L_1-norm of combinations of products of independent random variables

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Abstract

We show that L_1-norm of linear combinations (with scalar or vector coefficients) of products of i.i.d. nonnegative mean one random variables is comparable to l_1-norm of the coefficients.

1 Introduction and Main Results

Let $X, X_1, X_2, \ldots$ be i.i.d. nonnegative r.v.'s such that $\mathbb{E}X = 1$ and $\mathbb{P}(X = 1) < 1$. Define

$$R_0 := 1 \quad \text{and} \quad R_i := \prod_{j=1}^{i} X_j \ \text{ for } i = 1, 2, \ldots. \tag{1}$$

Obviously $\mathbb{E}R_i = 1$ and therefore for any $a_0, a_1, \ldots, a_n$,

$$\mathbb{E}\Big|\sum_{i=0}^{n} a_i R_i\Big| \le \sum_{i=0}^{n} |a_i|. \tag{2}$$

If $a_i = r^i$ for some $r \in \mathbb{R}$, then $\sum_{i=0}^{n} a_i R_i$ has the same distribution as the Markov chain M_n defined by the random difference equation $M_0 = 1$, $M_n = rX_n M_{n-1} + 1$, $n = 1, 2, \ldots$. Markov chains of such type are particular examples of perpetuities. Perpetuities play an important role in applied probability and since the seminal paper of Kesten [2] attracted the attention of many researchers.

Michał Wojciechowski (personal communication) asked whether inequality (2) may be reversed in the case when $X = 1 + \cos(Y)$, where Y has the uniform distribution on $[0, 2\pi]$. In [4] he showed that for such variables

*Research supported by the NCN grant DEC-2012/05/B/ST1/00412

there exist sequences (a_i) such that $|a_i| \leq 1$, $|\sum_{i=0}^{k} a_i| \leq C$ for all $k \leq n$ and $\mathbb{E}|\sum_{i=0}^{n} a_i R_i| \geq cn$. Recently he posed a more general problem.

Problem. Is it true that for any i.i.d. sequence as above estimate (2) may be reversed, i.e. there exists a constant $c > 0$ that depends only on the distribution of X such that

$$\mathbb{E}\Big|\sum_{i=0}^{n} a_i R_i\Big| \geq c \sum_{i=0}^{n} |a_i| \quad \text{for any } a_0, \ldots, a_n?$$

The aim of this note is to give an affirmative answer to the Wojciechowski question even in the more general situation of coefficients in a normed space $(F, \| \|)$.

First we study a simpler case when X takes with positive probability values close to zero. We prove a more general result that does not require the identical distribution assumption. Namely we consider sequences (X_i) satisfying the following assumptions:

$$X_1, X_2, \ldots \text{ are independent, nonnegative r.v's with mean one,} \tag{3}$$

$$\mathbb{E}\sqrt{X_i} \leq \lambda < 1 \quad \text{and} \quad \mathbb{E}|X_i - 1| \geq \mu > 0 \quad \text{for all } i. \tag{4}$$

Notice that if X is a nondegenerate nonnegative random variable, then $\mathbb{E}\sqrt{X} < \sqrt{\mathbb{E}X}$ and $\mathbb{E}|X - 1| > 0$, hence (4) holds for i.i.d. mean one nonnegative sequences.

Theorem 1. *Let R_i be as in* (1)*, where $X_1, X_2, \ldots$ satisfy assumptions* (3) *and* (4)*. Then for any coefficients $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \|)$ we have*

$$\mathbb{E}\Big\|\sum_{i=0}^{n} v_i R_i\Big\| \geq c \sum_{i=0}^{n} \|v_i\|,$$

where

$$c = \frac{1}{64} \min\{\mu, 1\} \min_{1 \leq i \leq n} \mathbb{P}\Big(X_i \leq \frac{(1-\lambda)^2}{256} \min\{\mu, 1\}\Big).$$

Theorem 1 immediately yields the following.

Corollary 2. *Let $X, X_1, X_2, \ldots$ be an i.i.d. sequence of nonnegative r.v's such that $\mathbb{E}X = 1$ and $\mathbb{P}(X \leq \varepsilon) > 0$ for any $\varepsilon > 0$. Then there exists a constant c that depends only on the distribution of X such that for any $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \|)$,*

$$\mathbb{E}\Big\|\sum_{i=0}^{n} v_i R_i\Big\| \geq c \sum_{i=0}^{n} \|v_i\|.$$

Example In the case related to Riesz products, when $X_1, X_2, \ldots$ are independent with the same distribution as $1+\cos(Y)$ with Y uniformly distributed on $[0, 2\pi]$ we have

$$\lambda = \mathbb{E}\sqrt{1+\cos(Y)} = \sqrt{2}\mathbb{E}\Big|\cos\Big(\frac{Y}{2}\Big)\Big| = \frac{2\sqrt{2}}{\pi}, \quad \mu = \mathbb{E}|\cos(Y)| = \frac{2}{\pi}$$

and (since $\cos x \geq 1 - x^2/2$) for $0 < \varepsilon < 1/2$,

$$\mathbb{P}(X_i \leq \varepsilon) = \mathbb{P}(\cos(Y) \geq 1-\varepsilon) \geq \frac{\sqrt{2\varepsilon}}{\pi}.$$

