefore

$$\|X_i\|_q \leq \alpha^{\lceil \log_2(\frac{q}{p}) \rceil} \|X_i\|_p \leq \alpha \left(\frac{q}{p} \right)^{\log_2 \alpha} \|X_i\|_p \quad \text{for } q \geq p \geq 2.$$

Let $Y_i := |X_i|^{1/\log_2 \alpha} \operatorname{sgn} X_i$. Then $X_i = |Y_i|^{1/r} \operatorname{sgn} Y_i$ with $r := \frac{1}{\log_2 \alpha}$ and

$$(2.8) \quad \|Y_i\|_q \leq 2^{\frac{q}{p}} \|Y_i\|_p \quad \text{for } q \geq p \geq 2 \log_2 \alpha.$$

If $\alpha \leq 2$ we have

$$(2.9) \quad \|Y_i\|_q \leq 2^{\frac{q}{p}} \|Y_i\|_p \quad \text{for } q \geq p \geq 2.$$

Otherwise, take $2 \log_2 \alpha \geq q \geq p \geq 2$. Then by Hölder's inequality and (2.8) with exponents $\frac{p(q-1)}{p-1}$ and q we get

$$\|Y_i\|_q^q = \mathbb{E} |Y_i| |Y_i|^{q-1} \leq (\mathbb{E} |Y_i|^p)^{\frac{1}{p}} \left(\mathbb{E} |Y_i|^{\frac{p(q-1)}{p-1}} \right)^{\frac{p-1}{p}} \leq \|Y_i\|_p \|Y_i\|_q^{q-1} \left(2^{\frac{p(q-1)}{q(p-1)}} \right)^{q-1}.$$

Observe that

$$\left(2^{\frac{p(q-1)}{q(p-1)}} \right)^{q-1} \leq 4^{q-1} \leq \frac{1}{4} \alpha^4,$$

so

$$\|Y_i\|_q \leq \frac{1}{4} \alpha^4 \|Y_i\|_p \quad \text{for } 2 \log_2 \alpha \geq q \geq p \geq 2.$$

Thus for any value of α we get

$$\|Y_i\|_q \leq \max \left\{ 2, \frac{1}{2} \alpha^4 \right\} \frac{q}{p} \|Y_i\|_p \quad \text{for } q \geq p \geq 2.$$

Hence, by [10, Theorem 2.3] the variables Y_1, \dots, Y_n satisfy (2.1) (in fact for arbitrary, not only unconditional sets T) and the assertion follows by Proposition 2.1. \square

3. SYMMETRIZATION ARGUMENT

We will use the following proposition to prove that we may skip the unconditionality assumption in Corollary 2.2.

Proposition 3.1. *Let $(X_i)_{i=1}^n$ be a sequence of independent random variables with finite second moments and let $(\varepsilon_i)_{i=1}^n$ be a Bernoulli sequence independent of $(X_i)_{i=1}^n$. Then for any $T \subset \mathbb{R}^n$ and $p \geq 1$,*

$$(3.1) \quad \mathbb{E}_X \sup_{t \in T} \left(\mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i \varepsilon_i X_i \right|^p \right)^{1/p} \leq C \left[\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i X_i + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i \varepsilon_i X_i \right|^p \right)^{1/p} \right].$$

Proof. Since this is only a matter of normalization we may and will assume that $\mathbb{E}X_i^2 = 1$ for all i .

