Modified Paouris inequality

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Abstract

The Paouris inequality gives the large deviation estimate for Euclidean norms of log-concave vectors. We present a modified version of it and show how the new inequality may be applied to derive tail estimates of l_r-norms and suprema of norms of coordinate projections of isotropic log-concave vectors.

1 Introduction and Main Results

A random vector X is called *log-concave* if it has a logarithmically concave distribution, i.e. $\mathbb{P}(X \in \lambda K + (1-\lambda)L) \geq \mathbb{P}(X \in K)^{\lambda}\mathbb{P}(X \in L)^{1-\lambda}$ for all nonempty compact sets K, L and $\lambda \in [0,1]$. The result of Borell [4] states that a random vector with the full dimensional support is log-concave iff it has a logconcave density, i.e. a density of the form $e^{-h(x)}$, where h is a convex function with values in $(-\infty, \infty]$. A typical example of a log-concave vector is a vector uniformly distributed over a convex body. In recent years the study of log-concave vectors attracted attention of many researchers, cf. the forthcoming monograph [5].

The fundamental result of Paouris [8] gives the large deviation estimate for Euclidean norms of log-concave vectors. It may be stated, c.f. [1], in the form

$$(\mathbb{E}|X|^p)^{1/p} \leq C_1(\mathbb{E}|X| + \sigma_X(p)) \quad \text{for any } p \geq 1,$$

and any log-concave vector X, where here and in the sequel C_i denote universal constants, $|x|$ is the canonical Euclidean norm on $\mathbb{R}^n$ and

$$\sigma_X(p) := \sup_{|t|=1} (\mathbb{E}|\langle t, X\rangle|^p)^{1/p}, \quad p \geq 1.$$

In particular if X is additionally *isotropic*, i.e. it is centered and has identity covariance matrix then

$$(\mathbb{E}|X|^p)^{1/p} \leq C_1(\sqrt{n} + \sigma_X(p)) \quad \text{for } p \geq 1. \tag{1}$$

Together with Chebyshev's inequality this implies

$$\mathbb{P}(|X| \geq 2eC_1 t\sqrt{n}) \leq \exp(-\sigma_X^{-1}(t\sqrt{n})) \quad \text{for } t \geq 1. \tag{2}$$

In this note we show the following modification of the Paouris inequality.

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Theorem 1. *For any isotropic log-concave n-dimensional random vector X and $p \geq 1$,*

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^2 \mathbf{1}_{\{|X_i|\geq t\}}\right)^p \leq (C_2\sigma_X(p))^{2p} \quad \textit{for } t \geq C_2 \log\left(\frac{n}{\sigma_X(p)^2}\right). \quad (3)$$

Obviously $\sum_{i=1}^{n} X_i^2 \mathbf{1}_{\{|X_i|\geq t\}} \geq t^2 N_X(t)$, where

$$N_X(t) := \sum_{i=1}^{n} \mathbf{1}_{\{|X_i|\geq t\}}, \quad t > 0,$$

thus (3) generalizes the estimate derived in [2]:

$$\mathbb{E}(t^2 N_X(t))^p \leq (C\sigma_X(p))^{2p} \quad \text{for } t \geq C \log\left(\frac{n}{\sigma_X(p)^2}\right).$$

It is also not hard to see that Theorem 1 implies Paouris' inequality (1). To see this let $p' := \inf\{q \geq p\colon\ \sigma_X(q) \geq \sqrt{n}\}$. Then

$$(\mathbb{E}|X|^p)^{1/p} \leq (\mathbb{E}|X|^{2p'})^{1/2p'} \leq C_2\sigma_X(p') \leq C_2(\sqrt{n} + \sigma_X(p)),$$

where the second inequality follows by (3) aplied with $p = p'$ and $t = 0$.

In fact we may extend estimate (1) replacing the Euclidean norm by the l_r-norm, $\|x\|_r := (\sum_i |x_i|^r)^{1/r}$, $r \geq 2$.

Theorem 2. *For any $r \geq 2$ and any isotropic log-concave n-dimensional random vector X,*

$$(\mathbb{E}\|X\|_r^p)^{1/p} \leq C_3(rn^{1/r} + \sigma_X(p)) \quad \textit{for } p \geq 1. \quad (4)$$

Theorem 2 gives better bounds than presented in [6], since the constant does not explode for $r \to 2+$ and the parameter p is replaced by the smaller quantity $\sigma_X(p)$. Estimate (4) and Chebyshev's inequality imply for $t \geq 1$,

$$\mathbb{P}(\|X\|_r \geq 2eC_3trn^{1/r}) \leq \exp(-\sigma_X^{-1}(trn^{1/r})).$$

In general (4) is sharp up to a multiplicative constant, since for a random vector X with i.i.d. symmetric exponential coordinates with variance 1 we have $\sigma_X(p) \leq p\sigma_X(2) = p$ and

$$(\mathbb{E}\|X\|_r^p)^{1/p} \geq \max\{\mathbb{E}\|X\|_r, (\mathbb{E}|X_1|^p)^{1/p}\} \geq \frac{1}{C}\max\{rn^{1/r}, p\}.$$

However there are reasons to believe that the following stronger estimate may hold for log-concave vectors (c.f. [7])

$$(\mathbb{E}\|X\|_r^p)^{1/p} \leq C\left(\mathbb{E}\|X\|_r + \sup_{\|t\|_{r'}\leq 1}(\mathbb{E}|\langle t, X\rangle|^p)^{1/p}\right).$$

Another consequence of Theorem 3 is the uniform version of the Paouris inequality. For $I \subset \{1,\ldots,n\}$ by P_I we denote the coordinate projection from $\mathbb{R}^n$ into $\mathbb{R}^I$.