Thus the constant given by Theorem 1 in this case is $c \geq \frac{1}{256}\pi^{-5/2}(1-\frac{2\sqrt{2}}{\pi}) \geq 2 \cdot 10^{-5}$.

To treat the general case we need one more assumption that basically states that the most of the mass of X_i's lies in the interval $[0, A]$. Namely we will assume that there exists a nonnegative constant A such that

$$\mathbb{E}|X_i - 1|\mathbb{1}_{\{X_i \geq A\}} \leq \frac{1}{4}\mu \quad \text{for all } i. \tag{5}$$

Theorem 3. *Let $X_1, X_2, \ldots$ satisfy assumptions* (3), (4) *and* (5). *Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \ \|)$, we have*

$$\mathbb{E}\Big\|\sum_{i=0}^{n} v_i R_i\Big\| \geq \frac{1}{512k}\mu^3 \sum_{i=0}^{n} \|v_i\|,$$

where R_i are as in (1) *and k is a positive integer such that*

$$\frac{2^{17}}{(1-\lambda)^2} k \lambda^{2k-2} A \leq \mu^3. \tag{6}$$

Since in the i.i.d. case all assumptions are clearly satisfied we get the positive answer to Wojciechowski's question.

Theorem 4. *Let $X, X_1, X_2, \ldots$ be an i.i.d. sequence of nonnegative nondegenerate r.v's such that $\mathbb{E}X = 1$. Then there exists a constant c that depends only on the distribution of X such that for any $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \ \|)$,*

$$\mathbb{E}\Big\|\sum_{i=0}^{n} v_i R_i\Big\| \geq c \sum_{i=0}^{n} \|v_i\|.$$

In the symmetric case the similar estimate follows by conditioning.

Corollary 5. *Let $X, X_1, X_2, \ldots$ be an i.i.d. sequence of symmetric r.v's such that $\mathbb{E}|X| = 1$ and $\mathbb{P}(|X| = 1) < 1$. Then there exists a constant c that depends only on the distribution of X such that for any $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \ \|)$,*

$$\mathbb{E}\Big\| \sum_{i=0}^{n} v_i R_i \Big\| \geq c \sum_{i=0}^{n} \|v_i\|.$$

Proof. Let (ε_i) be a sequence of independent symmetric ± 1 r.v's independent of (X_i). Then by Theorem 4

$$\begin{aligned} \mathbb{E}\Big\| \sum_{i=0}^{n} v_i R_i \Big\| &= \mathbb{E}_\varepsilon \mathbb{E}_X \Big\| v_0 + \sum_{i=1}^{n} v_i \prod_{k=1}^{i} \varepsilon_k \prod_{k=1}^{i} |X_k| \Big\| \\ &\geq \mathbb{E}_\varepsilon c\Big(\|v_0\| + \sum_{i=1}^{n} \Big\| v_i \prod_{k=1}^{i} \varepsilon_k \Big\| \Big) = c \sum_{i=0}^{n} \|v_i\|. \end{aligned}$$

□

Example. Assumption $\mathbb{P}(|X| = 1) < 1$ is crucial since

$$\mathbb{E}\Big| \sum_{i=1}^{n} \prod_{k=1}^{i} \varepsilon_k \Big| = \mathbb{E}\Big| \sum_{i=1}^{n} \varepsilon_i \Big| \leq \Big(\mathbb{E}\Big| \sum_{i=1}^{n} \varepsilon_i \Big|^2 \Big)^{1/2} = n^{1/2}.$$

Let $(n_k)_{k \geq 1}$ be an increasing sequence of positive integers such that $n_{k+1}/n_k \geq 3$. Riesz products are defined by

$$\bar{R}_i(t) = \prod_{j=1}^{i} (1 + \cos(n_j t)), \quad i = 1, 2, \ldots. \tag{7}$$

It is well known that if n_k grow sufficiently fast then $\| \sum_{i=0}^{n} a_i \bar{R}_i \|_{L_1} \sim \mathbb{E}| \sum_{i=0}^{n} a_i R_i |$, where R_i are products of independent random variables distributed as $\bar{R}_1$. Here is the more quantitative result.