Let m be such an integer that $2m \leq p < 2(m+1)$. Then, by the symmetry of X_i, ε_i , and the independence of $X_1, \dots, X_n, \varepsilon_1, \dots, \varepsilon_n$ we have

$$\begin{aligned} \left\| \sum_{i=1}^n t_i \varepsilon_i X_i \right\|_p &\geq \left\| \sum_{i=1}^n t_i \varepsilon_i X_i \right\|_{2m} \\ &= \left(\sum_{i_1+\dots+i_n=m} c_{i_1, \dots, i_n} t_1^{2i_1} \dots t_n^{2i_n} \mathbb{E}X_1^{2i_1} \dots \mathbb{E}X_n^{2i_n} \right)^{1/2m} \\ &\geq \left(\sum_{i_1+\dots+i_n=m} c_{i_1, \dots, i_n} t_1^{2i_1} \dots t_n^{2i_n} \right)^{1/2m} \\ &= \left(\sum_{i_1+\dots+i_n=m} c_{i_1, \dots, i_n} t_1^{2i_1} \dots t_n^{2i_n} \mathbb{E}\varepsilon_1^{2i_1} \dots \mathbb{E}\varepsilon_n^{2i_n} \right)^{1/2m} = \left\| \sum_{i=1}^n t_i \varepsilon_i \right\|_{2m}, \end{aligned}$$

where

$$c_{i_1, \dots, i_n} = \frac{(2i_1 + \dots + 2i_n)!}{(2i_1)! \dots (2i_n)!}.$$

Moreover by the result of Hitzenko [4],

$$\left\| \sum_{i=1}^n t_i \varepsilon_i \right\|_{2m} \geq \frac{1}{C} \left[\sum_{i \leq 2m} t_i^* + \sqrt{2m} \sqrt{\sum_{i > 2m} |t_i^*|^2} \right],$$

where $(t_i^*)_{i=1}^n$ denotes the non-increasing rearrangement of $(|t_i|)_{i=1}^n$.

Therefore to establish (3.1) it is enough to show that

$$(3.2) \quad \mathbb{E} \sup_{t \in T} \left(\mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i \varepsilon_i X_i \right|^p \right)^{1/p} \leq C \left(\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i X_i + pa \right),$$

where

$$a := \frac{1}{p} \sup_{t \in T} \left(\sum_{i \leq p} t_i^* + \sqrt{p} \left(\sum_{i > p} |t_i^*|^2 \right)^{1/2} \right).$$

To this end observe that since

$$\begin{aligned} \left\| \sum_{i=1}^n u_i \varepsilon_i \right\|_p &\leq C \sqrt{p} \|u\|_2, \quad \left\| \sum_{i=1}^n u_i \varepsilon_i \right\|_p \leq \|u\|_1, \\ \text{and} \quad \left\| \sum_{i=1}^n u_i \varepsilon_i \right\|_p &= \left\| \sum_{i=1}^n |u_i| \varepsilon_i \right\|_p, \end{aligned}$$

we have

$$\left\| \sum_{i=1}^n u_i \varepsilon_i \right\|_p \leq \sum_{i=1}^n (|u_i| - a)_+ + C \sqrt{p} \left(\sum_{i=1}^n \min\{u_i^2, a^2\} \right)^{1/2}.$$

Thus

$$(3.3) \quad \begin{aligned} \mathbb{E}_X \sup_{t \in T} \left(\mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i \varepsilon_i X_i \right|^p \right)^{1/p} \\ \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n (|t_i X_i| - a)_+ + C \sqrt{p} \left(\mathbb{E} \sup_{t \in T} \sum_{i=1}^n \min\{(t_i X_i)^2, a^2\} \right)^{1/2}. \end{aligned}$$

To estimate the first term above observe that

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \sum_{i=1}^n (|t_i X_i| - a)_+ \\ \leq \sup_{t \in T} \mathbb{E} \sum_{i=1}^n (|t_i X_i| - a)_+ + \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \left((|t_i X_i| - a)_+ - \mathbb{E} (|t_i X'_i| - a)_+ \right), \end{aligned}$$

where $(X'_i)_i$ is a copy of $(X_i)_i$, independent of (ε_i) and (X_i) .

