

# On the almost sure boundedness of norms of some empirical operators <sup>\*†</sup>

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## Abstract

Let  $X_1, X_2, \dots$  be i.i.d. random variables and  $h$  be a symmetric measurable real function. We show that the norms of operators on  $l_2^n$  given by the matrix  $(\frac{1}{n}h(X_i, X_j)\delta_{i \neq j})_{1 \leq i, j \leq n}$  are a.s. bounded if and only if  $h$  is square integrable.

**Introduction.** Let  $(E, \mathcal{F}, \mu)$  be a probability space,  $\mu_2 = \mu \otimes \mu$ ,  $h : E \times E \rightarrow \mathbb{R}$  be a symmetric measurable function and let  $H$  be an operator on  $L_2(\mu) = L_2(E, \mathcal{F}, \mu)$  defined by the formula

$$Hg(x) = \int_E h(x, y)g(y)d\mu(y), \quad g \in L_2(\mu), x \in E.$$

Let  $X_1, Y_1, X_2, Y_2, \dots$  be i.i.d.  $(\mu)$  random variables with values in  $E$ . We may then define random operators  $H_n$  on  $l_2^n$ , defining their matrices in canonical basis by

$$H_n = \left(\frac{1}{n}\delta_{i \neq j}h(X_i, X_j)\right)_{1 \leq i, j \leq n}.$$

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Operators  $H_n$  can be seen as natural empirical counterparts to  $H$ , as it was explained in Kolchinski and Giné (1995).

In their paper V. Kolchinsky and E. Giné proved that if  $h \in L_2(\mu_2)$  (so in other words if  $H$  is a Hilbert-Schmidt operator) then the spectrum of  $H$  may be a.s. approximated in very strong sense by the spectrum of  $H_n$  (e.g.  $\|H_n\|_{HS} \rightarrow \|H\|_{HS}$  a.s.). In particular  $\sup \|H_n\| < \infty$  a.s.

They posed also the question if the converse is true - namely if  $\sup \|H_n\| < \infty$  a.s. imply that  $h \in L^2(\mu_2)$ . In this paper we give a positive answer to this question.

**Lemma 1** *Let  $Z_1, Z_2, \dots$  be a sequence of i.i.d. nonnegative random variables such that  $Z_i \leq 4n^2$  a.s. and  $EZ_i \geq n$ . Then*

$$P\left(\sum_{i=1}^n Z_i \geq \frac{1}{2}n^2\right) \geq \frac{1}{20}$$

**Proof.** We have

$$E\left(\sum_{i=1}^n Z_i\right)^2 \leq \left(\sum_{i=1}^n EZ_i\right)^2 + \sum_{i=1}^n EZ_i^2 \leq \left(\sum_{i=1}^n EZ_i\right)^2 + 4n^2\left(\sum_{i=1}^n EZ_i\right) \leq 5\left(\sum_{i=1}^n EZ_i\right)^2.$$

Hence by Paley-Zygmund inequality (cf. Kahane (1985), p.8)

$$\begin{aligned} P\left(\sum_{i=1}^n Z_i \geq \frac{1}{2}n^2\right) &\geq P\left(\sum_{i=1}^n Z_i \geq \frac{1}{2}\left(E\sum_{i=1}^n Z_i\right)\right) \geq \\ &\frac{1}{4} \frac{\left(E\sum_{i=1}^n Z_i\right)^2}{E\left(\sum_{i=1}^n Z_i\right)^2} \geq \frac{1}{20}. \end{aligned}$$

**Lemma 2** *Suppose that a measurable set  $A \subset E \times E$  satisfy the following conditions*

$$\forall_{x \in E} \mu(A_x) \leq n^{-1} \tag{1}$$

and

$$\forall_{y \in E} \mu(A^y) \leq n^{-1}. \tag{2}$$

Then

$$P(\exists_{1 \leq i, j \leq n} (X_i, Y_j) \in A) \geq \frac{1}{16} \min(1, n^2 \mu_2(A)).$$

**Proof.** Obviously we may assume that  $\mu_2(A) > 0$ . Let us first notice that

$$P(\exists_{1 \leq i, j \leq n} (X_i, Y_j) \in A) = P\left(\sum_{1 \leq i, j \leq n} I_A(X_i, Y_j) > 0\right).$$