Corollary 6. *Suppose that $(n_k)_{k \geq 1}$ is an increasing sequence of positive integers such that $n_{k+1}/n_k \geq 3$ and $\sum_{k=1}^{\infty} \frac{n_k}{n_{k+1}} < \infty$. Then for any coefficients $a_0, a_1, \ldots, a_n$,*

$$c \sum_{i=0}^{n} |a_i| \leq \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{i=0}^{n} a_i \bar{R}_i(t) \Big| dt \leq \sum_{i=0}^{n} |a_i|, \tag{8}$$

where $c > 0$ is a positive constant that depends only on the sequence (n_k).

Proof. We have $\bar{R}_i \geq 0$, so $\|\bar{R}_i\|_{L_1} = 1$ and the upper estimate is obvious. To show the opposite bound let $X_1, X_2, \ldots$ be independent random variables distributed as $1+\cos(Y)$, where Y is uniformly distributed on $[0, 2\pi]$ and R_i be as in (1). By the result of Y. Meyer [3], $\|\sum_{i=0}^{n} a_i \bar{R}_i\|_{L_1} \geq c'\mathbb{E}|\sum_{i=0}^{n} a_i R_i|$ and the lower estimate follows by Corollary 2. □

The condition $\sum_{k=1}^{\infty} \frac{n_k}{n_{k+1}} < \infty$ may be weakened to $\sum_{k=1}^{\infty} \frac{n_k^2}{n_{k+1}^2} < \infty$ [1], we do not however know whether lower estimate holds under more general assumptions.

Problem. Does the estimate (8) hold for all sequences of integers such that $n_{k+1}/n_k \geq 3$?

2 Proof of Theorem 1

In this section $(F, \| \ \|)$ denotes a normed space. To avoid the measurability questions we assume that F is finite dimensional, in particular it is separable. First we show few simple estimates.

Lemma 7. *Suppose that X is a nonnegative r.v. and $\mathbb{E}X = 1$. Then for any $u, v \in F$ we have*

$$\mathbb{E}\|uX + v\| \geq \frac{1}{2}\mathbb{E}|X - 1| \max\{\|u\|, \|v\|\}.$$

Proof. We have $\mathbb{E}\|uX + v\| \geq \|u\mathbb{E}X + v\| = \|u + v\|$. Moreover,

$$\begin{aligned}\mathbb{E}\|uX + v\| &= \mathbb{E}\|u(X - 1) + (u + v)\| \geq \|u\|\mathbb{E}|X - 1| - \|u + v\| \\ &\geq \|u\|\mathbb{E}|X - 1| - \mathbb{E}\|uX + v\|\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\|uX + v\| &= \mathbb{E}\|v(1 - X) + (u + v)X\| \geq \|v\|\mathbb{E}|X - 1| - \|u + v\|\mathbb{E}|X| \\ &\geq \|v\|\mathbb{E}|X - 1| - \mathbb{E}\|uX + v\|.\end{aligned}$$

□

Lemma 8. *Let $v \in F$ and Y be a random vector with values in F such that $\mathbb{P}(\|Y\| > \frac{\|v\|}{4}) \leq 1/4$. Then $\mathbb{E}\|Y + v\| \geq \mathbb{E}\|Y\| + \frac{\|v\|}{8}$.*

Proof. We have by the triangle inequality

$$\begin{aligned}
\mathbb{E}\|Y+v\| &\geq \mathbb{E}(\|Y\|-\|v\|)\mathbb{1}_{\{\|Y\|>\|v\|/4\}} + \mathbb{E}\Big(\|Y\|+\frac{\|v\|}{2}\Big)\mathbb{1}_{\{\|Y\|\leq\|v\|/4\}} \\
&= \mathbb{E}\|Y\| + \|v\|\Big(\frac{1}{2}\mathbb{P}\Big(\|Y\|\leq\frac{\|v\|}{4}\Big) - \mathbb{P}\Big(\|Y\|>\frac{\|v\|}{4}\Big)\Big) \\
&\geq \mathbb{E}\|Y\| + \frac{\|v\|}{8}.
\end{aligned}$$

□

Lemma 9. *Suppose that X_i are independent nonnegative r.v's such that $\mathbb{E}\sqrt{X_i} \leq \lambda < 1$ for all i. Then for any $v_0, \ldots, v_n \in F$,*

$$\mathbb{E}\Big\|\sum_{k=0}^{n} v_k R_k\Big\|^{1/2} \leq \sum_{k=0}^{n} \lambda^k \|v_k\|^{1/2} \tag{9}$$

and

$$\mathbb{P}\Big(\Big\|\sum_{k=0}^{n} v_k R_k\Big\| \geq \frac{t}{1-\lambda}\sum_{k=0}^{n}\lambda^k\|v_k\|\Big) \leq \frac{1}{\sqrt{t}} \quad \text{for } t \geq 1. \tag{10}$$