Observe that for any u and i

$$\mathbb{E} (|u X_i| - a)_+ \leq |u| \mathbb{E} |X_i| \leq |u| \|X_i\|_2 = |u|$$

and, by the Cauchy-Schwarz inequality and the Markov inequality

$$\begin{aligned} \mathbb{E} (|u X_i| - a)_+ &\leq |u| \mathbb{E} |X_i| I_{\{|X_i| \geq a/|u|\}} \leq |u| \|X_i\|_2 (\mathbb{P}(|X_i| \geq a/|u|))^{1/2} \\ &\leq |u| \|X_i\|_2^2 \frac{|u|}{a} = \frac{u^2}{a}. \end{aligned}$$

Hence for any $t \in T$

$$\sum_{i=1}^n \mathbb{E} (|t_i X_i| - a)_+ \leq \sum_{i \leq p} t_i^* + \frac{1}{a} \sum_{i > p} (t_i^*)^2 \leq 2pa.$$

Moreover, by the Jensen inequality

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \left((|t_i X_i| - a)_+ - \mathbb{E} (|t_i X'_i| - a)_+ \right) \\ \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \left((|t_i X_i| - a)_+ - (|t_i X'_i| - a)_+ \right) \\ = \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i \left((|t_i X_i| - a)_+ - (|t_i X'_i| - a)_+ \right) \\ \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i (|t_i X_i| - a)_+ + \mathbb{E} \sup_{t \in T} \sum_{i=1}^n -\varepsilon_i (|t_i X'_i| - a)_+ \\ = 2 \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i (|t_i X_i| - a)_+. \end{aligned}$$

Function $x \mapsto (|x| - a)_+$ is 1-Lipschitz, so Talagrand's comparison theorem for Bernoulli processes [13, Theorem 2.1] yields

$$\mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i (|t_i X_i| - a)_+ \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i X_i.$$

Therefore

$$(3.4) \quad \mathbb{E} \sup_{t \in T} \sum_{i=1}^n (|t_i X_i| - a)_+ \leq 2pa + 2 \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i X_i.$$

Now we turn our attention to the other term in (3.3). We have

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \min\{(t_i X_i)^2, a^2\} \\ \leq \sup_{t \in T} \mathbb{E} \sum_{i=1}^n \min\{(t_i X_i)^2, a^2\} + \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \left(\min\{(t_i X_i)^2, a^2\} - \mathbb{E} \min\{(t_i X_i)^2, a^2\} \right). \end{aligned}$$

We have

$$\sum_{i=1}^n \mathbb{E} \min\{(t_i X_i)^2, a^2\} \leq \sum_{i=1}^n \min\{a^2, t_i^2 \mathbb{E} X_i^2\} \leq p a^2 + \sum_{i>p} (t_i^*)^2 \leq 2p a^2.$$

Moreover, by the Jensen inequality

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \left(\min\{(t_i X_i)^2, a^2\} - \mathbb{E} \min\{(t_i X'_i)^2, a^2\} \right) \\ \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \left(\min\{(t_i X_i)^2, a^2\} - \min\{(t_i X'_i)^2, a^2\} \right) \\ = \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i \left(\min\{(t_i X_i)^2, a^2\} - \min\{(t_i X'_i)^2, a^2\} \right) \\ \leq 2 \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i \min\{(t_i X_i)^2, a^2\}. \end{aligned}$$

Function $x \mapsto \min\{x^2, a^2\}$ is $2a$ -Lipschitz, so using the comparison theorem for Bernoulli processes again we get

$$\mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i \min\{(t_i X_i)^2, a^2\} \leq 2a \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i X_i.$$

Thus

$$(3.5) \quad p \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \min\{(t_i X_i)^2, a^2\} \leq 2p^2 a^2 + 4pa \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i X_i \\ \leq \left(2pa + \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i X_i \right)^2.$$

Estimate (3.2) follows by (3.3)-(3.5). \square

Proof of Theorem 1.1. Since it is enough to consider $T \cup (-T)$ instead of T , we may and will assume that the set T is symmetric, i.e. $T = -T$.

