On Some Inequalities for Gaussian Measures

R. Latała*

Abstract

We review several inequalities concerning Gaussian measures - isoperimetric inequality, Ehrhard's inequality, Bobkov's inequality, S-inequality and correlation conjecture.

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1 Introduction

Gaussian random variables and processes always played a central role in the probability theory and statistics. The modern theory of Gaussian measures combines methods from probability theory, analysis, geometry and topology and is closely connected with diverse applications in functional analysis, statistical physics, quantum field theory, financial mathematics and other areas. Some examples of applications of Gaussian measures can be found in monographs [4, 18, 20] and [23].

In this note we present several inequalities of geometric nature for Gaussian measures. All of them have elementary formulations, but nevertheless yield many important and nontrivial consequences. We begin in section 2 with the already classical Gaussian isoperimetric inequality that inspired in the 70's and 80's the vigorous development of concentration inequalities and their applications in the geometry and local theory of Banach spaces (cf. [19, 24, 32]). In the sequel we review several more recent results and finish in section 6 with the discussion of the Gaussian correlation conjecture that remains unsolved more than 30 years.

*Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warszawa, Poland. E-mail: rlatala@mimuw.edu.pl

A probability measure μ on a real separable Banach space F is called *Gaussian* if for every functional $x^* \in F^*$ the induced measure $\mu \circ (x^*)^{-1}$ is a one-dimensional Gaussian measure $\mathcal{N}(a, \sigma^2)$ for some $a = a(x^*) \in \mathbb{R}$ and $\sigma = \sigma(x^*) \geq 0$. Throughout this note we only consider centered Gaussian measures that is the measures such that $a(x^*) = 0$ for all $x^* \in F^*$. A random vector with values in F is said to be Gaussian if its distribution is Gaussian. Every centered Gaussian measure on $\mathbb{R}^n$ is a linear image of the canonical Gaussian measure γ_n, that is the measure on $\mathbb{R}^n$ with the density $d\gamma_n(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)dx$, where $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. Infinite dimensional Gaussian measures can be effectively approximated by finite dimensional ones using the following series representation (cf. [18, Proposition 4.2]): If μ is a centered Gaussian measure on F and $g_1, g_2, \ldots$ are independent $\mathcal{N}(0,1)$ random variables then there exist vectors $x_1, x_2, \ldots$ in F such that the series $X = \sum_{i=1}^\infty x_i g_i$ is convergent almost surely and in every L^p, $0 < p < \infty$, and is distributed as μ.

We will denote by Φ the distribution function of the standard normal $\mathcal{N}(0,1)$ r.v., that is

$$\Phi(x) = \gamma_1(-\infty, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy, \ -\infty \leq x \leq \infty.$$

For two sets A, B in a Banach space F and $t \in \mathbb{R}$ we will write $tA = \{tx : x \in A\}$ and $A + B = \{x + y : x \in A, y \in B\}$. A set A in F is said to be *symmetric* if $-A = A$.

Many results presented in this note can be generalized to the more general case of Radon Gaussian measures on locally convex spaces. For precise definitions see [4] or [7].

2 Gaussian Isoperimetry

For a Borel set A in $\mathbb{R}^n$ and $t > 0$ let $A_t = A + tB_2^n = \{x \in \mathbb{R}^n : |x - a| < t \text{ for some } a \in A\}$ be the open *t-enlargement* of A, where B_2^n denotes the open unit Euclidean ball in $\mathbb{R}^n$. The classical isoperimetric inequality for the Lebesgue measure states that if $\mathrm{vol}_n(A) = \mathrm{vol}_n(rB_2^n)$ then $\mathrm{vol}_n(A_t) \geq \mathrm{vol}_n((r+t)B_2^n)$ for $t > 0$. In the early 70's C. Borell [6] and V.N. Sudakov and B.S. Tsirel'son [29] proved independently the isoperimetric property of Gaussian measures.