Proof. We have

$$\mathbb{E}\Big\|\sum_{k=0}^{n} v_k R_k\Big\|^{1/2} \leq \sum_{k=0}^{n}\mathbb{E}\|v_k R_k\|^{1/2} \leq \sum_{k=0}^{n}\lambda^k\|v_k\|^{1/2}.$$

By the Cauchy-Schwarz inequality

$$\Big(\sum_{k=0}^{n}\lambda^k\|v_k\|^{1/2}\Big)^2 \leq \sum_{k=0}^{n}\lambda^k\sum_{k=0}^{n}\lambda^k\|v_k\| \leq \frac{1}{1-\lambda}\sum_{k=0}^{n}\lambda^k\|v_k\|,$$

and the estimate (10) follows by (9) and Chebyshev's inequality. □

We are now ready to formulate a main technical result that will easily imply Theorem 1.

Proposition 10. *Let $X_1, X_2, \ldots$ satisfy assumptions* (3) *and* (4) *and $0 < \varepsilon < \frac{1}{8}$ be such that $\mathbb{P}(X_i \leq \varepsilon) \geq p > 0$ for all i. Then for any vectors $v_0, v_1, \ldots, v_n \in F$ we have*

$$\mathbb{E}\Big\|\sum_{i=0}^{n} v_i R_i\Big\| \geq \alpha\|v_0\| + \sum_{k=1}^{n}(\beta - c_k)\|v_k\|,$$

where

$$\alpha := \frac{1}{16}p, \quad \beta := \min\left\{\frac{\alpha}{2}, \frac{1}{32}\mu p\right\} \quad \textit{and} \quad c_k := \frac{4p\varepsilon}{1-\lambda}\sum_{i=0}^{k-1}\lambda^i.$$

Proof. We will proceed by induction on n. For $n = 0$ the assertion is obvious, since $\alpha \leq 1$.

Now suppose that the induction assertion holds for n, we will show it for $n + 1$. To this end we consider two cases. To shorten the notation we put

$$\tilde{R}_1 := 1 \quad \text{and} \quad \tilde{R}_k := \prod_{i=2}^{k} X_i \text{ for } k = 2, 3, \ldots.$$

Case 1. $\|v_0\| \leq \frac{64\varepsilon}{1-\lambda}\sum_{k=1}^{n+1}\lambda^{k-1}\|v_k\|$.

By the induction assumption (applied conditionally on X_1) we have

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\| &\geq \alpha\mathbb{E}\|v_0 + v_1X_1\| + \sum_{k=2}^{n+1}(\beta - c_{k-1})\mathbb{E}\|X_1v_k\| \\
&\geq \beta\|v_1\| + \sum_{k=2}^{n+1}(\beta - c_{k-1})\|v_k\| \\
&\geq \alpha\|v_0\| - \frac{4p\varepsilon}{1-\lambda}\sum_{k=1}^{n+1}\lambda^{k-1}\|v_k\| + \beta\|v_1\| + \sum_{k=2}^{n+1}(\beta - c_{k-1})\|v_k\| \\
&= \alpha\|v_0\| + \sum_{k=1}^{n+1}(\beta - c_k)\|v_k\|,
\end{aligned}$$

where the second inequality follows by Lemma 7.

Case 2. $\|v_0\| \geq \frac{64\varepsilon}{1-\lambda}\sum_{k=1}^{n+1}\lambda^{k-1}\|v_k\|$.

The induction assumption, applied conditionally on X_1, yields

$$\begin{aligned}
&\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\|\mathbb{1}_{\{X_1>\varepsilon\}} \\
&\qquad\geq \alpha\mathbb{E}\|v_0 + v_1X_1\|\mathbb{1}_{\{X_1>\varepsilon\}} + \sum_{k=2}^{n+1}(\beta - c_{k-1})\mathbb{E}\|X_1v_k\|\mathbb{1}_{\{X_1>\varepsilon\}}. \qquad (11)
\end{aligned}$$