Assume first that the variables X_i are also symmetric. Let $\varepsilon = (\varepsilon_i)_{i=1}^n$ be a Bernoulli sequence independent of $(X_i)_{i=1}^n$. Weak and strong moments of $(\varepsilon_i)_{i=1}^n$ are comparable

$$\left(\mathbb{E} \sup_{s \in S} \left| \sum_{i=1}^n s_i \varepsilon_i \right|^p \right)^{1/p} \leq C \left[\mathbb{E} \sup_{s \in S} \left| \sum_{i=1}^n s_i \varepsilon_i \right| + \sup_{s \in S} \left(\mathbb{E} \left| \sum_{i=1}^n s_i \varepsilon_i \right|^p \right)^{1/p} \right].$$

Hence the symmetry of X_i yields

$$(3.6) \quad \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} = \left(\mathbb{E}_X \mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right)^{1/p} \\ \leq 2C \left[\left(\mathbb{E}_X \left(\mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right| \right)^p \right)^{1/p} + \left(\mathbb{E}_X \sup_{t \in T} \mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right)^{1/p} \right],$$

since $(a+b)^p \leq 2^p(a^p + b^p)$.

Since T is symmetric, we have for $x \in \mathbb{R}^n$,

$$\mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^n t_i x_i \varepsilon_i \right| = \sup_{t \in T_1} \sum_{i=1}^n t_i x_i,$$

where

$$T_1 := \{(\mathbb{E}_\varepsilon s_i(\varepsilon)\varepsilon_i)_{i=1}^n : s: \{-1, 1\}^n \rightarrow T\}$$

is an unconditional subset of \mathbb{R}^n . Estimate (2.7) applied for T_1 instead of T yields

$$\begin{aligned} & \left(\mathbb{E}_X \left(\mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right) \right)^{1/p} \\ & \leq C(\alpha) \left[\mathbb{E}_X \mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right| + \sup_{t \in T_1} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right]. \end{aligned}$$

By the symmetry of X_i we have

$$\mathbb{E}_X \mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right| = \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i.$$

Moreover,

$$T_1 \subset S(T) := \text{conv} \{(\eta_i t_i)_{i=1}^n : \eta \in \{-1, 1\}^n, t \in T\},$$

hence

$$\sup_{t \in T_1} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq \sup_{t \in S(T)} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} = \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p}.$$

Thus

$$\begin{aligned} (3.7) \quad & \left(\mathbb{E}_X \left(\mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right) \right)^{1/p} \\ & \leq C(\alpha) \left[\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right]. \end{aligned}$$

Let $q = p/(p-1)$ be the Hölder's dual of p . For $x \in \mathbb{R}^n$ we have

$$\left(\sup_{t \in T} \mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i x_i \varepsilon_i \right|^p \right)^{1/p} = \sup_{t \in T_2} \sum_{i=1}^n t_i x_i,$$

where

$$T_2 = \{\mathbb{E}_\varepsilon t h(\varepsilon) : t \in T, h: \{-1, 1\}^n \rightarrow \mathbb{R}, \mathbb{E}_\varepsilon |h(\varepsilon)|^q \leq 1\}$$

is a unconditional subset of \mathbb{R}^n . Estimate (2.7) applied for T_2 instead of T yields

$$\begin{aligned} & \left(\mathbb{E}_X \sup_{t \in T} \mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right)^{1/p} \\ & \leq C(\alpha) \left[\mathbb{E}_X \left(\sup_{t \in T} \mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right)^{1/p} + \sup_{t \in T_2} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right]. \end{aligned}$$

Proposition 3.1 and the symmetry of X_i gives

$$\mathbb{E}_X \left(\sup_{t \in T} \mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right)^{1/p} \leq C \left[\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right].$$

Since $T_2 \subset \text{conv } T$ (recall that we assume the symmetry of T) we have

$$\sup_{t \in T_2} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \leq \sup_{t \in \text{conv } T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} = \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p}.$$