Theorem 2.1 *Let A be a Borel set in $\mathbb{R}^n$ and let H be an affine halfspace such that $\gamma_n(A) = \gamma_n(H) = \Phi(a)$ for some $a \in \mathbb{R}$. Then*

$$\gamma_n(A_t) \geq \gamma_n(H_t) = \Phi(a + t) \text{ for all } t \geq 0. \tag{1}$$

Theorem 2.1 has an equivalent differential analog. To state it let us define for a measure μ on $\mathbb{R}^n$ and any Borel set A the *boundary μ-measure* of A by the formula

$$\mu^+(A) = \liminf_{t \to 0+} \frac{\mu(A_t) - \mu(A)}{t}.$$

Moreover let $\varphi(x) = \Phi'(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ and let

$$I(t) = \varphi \circ \Phi^{-1}(t), \ t \in [0, 1]$$

be the *Gaussian isoperimetric function.*

The equivalent form of Theorem 2.1 is that for all Borel sets A in $\mathbb{R}^n$

$$\gamma_n^+(A) \geq I(\gamma_n(A)). \tag{2}$$

The equality in (2) holds for any affine halfspace.

For a probability measure μ on $\mathbb{R}^n$ we may define the *isoperimetric function* of μ by

$$\mathrm{Is}(\mu)(p) = \inf\{\mu^+(A) : \mu(A) = p\},\ 0 \leq p \leq 1.$$

Only few cases are known when one can determine exactly $\mathrm{Is}(\mu)$. For Gaussian measures (2) states that $\mathrm{Is}(\gamma_n) = I$.

Let us finish section 2 by an example of application of (1) (see [20, Lemma 3.1]).

Corollary 2.2 *Let X be a centered Gaussian random vector in a separable Banach space $(F, \|\cdot\|)$. Then for any $t > 0$*

$$\mathbf{P}(|\|X\| - \mathrm{Med}(\|X\|)| \geq t) \leq 2(1 - \Phi(\frac{t}{\sigma})) \leq e^{-t^2/2\sigma^2},$$

where

$$\sigma = \sup\{\sqrt{\mathbf{E}(x^*(X))^2} : x^* \in F^*, \|x^*\| \leq 1\}.$$

3 Ehrhard's Inequality

It is well known that the classical isoperimetric inequality for the Lebesgue measure in $\mathbb{R}^n$ follows by the Brunn-Minkowski inequality (cf. [25]), which states that for any Borel sets A and B in $\mathbb{R}^n$

$$\mathrm{vol}_n(\lambda A + (1-\lambda)B)) \geq (\mathrm{vol}_n(A))^\lambda (\mathrm{vol}_n(B))^{1-\lambda} \text{ for } \lambda \in [0,1].$$

Gaussian measures satisfy the similar log-concavity property, that is the inequality

$$\ln(\mu(\lambda A + (1-\lambda)B)) \geq \lambda \ln(\mu(A)) + (1-\lambda)\ln(\mu(B)),\ \lambda \in [0,1] \tag{3}$$

holds for any Gaussian measure μ on a separable Banach space F and any Borel sets A and B in F (cf. [5]). However the log-concavity of the measure does not imply the Gaussian isoperimetry.

In the early 80's A. Ehrhard [9] gave a different proof of the isoperimetric inequality (1) using a Gaussian symmetrization procedure similar to the Steiner symmetrization. With the same symmetrization tool Ehrhard established a new Brunn-Minkowski type inequality, stronger than (3), however only for convex sets.

Theorem 3.1(Ehrhard's inequality) *If μ is a centered Gaussian measure on a separable Banach space F and A, B are Borel sets in F, with at least one of them convex, then*

$$\Phi^{-1}(\mu(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\mu(A)) + (1-\lambda)\Phi^{-1}(\mu(B)) \text{ for } \lambda \in [0,1]. \tag{4}$$

For both sets A and B convex Ehrhard's inequality was proved in [9]. The generalization to the case when only one of the sets is convex was established in [16].