Theorem 3. *For any isotropic log-concave n-dimensional random vector X and $1 \le m \le n$ we have*

$$\left(\mathbb{E} \max_{|I|=m} |P_I X|^p\right)^{1/p} \le C_4 \left(\sqrt{m} \log\left(\frac{en}{m}\right) + \sigma_X(p)\right) \quad \textit{for } p \ge 1. \qquad (5)$$

Again the example of a vector with the product isotropic exponential distribution shows that in general estimate (5) is sharp. Theorem 3 and Chebyshev's inequality yield for $t \ge 1$,

$$\mathbb{P}\left(\max_{|I|=m} |P_I X| \ge 2eC_4 t\sqrt{m} \log\left(\frac{en}{m}\right)\right) \le \exp\left(-\sigma_X^{-1}\left(t\sqrt{m}\log\left(\frac{en}{m}\right)\right)\right),$$

which removes an exponential factor from Theorem 3.4 in [2].

The paper is organised as follows. In Section 2 we recall basic facts about log-concave vectors and prove Theorem 1. In Section 3 we show how to use (3) to get estimates for the joint distribution of order statistics of X and derive Theorems 2 and 3.

Notation. For a r.v. Y and $p > 0$ we set $\|Y\|_p := (\mathbb{E}|Y|^p)^{1/p}$. We write $|I|$ for the cardinality of a set I. By a letter C we denote absolute constants, value of C may differ at each occurence. Whenever we want to fix a value of an absolute constant we use letters $C_1, C_2, \ldots$.

2 Proof of Theorem 1

The result of Barlow, Marshall and Proschan [3] imply that for symmetric log-concave random variables Y, and $p \ge q > 0$, $\|Y\|_p \le \Gamma(p+1)^{1/p}/\Gamma(q+1)^{1/q}\|Y\|_q$. If Y is centered and log-concave and Y' is an independent copy of Y then $Y - Y'$ is symmetric and log-concave, hence for $p \ge q \ge 2$,

$$\|Y\|_p \le \|Y-Y'\|_p \le \frac{\Gamma(p+1)^{1/p}}{\Gamma(q+1)^{1/q}}\|Y-Y'\|_q \le 2\frac{\Gamma(p+1)^{1/p}}{\Gamma(q+1)^{1/q}}\|Y\|_q \le 2\frac{p}{q}\|Y\|_q.$$

Thus for isotropic log-concave vectors X,

$$\sigma_X(\lambda p) \le 2\lambda\sigma_X(p) \quad \text{and} \quad \sigma_X^{-1}(\lambda t) \ge \frac{\lambda}{2}\sigma_X^{-1}(t) \quad \text{for } p \ge 2,\ t, \lambda \ge 1.$$

In particular $\sigma_X(p) \le p$ for $p \ge 2$.

If Y is a log-concave r.v. (not necessarily centered) then for $p \ge 2$, $\|Y\|_p \le |\mathbb{E}Y| + \|Y - Y'\|_p \le (p+1)\|Y\|_2$ and Chebyshev's inequality yields $\mathbb{P}(|Y| \ge e(p+1)\|Y\|_2) \le e^{-p}$. Thus we obtain a Ψ_1-estimate for log-concave r.v's

$$\mathbb{P}(|Y| \ge t) \le \exp\left(2 - \frac{t}{2e\|Y\|_2}\right) \quad \text{for } t \ge 0. \qquad (6)$$

We start with a variant of Proposition 7.1 from [2].

Proposition 4. *There exists an absolute positive constant C_5 such that the following holds. Let X be an isotropic log-concave n-dimensional random vector, $A = \{X \in K\}$, where K is a convex set in $\mathbb{R}^n$ satisfying $0 < \mathbb{P}(A) \le 1/e$. Then for every $t \ge 1$,*

$$\sum_{i=1}^{n} \mathbb{E}X_i^2 \mathbf{1}_{A\cap\{X_i\ge t\}} \le C_5 \mathbb{P}(A)\left(\sigma_X^2(-\log(\mathbb{P}(A))) + nt^2e^{-t/C_5}\right) \tag{7}$$

and for every $t > 0$, $u \ge 1$,

$$\sum_{k=0}^{\infty} 4^k |\{i \le n\colon\ \mathbb{P}(A \cap \{X_i \ge 2^k t\}) \ge e^{-u}\mathbb{P}(A)\}| \le \frac{C_5u^2}{t^2}\left(\sigma_X^2(-\log(\mathbb{P}(A))) + n\mathbf{1}_{\{t\le uC_5\}}\right). \tag{8}$$

Proof. Let Y be a random vector defined by

$$\mathbb{P}(Y \in B) = \frac{\mathbb{P}(A \cap \{X \in B\})}{\mathbb{P}(A)} = \frac{\mathbb{P}(X \in B \cap K)}{\mathbb{P}(X \in K)},$$

i.e. Y is distributed as X conditioned on A. Clearly, for every measurable set B one has $\mathbb{P}(X \in B) \ge \mathbb{P}(A)\mathbb{P}(Y \in B)$. It is easy to see that Y is log-concave, but not necessarily isotropic.