Thus

$$(3.8) \quad \left(\mathbb{E}_X \sup_{t \in T} \mathbb{E}_\varepsilon \left| \sum_{i=1}^n t_i X_i \varepsilon_i \right|^p \right)^{1/p} \leq C(\alpha) \left[\mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i X_i + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right].$$

Estimate (1.4) follows (for symmetric X_i 's) by (3.6)-(3.8)

In the case when the variables X_i are centered, but not necessarily symmetric let (X'_1, \dots, X'_n) be an independent copy of (X_1, \dots, X_n) . Then $X_i - X'_i$ are symmetric. The Jensen inequality and the assumption on X_i imply that for any $p \geq 2$ we have

$$\|X_i - X'_i\|_{2p} \leq 2\|X_i\|_{2p} \leq 2\alpha\|X_i - \mathbb{E}X_i\|_p \leq 2\alpha\|X_i - X'_i\|_p.$$

Therefore, Theorem 1.1 applied to $(X_1 - X'_1, \dots, X_n - X'_n)$ implies

$$\begin{aligned} & \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \\ &= \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i (X_i - \mathbb{E}X'_i) \right|^p \right)^{1/p} \leq \left(\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i (X_i - X'_i) \right|^p \right)^{1/p} \\ &\leq C(2\alpha) \left[\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i (X_i - X'_i) \right| + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i (X_i - X'_i) \right|^p \right)^{1/p} \right] \\ &\leq 2C(2\alpha) \left[\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right| + \sup_{t \in T} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p} \right], \end{aligned}$$

what finishes the proof in the general case. \square

Remark 3.2. It follows by the proof of [10, Theorem 2.3] that if $(X_i)_{i=1}^n$ are symmetric, independent and for any i moments of X_i grow β -regularly (i.e. (2.9) holds with β instead of 2), then the comparison of weak and strong moments of suprema of linear combinations of variables X_i holds with a constant $C(\beta) = C\beta^{11}$. Therefore, we may follow the constants in the proofs above to obtain that Theorem 1.1 holds with $C(\alpha) = C^{\log_2^2 \alpha}$.

4. FROM COMPARISON OF WEAK AND STRONG MOMENTS TO COMPARISON OF WEAK AND STRONG TAILS

In this Section we prove Corollary 1.3 and Corollary 1.4. To this end we need the following lemma.

Lemma 4.1. *Assume X_1, X_2, \dots satisfy the assumptions of Theorem 1.1. Then for any $t \in \mathbb{R}^n$,*

$$(4.1) \quad \left\| \sum_{i=1}^n t_i X_i \right\|_p \leq C(\alpha) \left(\frac{p}{q} \right)^{\max\{1/2, \log_2 \alpha\}} \left\| \sum_{i=1}^n t_i X_i \right\|_q \quad \text{for } p \geq q \geq 2.$$

Proof. Let $\beta := \max\{1/2, \log_2 \alpha\}$. It is enough to show that for positive integers $k \leq l$ we have

$$\left\| \sum_{i=1}^n t_i X_i \right\|_{2k} \leq C\alpha \left(\frac{k}{l} \right)^\beta \left\| \sum_{i=1}^n t_i X_i \right\|_{2l}.$$

A standard symmetrization argument shows that we may assume that the random variables X_i are symmetric (see the proof of Theorem 1.1 in the non-symmetric case).

Using the hypercontractivity method [5, Section 3.3], it is enough to show that for $1 \leq i \leq n$,

$$\left\| s + \frac{t}{2\sqrt{2e\alpha}} \left(\frac{l}{k}\right)^\beta X_i \right\|_{2k} \leq \|s + tX_i\|_{2l} \quad \text{for all } s, t \in \mathbb{R}.$$

This reduces to the following claim.