It is not hard to see that Theorem 3.1 implies the isoperimetric inequality (1). Indeed we have for any Borel set A in $\mathbb{R}^n$

$$\begin{aligned}\Phi^{-1}(\gamma_n(A_t)) &= \Phi^{-1}(\gamma_n(\lambda(\lambda^{-1}A) + (1-\lambda)((1-\lambda)^{-1}tB_2^n))) \\ &\geq \lambda\Phi^{-1}(\gamma_n(\lambda^{-1}A)) + (1-\lambda)\Phi^{-1}(\gamma_n((1-\lambda)^{-1}tB_2^n)) \stackrel{\lambda\to 1-}{\longrightarrow} \Phi^{-1}(\gamma_n(A)) + t.\end{aligned}$$

Conjecture 3.1 *Inequality (4) holds for any Borel sets in F.*

Ehrhard's symmetrization procedure enables us to reduce Conjecture 3.1 to the case $F = \mathbb{R}$ and $\mu = \gamma_1$. We may also assume that A and B are finite unions of intervals. At the moment the conjecture is known to hold when A is a union of at most 3 intervals.

Ehrhard's inequality has the following Prekopa-Leindler type functional version. Suppose that $\lambda \in (0,1)$ and $f, g, h : \mathbb{R}^n \to [0,1]$ are such that

$$\forall_{x,y\in\mathbb{R}^n}\ \Phi^{-1}(h(\lambda x + (1-\lambda)y)) \geq \lambda\Phi^{-1}(f(x)) + (1-\lambda)\Phi^{-1}(g(y))$$

then

$$\Phi^{-1}(\int_{\mathbb{R}^n} h d\gamma_n) \geq \lambda\Phi^{-1}(\int_{\mathbb{R}^n} f d\gamma_n) + (1-\lambda)\Phi^{-1}(\int_{\mathbb{R}^n} g d\gamma_n). \tag{5}$$

We use here the convention $\Phi^{-1}(0) = -\infty, \Phi^{-1}(1) = \infty$ and $-\infty + \infty = -\infty$. At the moment the above functional inequality is known to hold under the additional assumption that at least one of the functions $\Phi^{-1}(f), \Phi^{-1}(g)$ is convex. When one takes $f = 1_A$, $g = 1_B$ and $h = 1_{\lambda A + (1-\lambda)B}$ the inequality (5) immediately implies (4). On the other hand if we put $A = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \leq \Phi^{-1}(f(x))\}$ and $B = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \leq \Phi^{-1}(g(x))\}$ then $\lambda A + (1-\lambda)B \subset \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y \leq \Phi^{-1}(h(x))\}$, so Ehrhard's inequality in $\mathbb{R}^{n+1}$ implies (5) in $\mathbb{R}^n$. It is easy to show the inductive step in the proof of (5). Unfortunately the case $n = 1$ in the functional inequality seems to be much more complicated than the case $\mu = \gamma_1$ in Ehrhard's inequality.

4 Bobkov's Inequality

Isoperimetric inequality for the Lebesgue measure has an equivalent analytic form - the Sobolev inequality (cf. [25]). L. Gross [10] showed that the Gaussian measures γ_n satisfy the logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} g^2 \log g^2 d\gamma_n - \int_{\mathbb{R}^n} g^2 d\gamma_n \log(\int_{\mathbb{R}^n} g^2 d\gamma_n) \leq 2\int_{\mathbb{R}^n} |\nabla g|^2 d\gamma_n \tag{6}$$

for all smooth functions $g : \mathbb{R}^n \to \mathbb{R}$. Using the so-called Herbst argument one can show (cf. [19, Sect. 5.1]) that (6) implies the concentration inequality

$$\gamma_n(\{h \geq \int_{\mathbb{R}^n} h d\gamma_n + t\}) \leq e^{-t^2/2},\ t \geq 0$$

valid for all Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}$ with the Lipschitz seminorm $\|h\|_{\mathrm{Lip}} = \sup\{|h(x) - h(y)| : x, y \in \mathbb{R}^n\} \leq 1$. However the logarithmic Sobolev inequality does not imply the isoperimetric inequality.

The formulation of the functional form of Gaussian isoperimetry was given by S.G. Bobkov [2].

Theorem 4.1 *For any locally Lipschitz function $f : \mathbb{R}^n \to [0, 1]$ and $\mu = \gamma_n$ we have*

$$I(\int_{\mathbb{R}^n} f d\mu) \leq \int_{\mathbb{R}^n} \sqrt{I(f)^2 + |\nabla f|^2} d\mu. \tag{7}$$

Theorem 4.1 easily implies the isoperimetric inequality (2) by approximating the indicator function I_A by Lipschitz functions. On the other hand if we apply (2) to the set $A = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : \Phi(y) < f(x)\}$ in $\mathbb{R}^{n+1}$ we get (7). It is also not hard to derive the logarithmic Sobolev inequality (6) as a limit case of Bobkov's inequality (cf. [1]): one should use (7) for $f = \varepsilon g^2$ (with g bounded) and let ε tend to 0 ($I(t) \sim t\sqrt{2\log(1/t)}$ as $t \to 0+$).

