On the Equivalence Between Geometric and Arithmetic Means for Log-Concave Measures

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Abstract. Let $X$ be a random vector with log-concave distribution in some Banach space. We prove that $\|X\|_p \leq C_p \|X\|_0$ for any $p > 0$, where $\|X\|_p = (E\|X\|^p)^{1/p}$, $\|X\|_0 = \exp E \ln \|X\|$ and $C_p$ are constants depending only on $p$. We also derive some estimates of log-concave measures of small balls.

Introduction. Let $X$ be a random vector with log-concave distribution (for precise definitions see below). It is known that for any measurable seminorm and $p, q > 0$ the inequality

$$\|X\|_p \leq C_{p,q} \|X\|_q$$

holds with constants $C_{p,q}$ depending only on $p$ and $q$ (see [4], Appendix III). In this paper we show that the above constants can be made independent of $q$, which is equivalent to the inequality

$$\|X\|_p \leq C_p \|X\|_0,$$  \hspace{1cm} (1)

where $\|X\|_0$ is the geometric mean of $\|X\|$. In the particular case in which $X$ is uniformly distributed on some convex compact set in $\mathbb{R}^n$ and the seminorm is given by some functional, inequality (1) was established by V. D. Milman and A. Pajor [3]. As a consequence of (1) we prove the result of Ullrich [6] concerning the equivalence of means for sums of independent Steinhaus random variables with vector coefficients, even though these random variables are not log-concave (Corollary 2).

To prove (1) we derive some estimates of log-concave measures of small balls (Corollary 1), which are of independent interest. In the case of Gaussian random variables they were formulated and established in a weaker version in [5] and completely proved in [2].
Definitions and Notation. Let $E$ be a complete, separable, metric vector space endowed with its Borel $\sigma$–algebra $\mathcal{B}_E$. By $\mu$ we denote a log-concave probability measure on $(E, \mathcal{B}_E)$ (for some characterizations, properties and examples, see [1]) i.e. a probability measure with the property that for any Borel subsets $A, B$ and all $0 < \lambda < 1$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$ 

We say that a random vector $X$ with values in $E$ is log-concave if the distribution of $X$ is log-concave. For a random vector $X$ and a measurable seminorm $\|\|$ on $E$ (i.e. Borel measurable, nonnegative, subadditive and positively homogeneous function on $E$) we define

$$\|X\|_p = (E\|X\|^p)^{1/p} \text{ for } p > 0$$

and

$$\|X\|_0 = \lim_{p \to 0^+} \|X\|_p = \exp(E\ln\|X\|).$$

Let us begin with the following Lemma from [1].

**Lemma 1.** For any convex, symmetric Borel set $B$ and $k \geq 1$ we have

$$\mu((kB)^c) \leq \mu(B)^{(1 - \mu(B))^{(k+1)/2}}.$$

**Proof.** The statement follows immediately from the log-concavity of $\mu$ and the inclusion

$$\frac{k - 1}{k + 1} B + \frac{2}{k + 1} (kB)^c \subset B^c. \quad \square$$

**Lemma 2.** If $B$ is a convex, symmetric Borel set, with $\mu(KB) \geq (1 + \delta)\mu(B)$ for some $K > 1$ and $\delta > 0$ then

$$\mu(tB) \leq Ct\mu(B) \text{ for any } t \in (0, 1),$$

where $C = C(K/\delta)$ is a constant depending only on $K/\delta$.

**Proof.** Obviously it’s enough to prove the result for $t = 1/2n$, $n = 1, 2, \ldots$. So let us fix $n$ and define, for $u \geq 0$,

$$P_u = \{x : \|x\|_B \in (u - 1/2n, u + 1/2n)\},$$

where

$$\|x\|_B = \inf\{t > 0 : x \in tB\}.$$ 

By simple calculation $\lambda P_u + (1 - \lambda)(2n)^{-1}B \subset P_{\lambda u}$, so

$$\mu(P_{\lambda u}) \geq \mu(P_u)^\lambda \mu((2n)^{-1}B)^{1-\lambda} \text{ for } \lambda \in (0, 1). \quad (2)$$