Claim. Suppose that Y is a symmetric random variable such that $\|Y\|_{2p} \leq \alpha\|Y\|_p$ for some $\alpha \geq 1$ and every $p \geq 2$. Let $k \geq l$ be positive integers. Then

$$\|1 + \sigma Y\|_{2k} \leq \|1 + Y\|_{2l}, \quad \text{where } \sigma := \frac{1}{2\sqrt{2e\alpha}} \left(\frac{l}{k}\right)^\beta.$$

To show the claim observe first that

$$(4.2) \quad \|Y\|_q \leq \alpha \left(\frac{q}{p}\right)^{\log_2 \alpha} \|Y\|_p \leq \alpha \left(\frac{q}{p}\right)^\beta \|Y\|_p \quad \text{for } q \geq p \geq 2.$$

Moreover we have

$$\begin{aligned} \mathbb{E}|1 + \sigma Y|^{2k} &= 1 + \sum_{j=1}^k \binom{2k}{2j} \mathbb{E}|\sigma Y|^{2j} \leq 1 + \sum_{j=1}^k \left(\frac{ek}{j}\sigma\|Y\|_{2j}\right)^{2j} \\ &\leq 1 + \sup_{1 \leq j \leq k} \left(\frac{\sqrt{2}ek}{j}\sigma\|Y\|_{2j}\right)^{2j}, \end{aligned}$$

so it is enough to show that

$$(4.3) \quad 1 + \left(\frac{k^{1-\beta}l^\beta}{2j\alpha}\|Y\|_{2j}\right)^{2j} \leq \|1 + Y\|_{2l}^{2k} \quad \text{for } j = 1, 2, \dots, l.$$

To this end we will use the following deterministic inequality:

$$(4.4) \quad (1 + u)^p \geq \left(1 + \frac{p}{q}u\right)^q \geq 1 + \left(\frac{p}{q}u\right)^q \quad \text{for } p \geq q \geq 1 \text{ and } u \geq 0,$$

and a simple lower bound for $\|1 + Y\|_{2l}^{2l}$:

$$(4.5) \quad \mathbb{E}|1 + Y|^{2l} = 1 + \sum_{r=1}^l \binom{2l}{2r} \mathbb{E}|Y|^{2r} \geq 1 + \sum_{r=1}^l \left(\frac{l}{r}\|Y\|_{2r}\right)^{2r}.$$

Assume first that $1 \leq j \leq \frac{k}{l}$. Estimate (4.2) applied with $p = 2j$ and $q = 2$ yields

$$\frac{k^{1-\beta}l^\beta}{2j\alpha}\|Y\|_{2j} \leq \frac{k^{1-\beta}l^\beta}{j^{1-\beta}}\|Y\|_2 \leq \sqrt{\frac{kl}{j}}\|Y\|_2,$$

where the last inequality holds since $\beta \geq \frac{1}{2}$ and $k \geq jl$. Inequalities (4.5) and (4.4) (applied with $p = k/l$ and $q = j$) yield

$$\|1 + Y\|_{2l}^{2k} \geq (1 + (l\|Y\|_2)^2)^{k/l} \geq 1 + \left(\sqrt{\frac{kl}{j}}\|Y\|_2\right)^{2j}$$

so (4.3) holds for $j \leq \frac{k}{l}$.

If $j \geq \frac{k}{l}$ we choose $r = \lceil jl/k \rceil$, then $jl \leq kr \leq 2jl$. Since $1 \leq r \leq l$, the estimate (4.5) gives

$$\|1 + Y\|_{2l}^{2k} \geq \left(1 + \left(\frac{l}{r}\|Y\|_{2r}\right)^{2r}\right)^{k/l} \geq \left(1 + \left(\frac{l}{r}\|Y\|_{2r}\right)^{2r}\right)^{j/r} \geq 1 + \left(\frac{l}{r}\|Y\|_{2r}\right)^{2j},$$

where to get the last two inequalities we used $k/l \geq j/r$ and $j/r \geq 1$. Applying estimate (4.2) with $2j$ and $2r$ instead of p and q we get