The crucial point of the inequality (7) is its tensorization property. To state it precisely let us say that a measure μ on $\mathbb{R}^n$ *satisfies Bobkov's inequality* if the inequality (7) holds for all locally Lipschitz functions $f : \mathbb{R}^n \to [0, 1]$. Easy argument shows that if μ_i are measures on $\mathbb{R}^{n_i}$, $i = 1, 2$, that satisfy Bobkov's inequality then the measure $\mu_1 \otimes \mu_2$ also satisfies Bobkov's inequality.

The inequality (7) was proved by Bobkov in an elementary way, based on the following "two-point" inequality:

$$I(\frac{a+b}{2}) \leq \frac{1}{2}\sqrt{I(a)^2 + (\frac{a-b}{2})^2} + \frac{1}{2}\sqrt{I(b)^2 + (\frac{a-b}{2})^2} \tag{8}$$

valid for all $a, b \in [0, 1]$. In fact the inequality (8) is equivalent to Bobkov's inequality for $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and the discrete gradient instead of ∇f. Using the tensorization property and the central limit theorem Bobkov deduces (in the similar way as Gross in his proof of (6)) (7) from (8).

Using the co-area formula and Theorem 4.1 F. Barthe and M. Maurey [1] gave interesting characterization of all absolutely continuous measures that satisfy Bobkov's inequality.

Theorem 4.2 *Let $c > 0$ and μ be a Borel probability measure on the Riemannian manifold M, absolutely continuous with respect to the Riemannian volume. Then the following properties are equivalent*
(i) For every measurable $A \subset M$, $\mu^+(A) \geq cI(\mu(A))$;
(ii) For every locally Lipschitz function $f : M \to [0, 1]$

$$I(\int_M f d\mu) \leq \int_M \sqrt{I(f)^2 + \frac{1}{c^2}|\nabla f|^2} d\mu.$$

Theorem 4.2 together with the tensorization property shows that if $\mathrm{Is}(\mu_i) \geq cI$, $i = 1, 2 \ldots$, then also $\mathrm{Is}(\mu_1 \otimes \ldots \otimes \mu_n) \geq cI$. In general it is not known how to estimate $\mathrm{Is}(\mu_1 \otimes \ldots \otimes \mu_n)$ in terms of $\mathrm{Is}(\mu_i)$ even in the case when all μ_i's are equal (another important special case of this problem was solved in [3]) .

5 S-Inequality

In many problems arising in probability in Banach spaces one needs to estimate the measure of balls in some Banach space F. In particular one may ask what is the slowest possible grow of the Gaussian measure of balls in F or more general of some fixed convex symmetric closed set under dilations. The next theorem, proved by R. Latała and K. Oleszkiewicz [17], gives the positive answer to the conjecture posed in an unpublished manuscript of L. A. Shepp (1969).

Theorem 5.1(S-inequality) *Let μ be a centered Gaussian measure on a separable Banach space F. If A is a symmetric, convex, closed subset of F and $P \subset F$ is a symmetric strip, that is $P = \{x \in F : |x^*| \leq 1\}$ for some $x^* \in F^*$, such that $\mu(A) = \mu(P)$ then*

$$\mu(tA) \geq \mu(tP) \text{ for } t \geq 1$$

and

$$\mu(tA) \leq \mu(tP) \text{ for } 0 \leq t \leq 1.$$

A simple approximation argument shows that it is enough to prove Theorem 5.1 for $F = \mathbb{R}^n$ and $\mu = \gamma_n$. The case $n \leq 3$ was solved by V.N. Sudakov and V.A. Zalgaller [30]. Under the additional assumptions of symmetry of A in $\mathbb{R}^n$ with respect to each coordinate, Theorem 5.1 was proved by S. Kwapień and J. Sawa [15].