We have

$$E \sum_{1 \leq i, j \leq n} I_A(X_i, Y_j) = n^2 E I_A(X_1, Y_1) = n^2 \mu_2(A)$$

and

$$\begin{aligned} E\left(\sum_{1 \leq i, j \leq n} I_A(X_i, Y_j)\right)^2 &= n^2 E I_A(X_1, Y_1) + n^2(n-1)^2 E I_A(X_1, Y_1) I_A(X_2, Y_2) \\ &\quad + n^2(n-1) E I_A(X_1, Y_1) I_A(X_1, Y_2) + n^2(n-1) E I_A(X_1, Y_1) I_A(X_2, Y_1). \end{aligned}$$

But

$$\begin{aligned} E I_A(X_1, Y_1) I_A(X_2, Y_2) &= \mu_2^2(A), \\ E I_A(X_1, Y_1) I_A(X_1, Y_2) &= E \mu^2(A_{X_1}) \leq n^{-1} E \mu(A_{X_1}) = n^{-1} \mu_2(A) \end{aligned}$$

and

$$E I_A(X_1, Y_1) I_A(X_2, Y_1) = E \mu^2(A^{Y_1}) \leq n^{-1} E \mu(A^{Y_1}) = n^{-1} \mu_2(A).$$

Hence

$$E\left(\sum_{1 \leq i, j \leq n} I_A(X_i, Y_j)\right)^2 \leq 3n^2 \mu_2(A) + n^4 \mu_2^2(A).$$

So by Paley-Zygmund inequality

$$\begin{aligned} P\left(\sum_{1 \leq i, j \leq n} I_A(X_i, Y_j) \geq \frac{1}{2} E \sum_{1 \leq i, j \leq n} I_A(X_i, Y_j)\right) &\geq \frac{1}{4} \frac{(E \sum_{1 \leq i, j \leq n} I_A(X_i, Y_j))^2}{E(\sum_{1 \leq i, j \leq n} I_A(X_i, Y_j))^2} \\ &\geq \frac{1}{16} \min(1, n^2 \mu_2(A)). \end{aligned}$$

**Theorem 1** *Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables with values in  $(E, \mathcal{F})$  and  $h$  be a symmetric measurable function on  $E \times E$ . If a sequence*

$$\left\| \frac{1}{n} (h(X_i, X_j) \delta_{i \neq j})_{1 \leq i, j \leq n} \right\|_{l_n^2 \rightarrow l_n^2}$$

*is a.s. bounded then*

$$E h^2(X_1, X_2) < \infty.$$

**Proof.** Let  $Y_1, Y_2, \dots$  be an independent copy of  $X_i$ . Let  $I_l = \{2^l + 1, 2^l + 2, \dots, 2^{l+1}\}$ . Let us notice that

$$\|2^{-l}(h(X_i, X_j)\delta_{i \neq j})_{1 \leq i, j \leq 2^l}\|_{l^2_{2^l} \rightarrow l^2_{2^l}} \geq \|2^{-l}(h(X_i, X_j)\delta_{i \neq j})_{i, j \in I_{l-1}}\|_{l^2_{2^{l-1}} \rightarrow l^2_{2^{l-1}}}$$

and random variables

$$\|2^{-l}(h(X_i, X_j)\delta_{i \neq j})_{i, j \in I_{l-1}}\|_{l^2_{2^{l-1}} \rightarrow l^2_{2^{l-1}}}, \quad l = 1, 2, \dots$$

are independent. Hence by Borel-Cantelli Lemma we get that for some  $C < \infty$

$$\sum_{l=1}^{\infty} P(\|2^{-l}(h(X_i, X_j)\delta_{i \neq j})_{i, j \in I_{l-1}}\|_{l^2_{2^{l-1}} \rightarrow l^2_{2^{l-1}}} \geq C) < 2^{-4}.$$

But

$$\begin{aligned} \|(h(X_i, X_j)\delta_{i \neq j})_{i, j \in I_l}\|_{l^2_{2^l} \rightarrow l^2_{2^l}} &\geq \max_{i \in I_l} \left( \sum_{j \in I_l, j \neq i} h^2(X_i, X_j) \right)^{1/2} \geq \\ &\max_{2^l < i \leq 2^{l+2^{l-1}}} \left( \sum_{j=2^l+2^{l-1}+1}^{2^{l+1}} h^2(X_i, X_j) \right)^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} P(\|2^{-l-2}(h(X_i, X_j)\delta_{i \neq j})_{i, j \in I_{l+1}}\|_{l^2_{2^{l+1}} \rightarrow l^2_{2^{l+1}}} \geq C) \\ \geq P(\max_{1 \leq i \leq 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \geq C^2 2^{2l+4}). \end{aligned}$$