$$\frac{k^{1-\beta}l^\beta}{2j\alpha} \|Y\|_{2j} \leq \frac{k^{1-\beta}l^\beta}{2j} \left(\frac{j}{r}\right)^\beta \|Y\|_{2r} \leq \frac{k}{2j} \|Y\|_{2r} \leq \frac{l}{r} \|Y\|_{2r},$$

which completes the proof of the claim in the remaining case. \square

Proof of Corollary 1.3. Let

$$S := \sup_{t \in T} \left| \sum_{i=1}^n t_i X_i \right|.$$

By the Paley-Zygmund inequality and (4.1) we have for $t \in T$,

$$\begin{aligned} \mathbb{P} \left(\left| \sum_{i=1}^n t_i X_i \right| \geq \frac{1}{2} \left\| \sum_{i=1}^n t_i X_i \right\|_p \right) &= \mathbb{P} \left(\left| \sum_{i=1}^n t_i X_i \right|^p \geq 2^{-p} \mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right) \\ (4.6) \qquad \qquad \qquad &\geq (1 - 2^{-p})^2 \left(\frac{\left\| \sum_{i=1}^n t_i X_i \right\|_p}{\left\| \sum_{i=1}^n t_i X_i \right\|_{2p}} \right)^{2p} \geq e^{-C_4(\alpha)p}. \end{aligned}$$

In order to show (1.7) we consider 3 cases.

Case 1. $2u < \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\|_2$. Then by (4.6)

$$\sup_{t \in T} \mathbb{P} \left(\left| \sum_{i=1}^n t_i X_i \right| \geq u \right) \geq e^{-2C_4(\alpha)}$$

and (1.7) obviously holds if $C_2(\alpha) \geq \exp(2C_4(\alpha))$.

Case 2. $\sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\|_2 \leq 2u < \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\|_\infty$. Let us then define

$$p := \sup \left\{ q \geq 2C_4(\alpha) : \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\|_{q/C_4(\alpha)} \leq 2u \right\}.$$

By (4.6) we have

$$\sup_{t \in T} \mathbb{P} \left(\left| \sum_{i=1}^n t_i X_i \right| \geq u \right) \geq e^{-p}.$$

By (4.1) we have $\sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\|_p \leq C(\alpha)u$, so by Theorem 1.1 and Chebyshev's inequality we have

$$\mathbb{P}(S \geq C_1(\alpha)(\mathbb{E}S + u)) \leq \mathbb{P}(S \geq e\|S\|_p) \leq e^{-p}$$

for $C_1(\alpha)$ large enough. Thus (1.7) holds in this case.

Case 3. $u > \sup_{t \in T} \left\| \sum_{i=1}^n t_i X_i \right\|_\infty = \|S\|_\infty$. Then $\mathbb{P}(S \geq u) = 0$ and (1.7) holds for any $C_1(\alpha) \geq 1$. \square

Proof of Corollary 1.4. The result is an immediate consequence of Theorem 1.1, (4.1) and (1.1) used with q instead of p . \square

5. COMPARISON OF WEAK AND STRONG MOMENTS OF SUPREMA IMPLIES COMPARISON OF MOMENTS p AND $2p$

Proof of Theorem 1.2. We will use the assumption (1.5) for T containing all vectors of the standard base of \mathbb{R}^n and their negatives, i.e. we will use only the inequality

$$(5.1) \qquad \left(\mathbb{E} \sup_{1 \leq i \leq n} |X_i|^p \right)^{1/p} \leq L \left[\mathbb{E} \sup_{1 \leq i \leq n} |X_i| + \|X_1\|_p \right].$$

Fix $p \geq 2$ and let $n := \lfloor (4L)^{2p} \rfloor + 1$, $A := n^{1/p} \|X_1\|_p$. If $A \geq \|X_1\|_{2p}$, then (1.6) holds with $\alpha = (4L)^2 + 1$. Hence we may and will assume $A \leq \|X_1\|_{2p}$.