Let Y have the same distribution as $\sum_{i=1}^{n+1} v_i R_i = X_1 \sum_{i=1}^{n+1} v_i \tilde{R}_i$ conditioned on the set $\{X_1 \le \varepsilon\}$. Then

$$\mathbb{P}\Big(\|Y\| > \frac{1}{4}\|v_0\|\Big) \le \mathbb{P}\Big(\varepsilon\Big\|\sum_{i=1}^{n+1} v_i \tilde{R}_i\Big\| > \frac{1}{4}\|v_0\|\Big)$$
$$\le \mathbb{P}\Big(\Big\|\sum_{i=1}^{n+1} v_i \tilde{R}_i\Big\| > \frac{16}{1-\lambda}\sum_{k=1}^{n+1}\lambda^{k-1}\|v_k\|\Big) \le \frac{1}{4}$$

by Lemma 9. Thus we may apply Lemma 8 and get

$$\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\|\mathbb{1}_{\{X_1\le\varepsilon\}} = \mathbb{P}(X_1 \le \varepsilon)\mathbb{E}\|v_0 + Y\| \ge \mathbb{P}(X_1 \le \varepsilon)\Big(\mathbb{E}\|Y\| + \frac{\|v_0\|}{8}\Big)$$
$$= \mathbb{E}\Big\|\sum_{i=1}^{n+1} v_i R_i\Big\|\mathbb{1}_{\{X_1\le\varepsilon\}} + \frac{\|v_0\|}{8}\mathbb{P}(X_1 \le \varepsilon).$$

By the induction assumptions we get

$$\mathbb{E}\Big\|\sum_{i=1}^{n+1} v_i R_i\Big\|\mathbb{1}_{\{X_1\le\varepsilon\}} \ge \alpha\mathbb{E}\|v_1 X_1\|\mathbb{1}_{\{X_1\le\varepsilon\}} + \sum_{k=2}^{n+1}(\beta - c_{k-1})\mathbb{E}\|v_k X_1\|\mathbb{1}_{\{X_1\le\varepsilon\}}$$
$$\ge \alpha\mathbb{E}\|v_0 + v_1 X_1\|\mathbb{1}_{\{X_1\le\varepsilon\}} - \alpha\|v_0\| + \sum_{k=2}^{n+1}(\beta - c_{k-1})\mathbb{E}\|v_k X_1\|\mathbb{1}_{\{X_1\le\varepsilon\}}.$$

The above inequalities and our choice of α imply

$$\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\|\mathbb{1}_{\{X_1\le\varepsilon\}}$$
$$\ge \alpha\mathbb{E}\|v_0 + v_1 X_1\|\mathbb{1}_{\{X_1\le\varepsilon\}} + \sum_{k=2}^{n+1}(\beta - c_{k-1})\mathbb{E}\|v_k X_1\|\mathbb{1}_{\{X_1\le\varepsilon\}} + \alpha\|v_0\|.$$

Together with (11) this gives

$$\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\| \ge \alpha\|v_0\| + \alpha\mathbb{E}\|v_0 + v_1 X_1\| + \sum_{k=2}^{n+1}(\beta - c_{k-1})\|v_k\|$$
$$\ge \alpha\|v_0\| + \beta\|v_1\| + \sum_{k=2}^{n+1}(\beta - c_{k-1})\|v_k\|$$
$$\ge \alpha\|v_0\| + \sum_{k=1}^{n+1}(\beta - c_k)\|v_k\|,$$

where the second inequality follows by Lemma 7. □

Proof of Theorem 1. We apply Proposition 10 with $\varepsilon := \frac{(1-\lambda)^2}{256}\min\{\mu,1\}$ and $p := \min_i \mathbb{P}(X_i \le \varepsilon)$. Notice that then $\beta = \frac{1}{32}\min\{\mu,1\}p \le \alpha$ and we get

$$\mathbb{E}\Big\|\sum_{i=0}^{n} v_i R_i\Big\| \ge \alpha\|v_0\| + \sum_{i=0}^{n}(\beta - c_i)\|v_i\| \ge \Big(\beta - \frac{4p\varepsilon}{(1-\lambda)^2}\Big)\sum_{i=0}^{n}\|v_i\|$$
$$\ge \frac{\beta}{2}\sum_{i=0}^{n}\|v_i\|.$$

□

3 Proof of Theorem 3

We start with a few refinements of lemmas from the previous section.

Lemma 11. *Suppose that X is nonnegative, $\mathbb{E}X = 1$, $\mathbb{E}|X-1| \ge \mu$ and $\mathbb{E}|X-1|\mathbb{1}_{\{X>A\}} \le \frac{1}{4}\mu$. Then*

$$\mathbb{E}\|uX + v\|\mathbb{1}_{\{X\le A\}} \ge \frac{1}{8}\mu\|v\| \quad \text{for any } u, v \in F.$$

Proof. Let Y have the same distribution as X conditioned on the set $\{X \le A\}$. Then $p := \mathbb{E}Y \le \mathbb{E}X = 1$ and

$$\mathbb{E}\|uX + v\|\mathbb{1}_{\{X\le A\}} = \mathbb{P}(X \le A)\mathbb{E}\|uY + v\| \ge \mathbb{P}(X \le A)\|up + v\|.$$