We may obviously assume that  $C^2 = 2^{-5}$  and hence get

$$\sum_{l=1}^{\infty} P(\max_{1 \leq i \leq 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \geq 2^{2l-1}) < 2^{-4}.$$

Let us define the following sets

$$\begin{aligned} A_l &= \{(x, y) : 2^{2l+2} > h^2(x, y) \geq 2^{2l}\}, \\ B_l &= \{(x, y) \in A_l : \mu((A_l)_x) \geq 2^{-l}\}, \\ C_l &= \{(x, y) \in A_l : \mu((A_l)_y) \geq 2^{-l}\} \end{aligned}$$

and

$$D_l = A_l \setminus (B_l \cup C_l).$$

Then

$$\forall_{x,y \in E} \mu((D_l)_x), \mu((D_l)_y) \leq 2^{-l}. \quad (3)$$

Let

$$H_1(x, y) = \sum_{l=1}^{\infty} h^2(x, y) I_{\{(x,y) \in B_l\}},$$

$$H_2(x, y) = \sum_{l=1}^{\infty} h^2(x, y) I_{\{(x,y) \in C_l\}}$$

and

$$H_3(x, y) = \sum_{l=1}^{\infty} h^2(x, y) I_{\{(x,y) \in D_l\}}.$$

Then

$$Eh^2(X, Y) \leq 4 + EH_1(X, Y) + EH_2(X, Y) + EH_3(X, Y)$$

and by the symmetry of  $h^2$  we have  $EH_1(X, Y) = EH_2(X, Y)$ .

Let us notice that by the definition of  $A_l$

$$EH_3(X, Y) \leq \sum_{l=1}^{\infty} 2^{2l+2} \mu(D_l).$$

By Lemma 2 and (3)

$$2^{-4} \min(1, 2^{2l} \mu(D_l)) \leq P(\exists_{1 \leq i, j \leq 2^l} (X_i, Y_j) \in D_l) \leq P(\max_{1 \leq i, j \leq 2^l} h^2(X_i, Y_j) \geq 2^{2l}) \leq P(\max_{1 \leq i \leq 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \geq 2^{2l}).$$

Hence

$$EH_3(X, Y) \leq 2^6 \sum_{l=1}^{\infty} P(\max_{1 \leq i \leq 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \geq 2^{2l-1}) < \infty.$$

To estimate  $EH_1(X, Y)$  let us define

$$h_1(x) = E_Y H_1(x, Y)$$

and

$$V_l = \{x : 2^l \leq h_1(x) < 2^{l+1}\}.$$

Let us notice that by our definition of  $H_1(x, y)$  if for some  $x, y$  we have  $2^{2k} \leq H_1(x, y) < 2^{2k+2}$  then  $P_Y(H_1(x, Y) \geq 2^{2k}) \geq 2^{-k}$ , so  $h_1(x) \geq 2^k$ . Therefore

$$H_1(x, y) \leq 2^{2l+2} \text{ for } x \in V_l.$$

Now by Lemma 1 we obtain that

$$P_Y\left(\sum_{j=1}^{2^l} H_1(x, Y_j) \geq 2^{2l-1}\right) \geq \frac{1}{20} \text{ for } x \in V_l.$$

Since  $H_1 \leq h^2$  and

$$P(\exists_{1 \leq i \leq 2^l} X_i \in V_l) \geq \frac{1}{2} \min(1, 2^l \mu(V_l))$$

we get that

$$P\left(\max_{1 \leq i \leq 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \geq 2^{2l-1}\right) \geq \frac{1}{40} \min(1, 2^l \mu(V_l)).$$

But this means that

$$EH_1(X, Y) = Eh_1(X) \leq 2 + \sum_{l=1}^{\infty} 2^{l+1} \mu(V_l) \leq$$

$$2 + 80P\left(\max_{1 \leq i \leq 2^l} \sum_{j=1}^{2^l} h^2(X_i, Y_j) \geq 2^{2l-1}\right) < \infty.$$

From Theorem 1 and the results of Kolchinsky and Giné (1995) immediately follows the following Corollary, which may be seen as some type of law of large numbers for  $U$ -statistics.

**Corollary 1** *If  $h \in L^2(\mu_2)$  then  $\lim_{n \rightarrow \infty} \|H_n\| = \|H\|$  a.s.. Conversely if  $\limsup_{n \rightarrow \infty} \|H_n\| < \infty$  a.s. then  $h \in L^2(\mu_2)$ .*

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