S-inequality can be equivalently expressed as

$$\Psi^{-1}(\mu(tA)) \geq t\Psi^{-1}(\mu(A)) \text{ for } t \geq 1,$$

where Ψ^{-1} denotes the inverse of

$$\Psi(x) = \gamma_1(-x, x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-y^2/2} dy.$$

The crucial tool in the proof of S-inequality is the new modified isoperimetric inequality. Let us first define for a convex symmetric set A in $\mathbb{R}^n$

$$w(A) = 2 \sup\{r : B(0, r) \subset A\}.$$

It is easy to see that for a symmetric strip P, $w(P)$ is equal to the width of P and for a symmetric convex set A

$$w(A) = \inf\{w(P) : A \subset P, P \text{ is a symmetric strip in } \mathbb{R}^n\}. \tag{9}$$

Thus $w(A)$ can be considered as the width of the set A. The following isoperimetric-type theorem holds true.

Theorem 5.2 *If $\gamma_n(A) = \gamma_n(P)$, where P is a symmetric strip and A is a convex symmetric set in $\mathbb{R}^n$, then*

$$w(A)\gamma_n^+(A) \geq w(P)\gamma_n^+(P). \tag{10}$$

The main advantage of the inequality (10) is that one may apply here the symmetrization procedure and reduce Theorem 5.2 to the similar statement for 2-dimensional convex sets symmetric with respect to some axis.

It is not hard to see that Theorem 5.2 implies Theorem 5.1. Indeed, let us define for any measurable set B in $\mathbb{R}^n$, $\gamma_B(t) = \gamma_n(tB)$ for $t > 0$. Taking the derivatives of both sides of the inequalities in Theorem 5.1 one can see that it is enough to show

$$\gamma_n(A) = \gamma_n(P) \Rightarrow \gamma_A'(1) \geq \gamma_P'(1) \tag{11}$$

for any symmetric convex closed set A and a symmetric strip $P = \{|x_1| \leq p\}$. Let $w = w(A)$, so $B(0,w) \subset A$. Then for $t > 1$ and $x \in A$ we have $B(t^{-1}x, (t-1)w/t) = t^{-1}x + (1-t^{-1})B(0,w) \subset A$, so $B(x,(t-1)w) \subset tA$. Hence $A_{(t-1)w} \subset tA$ and

$$\gamma_A'(1) \geq w\gamma_n^+(A) = w(A)\gamma_n^+(A).$$

However for the strip P

$$\gamma_P'(1) = \sqrt{\frac{2}{\pi}} p e^{-p^2/2} = w(P)\gamma_n^+(P)$$

and the inequality (11) follows by Theorem 5.2.

It is not clear if the convexity assumption for the set A in Theorem 5.2 is necessary (obviously $w(A)$ for nonconvex symmetric sets A should be defined by (9)). One may also ask if the symmetry assumption can be released (with the suitable modification of the definition of the width for nonsymmetric sets). Also functional versions of Theorems 5.1 and 5.2 are not known.

As was noticed by S. Szarek S-inequality implies the best constants in comparison of moments of Gaussian vectors (cf. [17]).

Corollary 5.3 *If X is a centered Gaussian vector in a separable Banach space $(F, \|\cdot\|)$ then*

$$(\mathbf{E}\|X\|^p)^{1/p} \leq \frac{c_p}{c_q}(\mathbf{E}\|X\|^q)^{1/q} \text{ for any } p \geq q \geq 0,$$

where

$$c_p = (\mathbf{E}|g_1|^p)^{1/p} = \sqrt{2}\left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\right)^{1/p}.$$

Another interesting problem connected with the S-inequality was recently posed by W. Banaszczyk (private communication): Is it true that under the assumptions of Theorem 5.1

$$\mu(s^\lambda t^{1-\lambda}A) \geq \mu(sA)^\lambda \mu(tA)^{1-\lambda},\ \lambda \in [0,1] \tag{12}$$

for any closed convex symmetric set A in F and $s, t > 0$? Combining the facts that the function $\Phi^{-1}(\mu(tA))$ is concave (Theorem 3.1) and the function $\frac{1}{t}\Psi^{-1}(\mu(tA))$ is nondecreasing (Theorem 5.1) one can show that (12) holds if $\mu(sA), \mu(tA) \geq c$, where $c < 0.85$ is some absolute constant.