We have $\mathbb{E}(X-1)_+ = \mathbb{E}(X-1)_- \ge \frac{1}{2}\mu$, so

$$\mathbb{P}(X \le A)\mathbb{E}|Y - p| = \mathbb{E}|X - p|\mathbb{1}_{\{X\le A\}} \ge \mathbb{E}(X-1)_+\mathbb{1}_{\{X\le A\}} \ge \frac{1}{4}\mu$$

and

$$\mathbb{E}\|uY + v\| = \frac{1}{p}\mathbb{E}\|v(p - Y) + (pu + v)Y\| \ge \|v\|\frac{1}{p}\mathbb{E}|Y - p| - \|pu + v\|\frac{1}{p}\mathbb{E}Y$$
$$\ge \frac{1}{4\mathbb{P}(X \le A)}\mu\|v\| - \mathbb{E}\|uY + v\|.$$

□

Lemma 12. *Let Y and Z be random vectors in F such that*

$$\mathbb{E}\|Z\|\mathbb{1}_{\{\|Y\| > \frac{1}{8}\mathbb{E}\|Z\|\}} \le \frac{1}{8}\mathbb{E}\|Z\|.$$

Then $\mathbb{E}\|Y + Z\| \ge \mathbb{E}\|Y\| + \frac{1}{2}\mathbb{E}\|Z\|$.

Proof. We have

$$\begin{aligned}\mathbb{E}\|Y+Z\| &\geq \mathbb{E}(\|Y\|+\|Z\|-2\|Z\|)\mathbb{1}_{\{\|Y\|>\frac{1}{8}\mathbb{E}\|Z\|\}} \\ &\quad + \mathbb{E}(\|Y\|+\|Z\|-2\|Y\|)\mathbb{1}_{\{\|Y\|\leq\frac{1}{8}\mathbb{E}\|Z\|\}} \\ &= \mathbb{E}\|Y\|+\mathbb{E}\|Z\|-2\mathbb{E}\|Z\|\mathbb{1}_{\{\|Y\|>\frac{1}{8}\mathbb{E}\|Z\|\}}-2\mathbb{E}\|Y\|\mathbb{1}_{\{\|Y\|\leq\frac{1}{8}\mathbb{E}\|Z\|\}} \\ &\geq \mathbb{E}\|Y\|+\mathbb{E}\|Z\|-\frac{2}{8}\mathbb{E}\|Z\|-\frac{2}{8}\mathbb{E}\|Z\|=\mathbb{E}\|Y\|+\frac{1}{2}\mathbb{E}\|Z\|.\end{aligned}$$

□

Lemma 13. *Suppose that $X_1,\ldots,X_n$ are independent, nonnegative and $\mathbb{E}|X_i-1|\geq\mu$ for all i. Then for any vectors $v_0,\ldots,v_n\in F$,*

$$\mathbb{E}\Big\|\sum_{i=0}^n v_iR_i\Big\|\geq\frac{1}{4}\mu^2\max\{\|v_0\|,\ldots,\|v_n\|\}.$$

In particular

$$\mathbb{E}\Big\|\sum_{i=0}^k v_iR_i\Big\|\geq\frac{1}{4k}\mu^2\sum_{i=1}^k\|v_i\|.$$

Proof. We have for any $0\leq j\leq n$, $\sum_{i=0}^n v_iR_i=Y+X_j(v_jR_{j-1}+X_{j+1}Z)$, where variables Y and Z are independent of X_j and X_{j+1}. So Lemma 7 applied conditionally yields

$$\begin{aligned}\mathbb{E}\Big\|\sum_{i=0}^n v_iR_i\Big\| &\geq \frac{1}{2}\mathbb{E}|X_j-1|\mathbb{E}\|v_jR_{j-1}+X_{j+1}Z\| \\ &\geq \frac{1}{2}\mathbb{E}|X_j-1|\frac{1}{2}\mathbb{E}|X_{j+1}-1|\mathbb{E}\|v_jR_{j-1}\|\geq\frac{1}{4}\mu^2\|v_j\|.\end{aligned}$$

□

Next statement is a variant of Proposition 10.