The Paouris inequality (2) (applied for the isotropic vector P_IX) implies that for any $\emptyset \ne I \subset \{1,\dots,n\}$ and $t \ge (2eC_1)^2|I|$,

$$\mathbb{P}\left(\sum_{i\in I} X_i^2 \ge t\right) = \mathbb{P}(|P_IX| \ge \sqrt{t}) \le \exp\left(-\sigma_X^{-1}\left(\frac{1}{2eC_1}\sqrt{t}\right)\right). \tag{9}$$

Let

$$I := \{i \le n\colon\ \mathbb{E}Y_i^2 \ge 2(2eC_1)^2\}.$$

Log-concavity of Y (and as a consequence also of P_IY) yields $\mathbb{E}|P_IY|^4 \le C(\mathbb{E}|P_IY|^2)^2$. The Paley-Zygmund inequality implies

$$\mathbb{P}\left(\sum_{i\in I} Y_i^2 \ge \frac{1}{2}\sum_{i\in I}\mathbb{E}Y_i^2\right) \ge \frac{1}{4}\frac{(\mathbb{E}\sum_{i\in I}Y_i^2)^2}{\mathbb{E}(\sum_{i\in I}Y_i^2)^2} \ge \frac{1}{C_6}.$$

Therefore

$$\mathbb{P}\left(\sum_{i\in I} X_i^2 \ge \frac{1}{2}\sum_{i\in I}\mathbb{E}Y_i^2\right) \ge \mathbb{P}(A)\,\mathbb{P}\left(\sum_{i\in I} Y_i^2 \ge \frac{1}{2}\sum_{i\in I}\mathbb{E}Y_i^2\right) \ge \frac{1}{C_6}\mathbb{P}(A).$$

Together with (9) this gives

$$\frac{1}{C_6}\mathbb{P}(A) \le \exp\left(-\sigma_X^{-1}\left(\frac{1}{2eC_1}\sqrt{\frac{1}{2}\sum_{i\in I}\mathbb{E}Y_i^2}\right)\right),$$

hence

$$\sum_{i\in I}\mathbb{E}Y_i^2 \le C\sigma_X^2(-\log\mathbb{P}(A)).$$

Moreover if $i \notin I$, i.e. $\mathbb{E}Y_i^2 \leq 2(2eC_1)^2$ then (6) yields $\mathbb{E}Y_i^2 \mathbf{1}_{\{|Y_i| \geq t\}} \leq Ct^2 e^{-t/C}$ for $t \geq 1$. Therefore

$$\begin{aligned}\sum_{i=1}^n \mathbb{E}X_i^2 \mathbf{1}_{A \cap \{|X_i| \geq t\}} &= \mathbb{P}(A) \sum_{i=1}^n \mathbb{E}Y_i^2 \mathbf{1}_{\{|Y_i| \geq t\}} \\ &\leq \mathbb{P}(A) \left(\sum_{i \in I} \mathbb{E}Y_i^2 + nCt^2 e^{-t/C} \right) \\ &\leq C\mathbb{P}(A) \left(\sigma_X^2(-\log(\mathbb{P}(A))) + nt^2 e^{-t/C} \right).\end{aligned}$$

To show (8) note first that for every i the random variable Y_i is log-concave, hence for $s \geq 0$,

$$\frac{\mathbb{P}(A \cap \{X_i \geq s\})}{\mathbb{P}(A)} = \mathbb{P}(Y_i \geq s) \leq \exp\left(2 - \frac{t}{2e\|Y_i\|_2}\right).$$

Thus, if $\mathbb{P}(A \cap \{X_i \geq 2^k t\}) \geq e^{-u}\mathbb{P}(A)$ and $u \geq 1$ then $\|Y_i\|_2 \geq 2^k t/(2e(u+2)) \geq 2^k t/(6eu)$. In particular it cannot happen if $i \notin I$, $k \geq 0$ and $u \leq t/C_5$ with C_5 large enough.

Therefore

$$\begin{aligned}&\sum_{k=0}^\infty 4^k |\{i \leq n\colon\ \mathbb{P}(A \cap \{X_i \geq 2^k t\}) \geq e^{-u}\mathbb{P}(A)\}| \\ &\qquad\leq \left(\sum_{i \in I} + \mathbf{1}_{\{t \leq uC_5\}} \sum_{i \notin I} \right) \sum_{k=0}^\infty 4^k \mathbf{1}_{\{(\mathbb{E}Y_i^2)^{1/2} \geq 2^k t/(6eu)\}} \\ &\qquad\leq \frac{2(6eu)^2}{t^2} \left(\sum_{i \in I} + \mathbf{1}_{\{t \leq uC_5\}} \sum_{i \notin I} \right) \mathbb{E}Y_i^2 \\ &\qquad\leq \frac{Cu^2}{t^2} \left(\sigma_X^2(-\log(\mathbb{P}(A))) + n\mathbf{1}_{\{t \leq uC_5\}} \right).\end{aligned}$$

□

We will also use the following simple combinatorial lemma (Lemma 11 in [6]).