From the assumptions it easily follows that there exists $u \geq 1$ such that $\mu(P_u) \geq \delta \mu(B)/Kn$. Let $\mu((2n)^{-1}B) = \kappa \mu(B)/n$. If $\kappa \leq 2\delta/K$ we are done, so we will
assume that $\kappa \geq 2\delta / K$. Then by (2) it follows that $\mu(P_1) \geq \delta \mu(B)/Kn$. The sets $P_{(n-1)/n}, P_{(n-2)/n}, \ldots, P_{1/n}, (2n)^{-1}B$ are disjoint subsets of $B$, and hence
\[ \mu(B) \geq \mu(P_{(n-1)/n}) + \cdots + \mu(P_{1/n}) + \mu((2n)^{-1}B). \]
Using our estimations of $\mu(P_1)$ and $\mu((2n)^{-1}B)$ we obtain by (2)
\[ \mu(B) \geq n^{-1}\mu(B)((\delta / K)^{(n-1)/n}\kappa^{1/n} + \cdots + (\delta / K)^{1/n}\kappa^{(n-1)/n} + \kappa) = \frac{\kappa}{n} \mu(B) \frac{1 - \delta / K \kappa}{1 - (\delta / K \kappa)^{1/n}} \geq \frac{\kappa}{2n} \mu(B) \frac{1}{1 - (\delta / K \kappa)^{1/n}}. \]
Therefore
\[ \kappa \leq 2n(1 - (\delta / K \kappa)^{1/n}) \leq 2\ln K \kappa / \delta, \]
so that $\kappa \leq C(K/\delta)$ and the lemma follows.

\textbf{Corollary 1.} For each $b < 1$ there exists a constant $C_b$ such that for every log-concave probability measure $\mu$ and every measurable convex, symmetric set $B$ with $\mu(B) \leq b$ we have
\[ \mu(tB) \leq C_b t \mu(B) \text{ for } t \in [0, 1]. \]

\textbf{Proof.} If $\mu(B) = 2/3$ then by Lemma 1 $\mu(3B) \geq 5/6 = (1 + 1/4)\mu(B)$, so by Lemma 2 for some constant $C_1$, $\mu(tB) \leq C_1 t \mu(B)$.

If $\mu(B) \in [1/3, 2/3]$ then obviously $\mu(tB) \leq 2C_1 t \mu(B)$.

If $\mu(B) < 1/3$, let $K$ be such that $\mu(KB) = 2/3$. By the above case $\mu(B) \leq C_1 K^{-1} \mu(KB)$, and hence
\[ K \leq 2C_1 \left( \frac{\mu(KB)}{\mu(B)} - 1 \right). \]
So Lemma 2 gives in this case that $\mu(tB) \leq C_2 t \mu(B)$ for some constant $C_2$.

Finally if $\mu(B) > 2/3$, but $\mu(B) \leq b < 1$ then by Lemma 1 for some $K_b < \infty$, $\mu(K_b^{-1}B) \leq 2/3$ and we can use the previous calculations.

\textbf{Theorem 1.} For any $p > 0$ there exists a universal constant $C_p$, depending only on $p$ such that for any sequence $X_1, \ldots, X_n$ of independent log-concave random vectors and any measurable seminorm $\|\cdot\|$ on $E$ we have
\[ \left\| \sum_{i=1}^{n} X_i \right\|_p \leq C_p \left\| \sum_{i=1}^{n} X_i \right\|_0. \]

\textbf{Proof.} Since a convolution of log-concave measures is also log-concave (see [1]) we may and do assume that $n = 1$. Let
\[ M = \inf \{ t : P(||X_1|| \geq t) \leq 2/3 \}. \]
Then by Lemma 1 (used for $B = \{ x \in E : ||x|| \leq M \}$) it follows easily that $||X_1||_p \leq a_p M$ for $p > 0$ and some constants $a_p$, depending only on $p$. By similar reasoning Corollary 1 yields $||X_1||_0 \geq a_0 M$. \hfill \Box
Corollary 2. Let $E$ be a complex Banach space and $X_1, \ldots, X_n$ be a sequence of independent random variables uniformly distributed on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Then for any sequence of vectors $v_1, \ldots, v_n \in E$ and any $p > 0$ the following inequality holds:

$$\left\| \sum v_k X_k \right\|_p \leq K_p \left\| \sum v_k X_k \right\|_0,$$

where $K_p$ is a constant depending only on $p$.

Proof. It is enough to prove Corollary for $p \geq 1$. Let $Y_1, \ldots, Y_n$ be a sequence of independent random variables uniformly distributed on the unit disc $\{z : |z| \leq 1\}$. By Theorem 1 we have

$$\left\| \sum v_k Y_k \right\|_p \leq C_p \left\| \sum v_k Y_k \right\|_0. \quad (3)$$

But we may represent $Y_k$ in the form $Y_k = R_k X_k$, where $R_k$ are independent, identically distributed random variables on $[0, 1]$ (with an appropriate distribution), which are independent of $X_k$. Hence, by taking conditional expectation we obtain

$$\left\| \sum v_k Y_k \right\|_p \geq (ER_1) \left\| \sum v_k X_k \right\|_p. \quad (4)$$

Finally let us observe that for any $u, v \in E$ the function $f(z) = \ln \| u + zv \|$ is subharmonic on $\mathbb{C}$, so $g(r) = E \ln \| u + r v X_1 \|$ is nondecreasing on $[0, \infty)$ and therefore

$$\left\| \sum v_k X_k \right\|_0 \geq \left\| \sum v_k Y_k \right\|_0. \quad (5)$$

The corollary follows from (3), (4) and (5).

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References


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