It is of interest if Theorem 5.1 can be extended to the more general class of measures. The following conjecture seems reasonable.

Conjecture 5.1 *Let ν be a rotationally invariant measure on $\mathbb{R}^n$, absolutely continuous with respect to the Lebesgue measure with the density of the form $f(|x|)$ for some nondecreasing function $f : \mathbb{R}_+ \to [0, \infty)$. Then for any convex symmetric set A in $\mathbb{R}^n$ and any symmetric strip P in $\mathbb{R}^n$ such that $\nu(A) = \nu(P)$ the inequality $\nu(\lambda A) \geq \nu(\lambda P)$ is satisfied for $\lambda \geq 1$.*

To show Conjecture 5.1 it is enough to establish the following conjecture concerning the volumes of the convex hulls of symmetric sets on the $n-1$-dimensional unit sphere S^{n-1}.

Conjecture 5.2 *Let σ_{n-1} be a Haar measure on S^{n-1}, A be a symmetric subset of S^{n-1} and $P = \{x \in S^{n-1} : |x_1| \leq t\}$ be a symmetric strip on S^{n-1} such that $\sigma_{n-1}(A) = \sigma_{n-1}(P)$, then $\mathrm{vol}_n(\mathrm{conv}(A)) \geq \mathrm{vol}_n(\mathrm{conv}(P))$.*

It is known that both conjectures hold for $n \leq 3$ (cf. [30]).

6 Correlation Conjecture

The following conjecture is an object of intensive efforts of many probabilists since more then 30 years.

Conjecture 6.1 *If μ is a centered Gaussian measure on a separable Banach space F then*

$$\mu(A \cap B) \geq \mu(A)\mu(B) \tag{13}$$

for all convex symmetric sets A, B in F.

Various equivalent formulations of Conjecture 6.1 and history of the problem can be found in [27]. Standard approximation argument shows that it is enough to show (13) for $F = \mathbb{R}^n$ and $\mu = \gamma_n$. For $n = 2$ the solution was given by L. Pitt [26], for $n \geq 3$ the conjecture remains unsettled, but a variety of special results are known. Borell [8] established (13) for sets A, B in a certain class of (not necessary convex) sets in $\mathbb{R}^n$, which for $n = 2$ includes all symmetric sets. A special case of (13), when one of the sets A, B is a symmetric strip of the form $\{x \in F : |x^*(x)| \leq 1\}$ for some $x^* \in F^*$, was proved independently by C. G. Khatri [14] and Z. Šidák [28] (see [11] for an extension to elliptically contoured distributions and [31] for the case when one of the sets is a nonsymmetric strip). Recently, the Khatri-Šidák result has been generalized by G. Hargé [12] to the case when one of the sets is a symmetric ellipsoid.

Theorem 6.1 *If μ is a centered Gaussian measure on $\mathbb{R}^n$, A is a symmetric convex set in $\mathbb{R}^n$ and B is a symmetric ellipsoid, that is the set of the form $B = \{x \in \mathbb{R}^n : \langle Cx, x\rangle \leq 1\}$ for some symmetric nonnegative matrix C, then*

$$\mu(A \cap B) \geq \mu(A)\mu(B).$$

The following weaker form of (13)

$$\mu(A \cap B) \geq \mu(\lambda A)\mu(\sqrt{1-\lambda^2}B), \ 0 \leq \lambda \leq 1$$

was established for $\lambda = \frac{1}{\sqrt{2}}$ in [27] and for general λ in [21]. The Khatri-Šidák result and the above inequality turn out to be very useful in the study of the so-called small ball probabilities for Gaussian processes (see [22] for a survey of results in this direction).

The correlation conjecture has the following functional form:

$$\int fg d\mu \geq \int f d\mu \int g d\mu \tag{14}$$

for all nonnegative even functions f, g such that the sets $\{f \geq t\}$ and $\{g \geq t\}$ are convex for all $t \geq 0$. Y. Hu [13] showed that the inequality (14) (that we would like to have for log-concave functions) is valid for even convex functions $f, g \in L^2(F, \mu)$.

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