Obviously

$$\mathbb{P}\left(\sup_{1 \leq i \leq n} |X_i| \geq t\right) \leq \min\{1, n\mathbb{P}(|X_1| \geq t)\}.$$

Moreover, if $\mathbb{P}(|X_1| \geq t) \leq \frac{1}{n}$,

$$\begin{aligned} \mathbb{P}\left(\sup_{1 \leq i \leq n} |X_i| \geq t\right) &= 1 - \mathbb{P}(|X_1| < t)^n = \mathbb{P}(|X_1| \geq t) \sum_{k=0}^{n-1} \mathbb{P}(|X_1| < t)^k \\ &\geq \mathbb{P}(|X_1| \geq t) \cdot n \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{n}{3} \mathbb{P}(|X_1| \geq t). \end{aligned}$$

Since $\mathbb{P}(|X_1| \geq A) \leq \frac{1}{n}$ (which follows by the Markov inequality) and $A \leq \|X_1\|_{2p}$, we have

$$\begin{aligned} \mathbb{E} \sup_{1 \leq i \leq n} |X_i|^{2p} &\geq 2p \int_A^\infty t^{2p-1} \mathbb{P}\left(\sup_{1 \leq i \leq n} |X_i| \geq t\right) dt \geq 2p \int_A^\infty t^{2p-1} \frac{n}{3} \mathbb{P}(|X_1| \geq t) dt \\ &= \frac{n}{3} \mathbb{E}(|X_1|^{2p} - A^{2p})_+ \geq \frac{n}{3} (\|X_1\|_{2p}^{2p} - A^{2p}) \geq \frac{n}{3} (\|X_1\|_{2p} - A)^{2p} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \sup_{1 \leq i \leq n} |X_i| &\leq A + \int_A^\infty \mathbb{P}\left(\sup_{1 \leq i \leq n} |X_i| \geq t\right) dt \leq A + n \int_A^\infty \mathbb{P}(|X_1| \geq t) dt \\ &\leq A + n \mathbb{E}(|X_1| \mathbf{1}_{\{|X_1| \geq A\}}) \leq A + n \|X_1\|_p \mathbb{P}(|X_1| \geq A)^{1-\frac{1}{p}} \\ &\leq A + n^{1/p} \|X_1\|_p, \end{aligned}$$

where in the last inequality we used again the fact that $\mathbb{P}(|X_1| \geq A) \leq \frac{1}{n}$.

Thus our choice of n and A , and (5.1) (applied to $2p$ instead of p) imply that

$$\begin{aligned} 2L \|X_1\|_{2p} &\leq \frac{1}{2} n^{\frac{1}{2p}} \|X_1\|_{2p} \leq \frac{1}{2} n^{\frac{1}{2p}} A + \left(\mathbb{E} \sup_{1 \leq i \leq n} |X_i|^{2p}\right)^{1/(2p)} \\ &\leq \frac{1}{2} n^{\frac{1}{2p}} A + L \left[\mathbb{E} \sup_{1 \leq i \leq n} |X_i| + \|X_1\|_{2p}\right] \\ &\leq \frac{1}{2} n^{\frac{1}{2p}} A + LA + Ln^{\frac{1}{p}} \|X_1\|_p + L \|X_1\|_{2p} \\ &\leq \|X_1\|_p \left(\frac{1}{2}(4L+1)n^{\frac{1}{p}} + 2Ln^{\frac{1}{p}}\right) + L \|X_1\|_{2p} \\ &\leq \left(4L + \frac{1}{2}\right) ((4L)^2 + 1) \|X_1\|_p + L \|X_1\|_{2p}. \end{aligned}$$

Thus

$$\|X_1\|_{2p} \leq \left(4 + \frac{1}{2L}\right) (16L^2 + 1) \|X_1\|_p.$$

□

Remark 5.1. It is clear from the proof above that we may take $\alpha(L) = CL^2$ in Theorem 1.2.

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