Proposition 14. *Let $X_1,X_2,\ldots$ satisfy assumption* (3)-(5) *and $k\geq 1$. Then for any vectors $v_0,v_1,\ldots,v_n\in F$ we have*

$$\mathbb{E}\Big\|\sum_{i=0}^n v_iR_i\Big\|\geq\alpha\|v_0\|+\sum_{i=1}^n(\beta-c_i)\|v_i\|,$$

where

$$\alpha:=\frac{1}{64}\mu,\quad \beta:=\frac{1}{4k}\mu^2\alpha,\quad c_i:=0 \textit{ for } 1\leq i\leq k-1$$

and

$$c_i := \frac{2^8 A}{1-\lambda}\sum_{j=k}^{i}\lambda^{j+k-2}, \qquad \textit{for } i = k, k+1, \ldots.$$

Proof. Observe that $\mu \le 2$, hence $\alpha \le \frac{1}{32}$ and $\beta \le \min\{\frac{1}{8k}\mu^2, \frac{\alpha}{2}\mu\}$. As before we will proceed by induction on n. Notice that by Lemmas 7 and 13 we have for $n \le k$,

$$\mathbb{E}\Big\|\sum_{i=0}^{n} v_i R_i\Big\| \ge \frac{1}{4}\mu\|v_0\| + \frac{1}{8k}\mu^2\sum_{i=1}^{n}\|v_i\| \ge \alpha\|v_0\| + \sum_{i=1}^{n}\beta\|v_i\|.$$

Now suppose that the induction assertion holds for $n \ge k$, we will show it for $n+1$. To this end we consider two cases. To shorten the notation we put

$$R_{k+1,k} := 1 \quad \text{and} \quad R_{k+1,l} := \prod_{i=k+1}^{l} X_i \text{ for } l \ge k+1.$$

Case 1. $\mu\|v_0\| \le \frac{2^{14}}{1-\lambda}A\sum_{i=k}^{n+1}\lambda^{i+k-2}\|v_i\|$.

By the induction assumption (applied conditionally on X_1) we have

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\| &\ge \alpha\mathbb{E}\|v_0 + v_1 X_1\| + \sum_{i=2}^{n+1}(\beta - c_{i-1})\mathbb{E}\|X_1 v_i\| \\
&\ge \beta\|v_1\| + \sum_{i=2}^{n+1}(\beta - c_{i-1})\|v_i\| \\
&\ge \alpha\|v_0\| - \frac{2^8 A}{1-\lambda}\sum_{i=k}^{n+1}\lambda^{i+k-2}\|v_i\| + \beta\|v_1\| + \sum_{i=2}^{n+1}(\beta - c_{i-1})\|v_i\| \\
&= \alpha\|v_0\| + \sum_{i=1}^{n+1}(\beta - c_i)\|v_i\|,
\end{aligned}$$

where the second inequality follows by Lemma 7.

Case 2. $\mu\|v_0\| \ge \frac{2^{14}}{1-\lambda}A\sum_{i=k}^{n+1}\lambda^{i+k-2}\|v_i\|$.

Define the event $A_k \in \sigma(X_1, \ldots, X_k)$ by

$$A_k := \{X_1 \le A,\ R_{2,k} \le 4\lambda^{2k-2}\}.$$

By the induction assumption (applied conditionally) we have

$$\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\|\mathbb{1}_{\Omega\setminus A_k} \ge \alpha\mathbb{E}\Big\|\sum_{i=0}^{k} v_i R_i\Big\|\mathbb{1}_{\Omega\setminus A_k} + \sum_{i=k+1}^{n+1}(\beta - c_{i-k})\mathbb{E}\|v_i R_k\|\mathbb{1}_{\Omega\setminus A_k}. \tag{12}$$

We have

$$\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\| \mathbb{1}_{A_k} = \mathbb{P}(A_k)\mathbb{E}\|Y+Z\|,$$

where Y has the same distribution as the random variable $\sum_{i=k}^{n+1} v_i R_i$ conditioned on the event A_k and Z has the same distribution as the random variable $\sum_{i=0}^{k-1} v_i R_i$ conditioned on the event A_k. Lemma 11 applied conditionally implies

$$\mathbb{E}\|Z\| \geq \frac{1}{\mathbb{P}(X_1 \leq A)} \frac{1}{8}\mu\|v_0\| \geq \frac{1}{8}\mu\|v_0\|.$$

Notice also that

$$\|Y\| = \|R_k Y'\| \leq 4A\lambda^{2k-2}\|Y'\|,$$

where Y' is independent of Z with the same distribution as $\sum_{i=k}^{n+1} v_i R_{k+1,i}$. Therefore

$$\begin{aligned}\mathbb{E}\|Z\|\mathbb{1}_{\{\|Y\|\geq\frac{1}{8}\mathbb{E}\|Z\|\}} &\leq \mathbb{E}\|Z\|\mathbb{1}_{\{64\|Y\|\geq\mu\|v_0\|\}} \leq \mathbb{E}\|Z\|\mathbb{1}_{\{256A\lambda^{2k-2}\|Y'\|\geq\mu\|v_0\|\}}\\ &= \mathbb{E}\|Z\|\mathbb{P}(256A\lambda^{2k-2}\|Y'\| \geq \mu\|v_0\|).\end{aligned}$$