Lemma 5. *Let $\ell_0 \geq \ell_1 \geq \ldots \geq \ell_s$ be a fixed sequence of positive integers and*

$$\mathcal{F} := \{f\colon \{1,2,\ldots,l_0\} \to \{0,1,2,\ldots,s\}\colon\ \forall_{1 \leq i \leq s}\ |\{r\colon f(r) \geq i\}| \leq l_i\}.$$

Then

$$|\mathcal{F}| \leq \prod_{i=1}^s \left(\frac{el_{i-1}}{l_i} \right)^{l_i}.$$

Proof of Theorem 1. We have by the Paouris estimate (1)

$$\mathbb{E}\left(\sum_{i=1}^n X_i^2 \mathbf{1}_{\{|X_i| \geq t\}} \right)^p \leq \mathbb{E}|X|^{2p} \leq (C_1(\sqrt{n} + \sigma_X(2p)))^{2p},$$

so the estimate (3) is obvious if $\sigma_X(p) \geq \frac{1}{8}\sqrt{n}$, we will thus assume that $\sigma_X(p) \leq \frac{1}{8}\sqrt{n}$.

Observe that for $l = 1, 2, \ldots$,

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^2 \mathbf{1}_{\{X_i \geq t\}}\right)^l \leq \mathbb{E}\left(\sum_{i=1}^{n}\sum_{k=0}^{\infty} 4^{k+1} t^2 \mathbf{1}_{\{X_i \geq 2^k t\}}\right)^l$$
$$= (2t)^{2l} \sum_{i_1,\ldots,i_l=1}^{n} \sum_{k_1,\ldots,k_l=0}^{\infty} 4^{k_1+\ldots+k_l} \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}),$$

where

$$B_{i_1,k_1\ldots,i_l,k_l} := \{X_{i_1} \geq 2^{k_1} t, \ldots, X_{i_l} \geq 2^{k_l} t_l\}.$$

Define a positive integer l by

$$p < l \leq 2p \quad \text{and} \quad l = 2^m \text{ for some positive integer } m.$$

Then $\sigma_X(p) \leq \sigma_X(l) \leq \sigma_X(2p) \leq 4\sigma_X(p)$. Since $-X$ is also isotropic log-concave and for any nonnegative r.v. Y, $(\mathbb{E}Y^p)^{1/p} \leq (\mathbb{E}Y^l)^{1/l}$, it is enough to show that

$$m(l) := \sum_{k_1,\ldots,k_l=0}^{\infty} \sum_{i_1,\ldots,i_l=1}^{n} 4^{k_1+\ldots+k_l} \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}) \leq \left(\frac{C\sigma_X(l)}{t}\right)^{2l} \quad (10)$$

provided that $t \geq C_2 \log(\frac{n}{\sigma_X(l)^2})$. Since $\sigma_X(l) \leq 4\sigma_X(p) \leq \frac{1}{2}\sqrt{n}$ this in particular implies that $t \geq C_2$.

We divide the sum in $m(l)$ into several parts. Define sets

$$I_0 := \left\{(i_1, k_1, , \ldots, i_l, k_l) \colon \ \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) > e^{-l}\right\},$$

and for $j = 1, 2, \ldots$,

$$I_j := \left\{(i_1, k_1, , \ldots, i_l, k_l) \colon \ \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) \in (e^{-2^j l}, e^{-2^{j-1} l}]\right\}.$$

Then $m(l) = \sum_{j \geq 0} m_j(l)$, where

$$m_j(l) := \sum_{(i_1,k_1,\ldots,i_l,k_l) \in I_j} 4^{k_1+\ldots+k_l} \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}).$$

To estimate $m_0(l)$ define for $1 \leq s \leq l$,

$$P_s I_0 := \{(i_1, k_1, \ldots, i_s, k_s) \colon (i_1, k_1, \ldots, i_l, k_l) \in I_0 \text{ for some } i_{s+1}, \ldots, k_l\}.$$

We have by (6) (if C_2 is large enough)

$$\mathbb{P}(B_{i_1,k_1\ldots,i_s,k_s}) \leq \mathbb{P}(B_{i_1,k_1}) \leq \exp(2 - 2^{k_1-1} t/e) \leq e^{-1}.$$