We have (by our assumptions on v_0)

$$\begin{aligned}\mathbb{P}(256A\lambda^{2k-2}\|Y'\| \geq \mu\|v_0\|) &\leq \mathbb{P}\Big(\|Y'\| \geq \frac{2^6}{1-\lambda}\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|\Big)\\ &= \mathbb{P}\Big(\Big\|\sum_{i=k}^{n+1} v_i R_{k+1,i}\Big\| \geq \frac{2^6}{1-\lambda}\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|\Big) \leq \frac{1}{8},\end{aligned}$$

where the last inequality follows by Lemma 9. Thus $\mathbb{E}\|Z\|\mathbb{1}_{\{\|Y\|\geq\frac{1}{8}\mathbb{E}\|Z\|\}} \leq \frac{1}{8}\mathbb{E}\|Z\|$ and by Lemma 12, $\mathbb{E}\|Z+Y\| \geq \mathbb{E}\|Y\| + \frac{1}{2}\mathbb{E}\|Z\|$, that is

$$\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\|\mathbb{1}_{A_k} \geq \frac{1}{2}\mathbb{E}\Big\|\sum_{i=0}^{k-1} v_i R_i\Big\|\mathbb{1}_{A_k} + \mathbb{E}\Big\|\sum_{i=k}^{n+1} v_i R_i\Big\|\mathbb{1}_{A_k}. \tag{13}$$

By Lemma 11

$$\mathbb{E}\Big\|\sum_{i=0}^{k-1} v_i R_i\Big\|\mathbb{1}_{A_k} \geq \frac{1}{8}\mu\|v_0\|\mathbb{P}(R_{2,k} \leq 4\lambda^{2k-2}) \geq \frac{1}{16}\mu\|v_0\| = 4\alpha\|v_0\|,$$

where the second inequality follows by the bound $\mathbb{E}\sqrt{R_{2,k}} = \prod_{i=2}^{k} \mathbb{E}\sqrt{X_i} \le \lambda^{k-1}$ and Chebyshev's inequality. Since $\alpha \le \frac{1}{4}$ we get

$$\frac{1}{2}\mathbb{E}\Big\|\sum_{i=0}^{k-1} v_i R_i\Big\|\mathbb{1}_{A_k} \ge \alpha\|v_0\| + \alpha\mathbb{E}\Big\|\sum_{i=0}^{k-1} v_i R_i\Big\|\mathbb{1}_{A_k}. \tag{14}$$

By the induction assumption

$$\mathbb{E}\Big\|\sum_{i=k}^{n+1} v_i R_i\Big\|\mathbb{1}_{A_k} \ge \alpha\mathbb{E}\|v_k R_k\|\mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1}(\beta - c_{i-k})\mathbb{E}\|v_i R_k\|\mathbb{1}_{A_k}. \tag{15}$$

By (13)-(15) we get

$$\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\|\mathbb{1}_{A_k} \ge \alpha\|v_0\| + \alpha\mathbb{E}\Big\|\sum_{i=0}^{k} v_i R_i\Big\|\mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1}(\beta - c_{i-k})\mathbb{E}\|v_i R_k\|\mathbb{1}_{A_k}.$$

Together with (12) this yields

$$\begin{aligned}\mathbb{E}\Big\|\sum_{i=0}^{n+1} v_i R_i\Big\| &\ge \alpha\|v_0\| + \alpha\mathbb{E}\Big\|\sum_{i=0}^{k} v_i R_i\Big\| + \sum_{i=k+1}^{n+1}(\beta - c_{i-k})\mathbb{E}\|v_i R_k\| \\ &\ge \alpha\|a_0\| + \beta\sum_{i=1}^{k}\|v_i\| + \sum_{i=k+1}^{n+1}(\beta - c_{i-k})\|v_i\| \ge \alpha\|v_0\| + \sum_{i=1}^{n+1}(\beta - c_i)\|v_i\|,\end{aligned}$$

where the second inequality follows by Lemma 13 and the definition of β. □

Proof of Theorem 3. Let α, β and c_i be as in Proposition 14. Observe that (6) yields

$$c_i \le \frac{2^8}{(1-\lambda)^2}\lambda^{2k-2}A \le 2^{-9}\frac{\mu^3}{k} = \frac{\beta}{2},$$

therefore $\alpha, \beta - c_i \ge \frac{1}{2}\beta = \frac{1}{512k}\mu^3$ for all i and the assertion follows by Proposition 14. □

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