Thus for $s = 1, \dots, l-1$,

$$\begin{aligned}
t^2 &\sum_{(i_1,\dots,k_{s+1})\in P_{s+1}I_0} 4^{k_1+\dots+k_{s+1}}\mathbb{P}(B_{i_1,\dots,k_{s+1}}) \\
&\le \sum_{(i_1,\dots,k_s)\in P_sI_0} 4^{k_1+\dots+k_s} \sum_{i_{s+1}=1}^{n}\sum_{k_{s+1}=0}^{\infty} 4^{k_{s+1}} t^2 \mathbb{P}(B_{i_1,\dots,k_s}\cap\{X_{i_{s+1}} \ge 2^{k_{s+1}}t\}) \\
&\le \sum_{(i_1,\dots,k_s)\in P_sI_0} 4^{k_1+\dots+k_s} \sum_{i_{s+1}=1}^{n} \mathbb{E}2X_{i_{s+1}}^2 \mathbf{1}_{B_{i_1,\dots,k_s}\cap\{X_{i_{s+1}}\ge t\}} \\
&\le 2C_5 \sum_{(i_1,\dots,k_s)\in P_sI_0} 4^{k_1+\dots+k_s}\mathbb{P}(B_{i_1,\dots,k_s})(\sigma_X^2(-\log\mathbb{P}(B_{i_1,\dots,k_s})) + nt^2e^{-t/C_5}),
\end{aligned}$$

where the last inequality follows by (7). Note that for $(i_1,\dots,k_s) \in P_sI_0$ we have $\mathbb{P}(B_{i_1,\dots,k_s}) \ge e^{-l}$ and, by our assumptions on t (if C_2 is sufficiently large) $nt^2e^{-t/C_5} \le ne^{-t/(2C_5)} \le \sigma_X^2(l)$. Therefore

$$\begin{aligned}
&\sum_{(i_1,\dots,k_{s+1})\in P_{s+1}I_0} 4^{k_1+\dots+k_{s+1}}\mathbb{P}(B_{i_1,\dots,k_{s+1}}) \\
&\qquad\qquad \le 4C_5t^{-2}\sigma_X^2(l) \sum_{(i_1,\dots,k_s)\in P_sI_0} 4^{k_1+\dots+k_s}\mathbb{P}(B_{i_1,\dots,k_s}).
\end{aligned}$$

By induction we get

$$\begin{aligned}
m_0(l) &= \sum_{(i_1,\dots,k_l)\in I_0} 4^{k_1+\dots+k_l}\mathbb{P}(B_{i_1,\dots,k_l}) \\
&\le (4C_5t^{-2}\sigma_X^2(l))^{l-1} \sum_{(i_1,k_1)\in P_1I_0} 4^{k_1}\mathbb{P}(B_{i_1,k_1}) \\
&\le (4C_5t^{-2}\sigma_X^2(l))^{l-1} t^{-2}\sum_{i=1}^{n} 2\mathbb{E}X_i^2\mathbf{1}_{\{X_i\ge t\}} \\
&\le (4C_5t^{-2}\sigma_X^2(l))^{l-1} nCe^{-t/C} \le \left(\frac{C\sigma_X(l)}{t}\right)^{2l},
\end{aligned}$$

where the last inequality follows from the assumptions on t.

Now we estimate $m_j(l)$ for $j > 0$. Fix $j > 0$ and define a positive integer r_1 by

$$2^{r_1-1} < \frac{t}{C_5} \le 2^{r_1}.$$

For all $(i_1,k_1,\dots,i_l,k_l) \in I_j$ define a function $f_{i_1,k_1,\dots,i_l,k_l}\colon \{1,\dots,\ell\} \to \{0,1,2,\dots\}$ by

$$f_{i_1,k_1,\dots,i_l,k_l}(s) := \begin{cases} 0 & \text{if } \frac{\mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\dots,i_{s-1},k_{s-1}})} > e^{-1}, \\ r & \text{if } e^{-2^r} < \frac{\mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\dots,i_{s-1},k_{s-1}})} \le e^{-2^{r-1}},\ r \ge 1. \end{cases}$$

Note that for every $(i_1,k_1,\dots,i_l,k_l) \in I_j$ one has

$$1 = \mathbb{P}(B_\emptyset) \ge \mathbb{P}(B_{i_1,k_1}) \ge \mathbb{P}(B_{i_1,k_1,i_2,k_2}) \ge \dots \ge \mathbb{P}(B_{i_1,k_1\dots,i_l,k_l}) > \exp(-2^jl).$$

Denote

$$\mathcal{F}_j := \{f_{i_1,k_1,\ldots,i_l,k_l}\colon\ (i_1,k_1,\ldots,i_l,k_l)\in I_j\}\,.$$

Then for $f = f_{i_1,k_1,\ldots,i_l,k_l} \in \mathcal{F}_j$ and $r \geq 1$ one has

$$\exp(-2^j l) < \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) = \prod_{s=1}^{\ell} \frac{\mathbb{P}(B_{i_1,k_1\ldots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\ldots,i_{s-1},k_{s-1}})}$$
$$\leq \exp(-2^{r-1}|\{s\colon\ f(s)\geq r\}|).$$

Hence for every $r \geq 1$ one has

$$|\{s\colon\ f(s)\geq r\}| \leq \min\{2^{j+1-r}l, l\} =: l_r. \tag{11}$$

In particular f takes values in $\{0,1,\ldots,j+1+\lfloor\log_2 l\rfloor\}$. Clearly, $\sum_{r\geq 1} l_r = (j+2)l$ and $l_{r-1}/l_r \leq 2$, so by Lemma 5

$$|\mathcal{F}_j| \leq \prod_{r=1}^{j+1+\lfloor\log_2 l\rfloor} \left(\frac{el_{r-1}}{l_r}\right)^{l_r} \leq e^{2(j+2)l}.$$

Now fix $f \in \mathcal{F}_j$ and define

$$I_j(f) := \{(i_1,k_1,\ldots,i_l,k_l)\colon\ f_{i_1,k_1,\ldots,i_l,k_l} = f\}$$

and for $s \leq l$,

$$I_{j,s}(f) := \{(i_1,k_1,\ldots,i_s,k_s)\colon\ f_{i_1,k_1,\ldots,i_l,k_l} = f \text{ for some } i_{s+1},k_{s+1}\ldots,i_l,k_l\}.$$

Recall that for $s\geq 1$, $\mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s}) \leq e^{-1}$, moreover for $s \leq l$,

$$\sigma_X(-\log\mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s})) \leq \sigma_X(-\log\mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l})) \leq \sigma_X(2^j l)$$
$$\leq 2^{j+1}\sigma_X(l).$$

Hence estimate (8) applied with $u = 2^{f(s+1)}$ implies for $1 \leq s \leq l-1$,

$$\sum_{(i_1,k_1,\ldots,i_{s+1},k_{s+1})\in I_{j,s+1}(f)} 4^{k_1+\ldots+k_{s+1}}\mathbb{P}(B_{i_1,k_1,\ldots,i_{s+1},k_{s+1}})$$
$$\leq g(f(s+1)) \sum_{(i_1,k_1,\ldots,i_s,k_s)\in I_{j,s}(f)} 4^{k_1+\ldots+k_s}\mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s}),$$

where

$$g(r) := \begin{cases} C_5 t^{-2}4^{j+1}\sigma_X^2(l) & \text{for } r=0,\\ C_5 t^{-2}4^{r+j+1}\sigma_X^2(l)\exp(-2^{r-1}) & \text{for } 1\leq r<r_1,\\ C_5 t^{-2}4^{r}(4^{j+1}\sigma_X^2(l)+n)\exp(-2^{r-1}) & \text{for } r\geq r_1.\end{cases}$$

Suppose that $(i_1,k_1)\in I_1(f)$ and $f(1) = r$ then

$$\exp(-2^r) \leq \mathbb{P}(X_{i_1}\geq 2^{k_1}t) \leq \exp(2-2^{k_1-1}t/e),$$

hence $2^{k_1}t \leq e2^{r+2}$. W.l.o.g. $C_5 > 4e$, therefore $r \geq r_1$. Moreover, $4^{k_1} \leq 16e^2 4^r t^{-2}$, hence

$$\sum_{(i_1,k_1)\in I_{j,1}(f)} 4^{k_1}\mathbb{P}(B_{i_1,k_1}) \leq n32e^2t^{-2}4^r\exp(-2^{r-1}) \leq g(r) = g(f(1)),$$

since we may assume that $C_5 \geq 32e^2$. Thus the easy induction shows that

$$m_j(f) := \sum_{(i_1,\ldots,k_l)\in I_j(f)} 4^{k_1+\ldots+k_l}\mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) \leq \prod_{s=1}^{l} g(f(s)) = \prod_{r=0}^{\infty} g(r)^{n_r},$$

where $n_r := |f^{-1}(r)|$.

Observe that

$$e^{-2^{j-1}l} \geq \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) = \prod_{s=1}^{l} \frac{\mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\ldots,i_{s-1},k_{s-1}})} \geq e^{-l} \prod_{s\colon f(s)\geq 1} e^{-2^{f(s)}},$$

therefore

$$\sum_{r=1}^{\infty} n_r 2^{r-1} = \frac{1}{2} \sum_{s\colon f(s)\geq 1} 2^{f(s)} \geq \frac{1}{2} l(2^{j-1} - 1).$$

Moreover $4^{j+1}\sigma_X^2(l) + n \leq 2 \cdot 4^{j+1} n$ and

$$\sum_{r\geq 1} r n_r \leq (j+1)l + \sum_{r\geq j+2} r l_r = (2j+4)l.$$

Hence

$$\prod_{r=0}^{\infty} g(r)^{n_r} \leq \left(\frac{C_5 4^{j+1}\sigma_X^2(l)}{t^2}\right)^{l} 4^{(2j+4)l} \left(\frac{2n}{\sigma_X^2(l)}\right)^{m} \exp\left(-\frac{l}{2}(2^{j-1}-1)\right).$$

where $m = \sum_{r\geq r_1} n_r \leq l_{r_1} \leq 2^{j+1-r_1} l$. By the assumption on l we have $(2n/\sigma_X^2(l)) \leq 2\exp(t/C_2) \leq \exp(2^{r_1-4})$ if C_2 is large enough with respect to C_5. Hence

$$m_j(l) \leq \left(\frac{\sqrt{e} C_5 4^{3j+5} \sigma_X^2(l)}{t^2}\right)^{l} \exp(-l2^{j-3})$$

and we get

$$m(l) = \sum_{j=0}^{\infty} m_j(l) \leq \left(\frac{C\sigma_X(l)}{t}\right)^{2l} + \sum_{j\geq 1} \left(\frac{\sqrt{e} C_5 4^{3j+5} \sigma_X^2(l)}{t^2}\right)^{l} \exp(-l2^{j-3})$$

and (10) easily follows. □

3 Estimates for joint distribution of order statistics

For a random vector $X = (X_1, \ldots, X_n)$ by $X_1^* \geq X_2^* \geq \ldots \geq X_n^*$ we denote the nonincreasing rearrangement of $|X_1|, \ldots, |X_n|$, in particular $X_1^* = \max\{|X_1|, \ldots, |X_n|\}$ and $X_n^* = \min\{|X_1|, \ldots, |X_n|\}$. The following consequence of Theorem 1 generalizes Theorem 3.3 from [2].

Theorem 6. *Let X be an isotropic log-concave vector, $0 = l_0 < l_1 < l_2 < \ldots < l_k \leq n$ and $t_1, \ldots, t_k \geq 0$ be such that*

$$t_r \geq C_7 \log\left(\frac{C_7^2 n}{\sum_{j=1}^s t_j^2 (l_j - l_{j-1})}\right) \qquad \text{for } 1 \leq r \leq k.$$

Then

$$\mathbb{P}\left(X_{l_1}^* \geq t_1, \ldots, X_{l_k}^* \geq t_k\right) \leq \exp\left(-\sigma_X^{-1}\left(\frac{1}{C_7}\sqrt{\sum_{j=1}^k t_j^2 (l_j - l_{j-1})}\right)\right).$$

Proof. Let $t := \min\{t_1, \ldots, t_k\}$, $u := (\sum_{j=1}^k t_j^2 (l_j - l_{j-1}))^{1/2}$ and $p := \sigma_X^{-1}(e^{-1/2} u / C_2)$. It is not hard to see that if C_7 is large enough then $u \geq \sqrt{e} C_2$, so $p \geq 2$. Assumptions imply (if C_7 is large enough) that $C_2 \log(n/\sigma_X^2(p)) = C_2 \log(enC_2^2/u^2) \leq t$. Therefore Chebyshev's inequality and Theorem 1 yield

$$\begin{aligned}\mathbb{P}\Big(X_{l_1}^* \geq t_1, \ldots, X_{l_k}^* \geq t_k\Big) &\leq \mathbb{P}\left(\sum_{i=1}^n X_i^2 \mathbf{1}_{\{|X_i| \geq t\}} \geq u^2\right) \\ &\leq u^{-2p}\mathbb{E}\left(\sum_{i=1}^n X_i^2 \mathbf{1}_{\{|X_i| \geq t\}}\right)^p \leq \left(\frac{C_2 \sigma_X(p)}{u}\right)^{2p} \leq e^{-p}.\end{aligned}$$

□

Corollary 7. *Let X be an isotropic log-concave vector and*

$$Y_j := \Big(X_{2^{j-1}}^* - C_7 \log(4n2^{-j})\Big)_+, \quad 1 \leq j \leq 1 + \log_2 n.$$

Then for any $1 \leq s \leq 1 + \log_2 n$ and $u_1, \ldots, u_s \geq 0$ with $\sum_{j=1}^s u_s > 0$ we have

$$\mathbb{P}(Y_1 \geq u_1, \ldots, Y_s \geq u_s) \leq \exp\left(-\sigma_X^{-1}\left(\frac{1}{2C_7}\sqrt{\sum_{j=1}^s 2^j u_j^2}\right)\right).$$

Proof. Let

$$I = \{j \geq 0\colon\ u_j > 0\} = \{i_1 < \ldots < i_k\}.$$

By our assumptions $I \neq \emptyset$, hence $k \geq 1$. Let $l_0 = 0$, $l_j = 2^{i_j - 1}$, $t_j := C_7 \log(4n2^{-i_j}) + u_{i_j}$ for $1 \leq j \leq k$ and $u := (\sum_{j=1}^k (l_j - l_{j-1}) t_j^2)^{1/2}$. Then for $1 \leq j \leq k$, $u^2 \geq C_7^2 2^{i_j - 2}$ therefore $t_j \geq C_7 \log(C_7^2 n/u^2)$ for all j and we may apply Theorem 6 and get

$$\begin{aligned}\mathbb{P}(Y_1 \geq u_1, \ldots, Y_s \geq u_s) = \mathbb{P}(X_{l_1}^* \geq t_1, \ldots, X_{l_k}^* \geq t_k) &\leq \exp\left(-\sigma_X^{-1}\left(\frac{1}{C_7}u\right)\right) \\ &\leq \exp\left(-\sigma_X^{-1}\left(\frac{1}{2C_7}\sqrt{\sum_{j=1}^s 2^j u_j^2}\right)\right).\end{aligned}$$

□

Lemma 8. *For nonnegative r.v.'s $Y_1, \ldots, Y_s$ and $u > 0$ we have*

$$\mathbb{P}\left(\sum_{i=1}^{s} Y_i \geq u\right) \leq \sum_{(k_1,\ldots,k_s)\in I_s} \mathbb{P}\left(Y_1 \geq \frac{k_1 u}{2s}, \ldots, Y_n \geq \frac{k_s u}{2s}\right),$$

where

$$I_s := \{k_1, \ldots, k_s \in \{0, 1, \ldots, s\}^s : \ k_1 + \ldots + k_s = s\}.$$

Proof. It is enough to observe that if $y_1 + \ldots + y_s \geq u$ and we set $l_i := \lfloor 2sy_i/u \rfloor$ then $y_i \geq l_i u/(2s)$ and $\sum_{i=1}^{s} l_i \geq \sum_{i=1}^{s} (2sy_i/u - 1) \geq s$. □

Proof of Theorem 2. Let $s := 1 + \lfloor \log_2 n \rfloor$ and Y_j, $1 \leq j \leq s$ be as in Corollary 7. We have

$$\|X\|_r^r = \sum_{i=1}^{n} |X_i^*|^r \leq \sum_{j=1}^{s} 2^{j-1} |X_{2^{j-1}}^*|^r \leq \sum_{j=1}^{s} 2^{r+j-1} (Y_j^r + C_7^r \log^r(4n2^{-j}))$$
$$\leq (C_8 r)^r n + \sum_{j=1}^{s} 2^{r+j-1} Y_j^r.$$

By Lemma 8

$$\mathbb{P}\left(\sum_{j=1}^{s} 2^{r+j-1} Y_j^r \geq u^r\right) \leq \sum_{(k_1,\ldots,k_s)\in I_s} \mathbb{P}\left(2Y_1^r \geq \frac{k_1 u^r}{s2^r}, \ldots, 2^s Y_s^r \geq \frac{k_s u^r}{s2^r}\right).$$

Moreover for any $(k_1, \ldots, k_s) \in I_s$,

$$\sum_{j=1}^{s} 2^{j-2j/r} \left(\frac{k_j}{s}\right)^{2/r} \geq \sum_{j=1}^{s} \left(\frac{k_j}{s}\right)^{2/r} \geq \left(\sum_{j=1}^{s} \frac{k_j}{s}\right)^{2/r} = 1.$$

Therefore Corollary 7 yields

$$\mathbb{P}\left(\sum_{j=1}^{s} 2^{r+j-1} Y_j^r \geq u^r\right) \leq |I_s| \exp\left(-\sigma_X^{-1}\left(\frac{u}{4C_7}\right)\right).$$

However $|I_s| = \binom{2s-1}{s-1} \leq 2^{2s-2} \leq n^2$, so we obtain for $u \geq 2C_8 r n^{1/r}$,

$$\mathbb{P}(\|X\|_r \geq u) \leq n^2 \exp\left(-\sigma_X^{-1}\left(\frac{u}{8C_7}\right)\right).$$

Since $rn^{1/r} \geq e \log n$ and for $\lambda, s \geq 1$, $\sigma_X^{-1}(2\lambda s) \geq \lambda \sigma_X^{-1}(s)$ and $\sigma_X^{-1}(s) \geq s$ we get

$$\mathbb{P}(\|X\|_r \geq Ct) \leq \exp(-\sigma_X^{-1}(t)) \quad \text{for } t \geq rn^{1/r}.$$

Integration by parts easily yields (4). □

Proof of Theorem 3. Let $s := 1 + \lfloor \log_2 m \rfloor$ and Y_j, $1 \leq j \leq s$ be as in Corollary 7. Then

$$\sup_{|I|=m} |P_I X|^2 = \sum_{i=1}^{m} |X_i^*|^2 \leq \sum_{j=1}^{s} 2^{j-1} |X_{2^{j-1}}^*|^2 \leq \sum_{j=1}^{s} 2^j (C_7^2 \log^2(4n2^{-j}) + Y_j^2)$$
$$\leq C_9 m \log^2(en/m) + \sum_{j=1}^{s} 2^j Y_j^2.$$

Moreover,

$$\mathbb{P}\left(\sum_{j=1}^{s} 2^j Y_j^2 \geq u^2\right) \leq \sum_{(k_1,\ldots,k_s)\in I_s} \mathbb{P}\left(2Y_1^2 \geq \frac{k_1 u^2}{2s}, \ldots, 2^s Y_s^2 \geq \frac{k_s u^2}{2s}\right)$$
$$\leq |I_s| \exp\left(-\sigma_X^{-1}\left(\frac{u}{2\sqrt{2}C_7}\right)\right),$$

where the first inequality follows by Lemma 8 and the second one by Corollary 7. Observe that $|I_s| = \binom{2s-1}{s-1} \leq 2^{2s-2} \leq m^2$, thus we showed that for $u \geq \sqrt{2C_9 m}\log(en/m)$

$$\mathbb{P}\left(\max_{|I|=m} |P_I X| \geq u\right) \leq m^2 \exp\left(-\sigma_X^{-1}\left(\frac{u}{4C_7}\right)\right).$$

Since for $\lambda, s \geq 1$, $\sigma_X^{-1}(2\lambda s) \geq \lambda \sigma_X^{-1}(s)$ and $\sigma_X^{-1}(s) \geq s$ we easily get for $t \geq 1$,

$$\mathbb{P}\left(\max_{|I|=m} |P_I X| \geq Ct\sqrt{m}\log(en/m)\right) \leq \exp\left(-\sigma_X^{-1}\left(t\sqrt{m}\log(en/m)\right)\right).$$

Theorem 3 follows by integration by parts. □

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